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Concerning Transformations of Bases Associated with Unimodular diag$(1,-1,-1)$-Matrices

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Abstract: Considering a representation space for a group of unimodular diag$(1,-1,-1)$-matrices, we construct several bases whose elements are eigenfunctions of Casimir infinitesimal operators related to a reduction in the group to some one-parameter subgroups. Finding the kernels of base transformation integral operators in terms of special functions, we consider the compositions of some of these transformations. Since composition is a 'closed' operation on the set of base transformations, we obtain some integral relations for the special functions involved in the above kernels.

Keywords: Lie group; Casimir operators; eigenfunctions of Casimir infinitesimal operators; unimodular diag$(1,-1,-1)$-matrices; reduction of group; integral operators; composition of transformations; integral relations for special functions

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1. Introduction

Consider two distinct orthonormal bases on the Cartesian plane, denoted by $\{\overrightarrow{OA_1}, \overrightarrow{OB_1}\}$ and $\{\overrightarrow{OA_2}, \overrightarrow{OB_2}\}$, where O represents the origin. The composition of the basis transformations $\{\overrightarrow{OA_1}, \overrightarrow{OB_1}\} \longrightarrow \{\overrightarrow{OA_2}, \overrightarrow{OB_2}\}$ and $\{\overrightarrow{OA} = (1,0), \overrightarrow{OB} = (0,1)\} \longrightarrow \{\overrightarrow{OA_1}, \overrightarrow{OB_1}\}$ can be expressed as the multiplication of two 2×2-orthogonal matrices. This leads to the addition theorem for cosine and sine, as the matrix representing this composition concerning the basis $\{\overrightarrow{OA}, \overrightarrow{OB}\}$ is itself an orthogonal matrix. The kernels of the basis transformation integral operators in infinite-dimensional functional linear spaces involve intricate functions known as special functions in mathematical physics. Since these special functions are eigenfunctions of differential operators which are invariant under the associated Lie groups, we have a direct connection between these special functions, which constitute the kernels, and the representation of the corresponding Lie group.

In this manuscript, we establish several bases within a functional linear space in Section 3, analyze the kernels of basis transformation operators in Section 4, and derive integral relationships pertaining to specific instances of confluent hypergeometric functions ${}_1F_1$ in Section 5. These functions include Bessel $J_\nu(x)$, Hankel $H^{(1)}_\mu(x)$, and $H^{(2)}_\mu(x)$, Whittaker $W_{\mu,\nu}(x)$, Macdonald $K_\nu(x)$, and Coulomb $F_\mu(\rho;x)$ functions. In [1,2], we used another approach investigating the kernels of the restriction of representation integral operators to certain one-parameter groups expressed in various 'direct' or 'mixed' bases, thereby unveiling additional integral relationships. Let us remember that a group-theoretical approach to classical Bessel functions (in a wide sense) had been considered in monographs [3,4] and to some their multi-variable or multi-index analogues and generalizations have been presented, for example, in [5–7].

2. The Group G_0, Algebra $\mathfrak{g}$, and Space $\mathscr{L}$

Let a be an arbitrary square matrix. Let us call a matrix b of the same size an a-matrix if $b^{\mathrm{T}}ab = a$, where T represents the transpose of the matrix. For the case $\det a \neq 0$, we immediately obtain $|\det b| = 1$. By selecting different matrices denoted as a, we can derive diverse sets of well-known matrix classes: orthogonal ($a = \mathrm{diag}(1, \ldots, 1)$), symplectic, etc. The equalities

$$(b\hat{b})^{\mathrm{T}}a(b\hat{b}) = \hat{b}^{\mathrm{T}}(b^{\mathrm{T}}ab)\hat{b} = \hat{b}^{\mathrm{T}}a\hat{b} = a,$$
$$(b^{-1})^{\mathrm{T}}ab^{-1} = (b^{-1})^{\mathrm{T}}(b^{\mathrm{T}}ab)b^{-1} = (bb^{-1})^{\mathrm{T}}a(bb^{-1}) = a$$

indicate that the set of a-matrices form a subgroup $O(1,2)$ within the general linear group of order n, where $n \times n$ represents the dimensions of matrix a. Let the symbol G represent the intersection of $O(1,2)$ and the special linear subgroup $SL(3,\mathbb{R})$. From the given definition, it can be deduced that for any $b \in G$, the following equations hold:

$$b_{1i}b_{1j} - b_{2i}b_{2j} - b_{3i}b_{3j} = \mathrm{sign}\left(\frac{3}{2} - j\right)\delta_{ij}, \tag{1}$$
$$b_{i1}b_{j1} - b_{i2}b_{j2} - b_{i3}b_{j3} = \mathrm{sign}\left(\frac{3}{2} - j\right)\delta_{ij}. \tag{2}$$

Here, δ_{ij} is Kronecker delta: 0 if $i \neq j$; 1 if $i = j$.

It is demonstrable [8] that the expression for matrix b is given by:

$$b = \begin{pmatrix} b_{11} & A \\ B & C \end{pmatrix}, \tag{3}$$

where $A = C_1C_2$, matrix C_1 is orthogonal and C_2 ia a positive–definite matrix. The matrix b relies on b_{11} and independent parameters of C_1 and A (or B), therefore being defined by three parameters. From Equation (3) (see [8]), it is deduced that the determinant of C equals b_{11}. This implies that the mapping $\iota:\ G \longrightarrow \mathbb{U}_2 = \{1,-1\}$ defined by the formula

$$\iota(b) = \begin{cases} 1 & \text{if } b_{11} > 0, \\ -1 & \text{if } b_{11} < 0, \end{cases}$$

is a group epimorphism. Both cosets of the normal divisor $G_0 = \mathrm{Ker}\,\iota$ represent connected components in G.

Let $h_1(\varphi)$ be the matrix of the circle rotation in the plane Ox_2x_3 through angle φ:

$$h_1(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

Let $h_2(\varphi)$ and $h_3(\varphi)$ denote matrices of hyperbolic rotations in the planes Ox_1x_2 and Ox_1x_3, respectively, that is

$$h_2(\varphi) = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 \\ \sinh\varphi & \cosh\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_3(\varphi) = \begin{pmatrix} \cosh\varphi & 0 & \sinh\varphi \\ 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \end{pmatrix}.$$

Obviously, $h_1, h_2, h_3 \in G_0$. The vectors $e_{2,3} = \frac{\mathrm{d}h_1}{\mathrm{d}\varphi}|_{\varphi=0}$, $e_2 = \frac{\mathrm{d}h_2}{\mathrm{d}\varphi}|_{\varphi=0}$, and $e_3 = \frac{\mathrm{d}h_3}{\mathrm{d}\varphi}|_{\varphi=0}$ constitute a basis E for the tangent space of the group G_0, evaluated at the point id. The commuting relations of the corresponding Lie algebra $\mathfrak{g}$ can be expressed as follows:

$$[e_{2,3}, e_2] = e_3, \quad [e_{2,3}, e_3] = -e_2, \quad [e_2, e_3] = -e_{2,3}. \tag{4}$$

The matrices h_i (for each i) constitute a subgroup H_i within G_0. It is evident that the group G acts transitively on both the cone $X_0: \ x_1^2 - x_2^2 - x_3^2 = 0$ and the hyperboloid $X_1: \ x_1^2 - x_2^2 - x_3^2 = 1$.

Lemma 1. $G_0 = H_1 H_2 H_1 = H_1 H_3 H_1$.

Proof. Let $\operatorname{St}\tilde{y} \prec G$ be the stabilizer of the point $\tilde{y} = (1,0,0) \in X_1$. Given that for any $g \in \operatorname{St}\tilde{y}$, the equality $g_{11} = 1$ holds, based on (1) and (2), we can express g as:

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{pmatrix},$$

where $\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{pmatrix}$ is an orthogonal matrix. Thus, $\operatorname{St}\tilde{y} = H_1$. Let τ be the bijective mapping $G/H_1 \longrightarrow X_1$, such that $\tau(\tilde{g}H_1) = y$, where y is the image of $\tilde{y}$ for any transformation belonging to the coset $\tilde{g}\operatorname{St}\tilde{y}$. If $g \in G_0$, $g \in \tilde{g}\operatorname{St}\tilde{y}$ and $\tau(\tilde{g}\operatorname{St}\tilde{y}) = y$, then $g = \tilde{g}h_1$. Because g relies on three parameters and H_1 is a one-parameter subgroup, the generator $\tilde{g}$ of the coset $\tilde{g}\operatorname{St}\tilde{y}$ ought to be contingent upon two parameters. Let $y = (\cosh\xi, \sinh\xi\cos\mu, \sinh\xi\sin\mu)$. Let us show that $\tilde{g}$ can be written in the form $\tilde{g} = h_1(\mu)h_2(\xi)$:

$$h_2(\xi)\tilde{y} = (\cosh\xi, \sinh\xi, 0),$$
$$h_1(\mu)(\cosh\xi, \sinh\xi, 0) = y.$$

Therefore, $g = \tilde{g}h_1 = h_1(\mu)h_2(\xi)h_1(\nu)$. The second equality of the present lemma can be demonstrated using the same method. □

We denote by X_0^+ the subset of the cone X_0 that comprises points x, where $x_1 > 0$.

Lemma 2. *The semicone X_0^+ is invariant under the transformations of the group G_0.*

Proof. Based on Lemma 1, it is enough to confirm for h_1 and h_2. For $x \in X_0^+$ we have $x_1 > 0$ and $x_1^2 = x_2^2 + x_3^2$, therefore, $x_1 > |x_2|$. Since

$$h_1(\varphi)x = (x_1, x_2\cos\varphi - x_3\sin\varphi, x_2\sin\varphi + x_3\cos\varphi),$$

we find that $h_1(\varphi)x \in X_0^+$. Given that

$$h_1(\varphi)x = (x_1\cosh\varphi + x_2\sinh\varphi, x_1\sinh\varphi + x_2\cosh\varphi, x_3),$$

where $\cosh\varphi > |\sinh\varphi|$, we can derive that $x_1\cosh\varphi + x_2\sinh\varphi > 0$. This implies that $g_2(\varphi)x \in X_0^+$. □

Lemma 3. *The group G_0 acts transitively on the semicone X_0^+.*

Proof. We denote by γ_1 the circle $x_1 = 1$ belonging to X_0^+. Introducing polar coordinates on γ_1, we write any point $x \in X_0^*$ in the form

$$x = (r, r\cos\alpha, r\sin\alpha), \tag{5}$$

where $r > 0$. Since

$$h_1\left(\frac{3\pi}{2} + \alpha\right) h_3(\ln r)\,(1,0,1) = h_1\left(\frac{3\pi}{2} + \alpha\right)(r,0,r) = x, \tag{6}$$

and

$$h_1\left(\frac{3\pi}{2}+\alpha\right)h_3(\ln r)=\begin{pmatrix}\cosh\ln r & \dots & \dots\\ \dots & \dots & \dots\\ \dots & \dots & \dots\end{pmatrix},$$

in view of $\cosh\ln r>0$ the transformation $g=h_1\left(\frac{3\pi}{2}+\alpha\right)h_3(\ln r)$ belongs to G_0.

Let $\tilde{x}\in X_0^+$. It has been demonstrated that there exists a transformation denoted as $\tilde{g}\in G_0$, such that when applied, it satisfies the condition $\tilde{g}\,(1,0,0)=\tilde{x}$. We thus have the equality

$$(1,0,0)=\tilde{g}^{-1}\,\tilde{x}. \tag{7}$$

By substituting (7) into (6), the resultant equation becomes $g\tilde{g}^{-1}\,\tilde{x}=x$. $\square$

Let f be a function defined on the semicone X_0^+. Let us call this function infinitely differentiable, if the derivative $\frac{\partial^{k_1+k_2+k_3}f}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial x_3^{k_3}}$ exists at any point of X_0^+ and for any nonnegative integers k_1, k_2, k_3 such that $(k_1,k_2,k_3)\neq(0,0,0)$. Let $\sigma\in\mathbb{C}$. We call a function f σ-homogeneous if $f(\lambda x)=\lambda^\sigma f(x)$. Given $x_1>0$, this implies that $\lambda>0$.

Let $\mathscr{L}$ represent the linear space comprising σ-homogeneous infinitely differentiable functions on X_0^+. It can be readily verified that the mapping $T(g):\mathscr{L}\longrightarrow\mathscr{L}$, defined as $f\longmapsto f(gx)$, constitutes an automorphism of $\mathscr{L}$. Moreover, the function $G_0\longrightarrow\operatorname{Aut}\mathscr{L}$, where $g\longmapsto T(g)$, forms a representation of G_0.

3. Construction of Bases

For each vector belonging to the above basis E of tangent algebra $\mathfrak{g}$, we define the corresponding infinitesimal operator:

$$\mathfrak{d}_{2,3}=\mathbf{i}\lim_{t\to0}\frac{T\big(\exp(te_{2,3})\big)f(x)-f(x)}{t},\quad \mathfrak{d}_i=\mathbf{i}\lim_{t\to0}\frac{T\big(\exp(te_i)\big)f(x)-f(x)}{t}.$$

In this context and throughout, the symbol $\mathbf{i}$ denotes the purely imaginary number, which is the square root of -1. It is easy to find that

$$\mathfrak{d}_{2,3}=\mathbf{i}\left(x_2\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_2}\right),\ \ \mathfrak{d}_i=\mathbf{i}\left(x_1\frac{\partial}{\partial x_i}+x_i\frac{\partial}{\partial x_1}\right). \tag{8}$$

In polar coordinates on γ_1 we have $\mathfrak{d}_{2,3}=\mathbf{i}\frac{\mathrm{d}}{\mathrm{d}\alpha}$. Let $f_\downarrow(\alpha)$ be an eigenfunction of the operator $\mathfrak{d}_{2,3}$ with respect to the eigenfunction λ and, in addition, a restriction of $f\in\mathscr{L}$ to γ_1. From the equation $\mathbf{i}\frac{\mathrm{d}f_\downarrow}{\mathrm{d}\alpha}=\lambda\,f_\downarrow$ we have $f_\downarrow\in\operatorname{Span}\big(\mathrm{e}^{-\mathbf{i}\lambda\alpha}\big)$. From the condition that $f_\downarrow(\pi)=f_\downarrow(-\pi)$, we can deduce $\mathrm{e}^{-2\mathbf{i}\lambda\alpha}=1$, implying that $\lambda\in\mathbb{Z}$. By introducing $n=-\lambda$, we derive the basis within the space of function restrictions from $\mathscr{L}$ to γ_1, comprising a set of functions $\mathrm{e}^{\mathbf{i}n\alpha}$. Considering the σ-homogeneity property for any $x\in X_0^+$:

$$f(x)=x_1^\sigma\,f_\downarrow\left(1,\frac{x_2}{x_1},\frac{x_3}{x_1}\right),$$

therefore, the functions

$$f_n^{(1)}(x)=x_1^\sigma\,\mathrm{e}^{\mathbf{i}n\alpha}=x_1^\sigma\left(\frac{x_2}{x_1}+\mathbf{i}\frac{x_3}{x_1}\right)^n=x_1^{\sigma-n}\,(x_2+\mathbf{i}x_3)^n,$$

form the basis B_1 in the space $\mathscr{L}$.

We denote the hyperbola $x_3=\pm1$ on X_0^+ by symbol $\gamma_{2,\pm}$. Let $\gamma_2=\gamma_{2,+}\cup\gamma_{2,-}$. In hyperbolic coordinates

$$x_1=\cosh\alpha,\quad x_2=\sinh\alpha,\quad x_3=\pm1 \tag{9}$$

on γ_2, where $\alpha \in \mathbb{R}$, we have $\mathfrak{d}_2 = \mathbf{i}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}$. Let $f_\downarrow$ be an eigenfunction of $\mathfrak{d}_2$ with respect to value λ and a restriction of $f \in \mathscr{L}$ to γ_2. From the equality $\mathbf{i}\frac{\mathrm{d}f_\downarrow}{\mathrm{d}\alpha} = \lambda f_\downarrow$ we have $f_\downarrow \in \mathrm{Span}\big(\mathrm{e}^{-\mathbf{i}\lambda\alpha}\big)$, therefore, $\mathrm{e}^{\mathbf{i}\lambda\alpha}$, $\lambda \in \mathbb{R}$, form a basis in the space of function restrictions from $\mathscr{L}$ to γ_2.

Let us consider the value of f at the point $x \in X_0^+$ such that $x_3 \neq 0$. In case of $x_3 > 0$ we have

$$f(x) = x_3^\sigma\, f_{\downarrow,+}\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right),$$

where $f_{\downarrow,+}$ is the restriction to $\gamma_{2,+}$. In case of $x_3 < 0$ we can write

$$f(x) = |x_3|^\sigma\, f_{\downarrow,-}\left(\frac{x_1}{|x_3|}, \frac{x_2}{|x_3|}, -1\right).$$

It implies that

$$f(x) = |x_3|^\sigma\, \delta_{1,\mathrm{sign}\, x_3} \cdot f_{\downarrow,+}\left(\frac{x_1}{|x_2|}, \frac{x_2}{|x_3|}, 1\right) + |x_3|^\sigma\, \delta_{-1,\mathrm{sign}\, x_3}\, f_{\downarrow,-}\left(\frac{x_1}{|x_3|}, \frac{x_2}{|x_3|}, -1\right).$$

Using the generalized functions [9],

$$(s)_\pm^\nu = \begin{cases} |s|^\nu & \text{if } \pm s \geqslant 0, \\ 0 & \text{if } \pm s < 0, \end{cases}$$

we obtain

$$f(x) = (x_3)_\pm^\sigma\, f_{\downarrow,\pm}\left(\frac{x_1}{|x_3|}, \frac{x_2}{|x_3|}, \pm 1\right)$$

(double signs $\pm$ are consistently employed in same order here and in other instances). This implies that functions

$$\begin{aligned} f_{\lambda,\pm}^{(2)}(x) &= (x_3)_\pm^\sigma\, \mathrm{e}^{\mathbf{i}\lambda\alpha} = (x_3)_\pm^\sigma\, (\cosh\alpha + \sinh\alpha)^{\mathbf{i}\lambda} \\ &= (x_3)_\pm^\sigma \left(\frac{x_1}{|x_3|} + \frac{x_2}{|x_3|}\right)^{\mathbf{i}\lambda} = (x_3)_\pm^{\sigma - \mathbf{i}\lambda}\, (x_1 + x_2)^{\mathbf{i}\lambda} \end{aligned}$$

form a basis B_2 in the space $\mathscr{L}$.

By analogy, defining hyperbolas $\gamma_{3,\pm} :\ x_2 = \pm 1$, we obtain a basis B_3 in $\mathscr{L}$, consisting of functions

$$f_{\lambda,\pm}^{(3)}(x) = (x_2)_\pm^{\sigma - \mathbf{i}\lambda}\, (x_1 + x_3)^{\mathbf{i}\lambda}, \ \ \lambda \in \mathbb{R},$$

related to the contour $\gamma_3 = \gamma_{3,+} \cup \gamma_{3,-}$ on the semicone X_0^+.

Let us define the linear subspaces $\mathfrak{k} = \mathrm{Span}(e_{2,3})$ and $\mathfrak{p} = \mathrm{Span}(e_2, e_3)$ in $\mathfrak{g}$. As per (8), it follows that $\mathfrak{so}(1,2) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

In view of relations (8), the dimension of maximal commutative subalgebra $\mathfrak{a}$ in $\mathfrak{g}$ is equal to 1. Letting $\mathfrak{a} = \mathrm{Span}(e_2)$, we get the following matrix of the adjoint operator $\mathrm{ad}\, e_2$ in the above basis E:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finding the eigenvalues of the characteristic polynomial of this matrix, we obtain

$$\mathfrak{so}(1,2) = \mathrm{Ker}\, \mathrm{ad}\, e_2 + V_1 + V_{-1},$$

where the root linear subspace V_j consists of a zero vector and all eigenvectors of the operator $\operatorname{ad} e_2$ related to the value j. This implies that the maximal nilpotent subalgebra $\mathfrak{n}$ in $\mathfrak{g}$ can be defined by the equality $\mathfrak{n} = V_1$, that is, $\mathfrak{n} = \operatorname{Span}(e_{2,3} + e_3)$.

Let us define the subgroup $H_4 = \exp \mathfrak{n} = \{\exp\big(t(e_{2,3} + e_3)\big)\} = \{h_4(t)\}$ in G_0. It consists of matrices

$$\begin{aligned} h_4(t) =& \operatorname{diag}(1,1,1) + t\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \\ &+ \frac{t^2}{2!}\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^3}{3!}\operatorname{diag}(0,0,0) = \frac{1}{2}\begin{pmatrix} 2+t^2 & t^2 & 2t \\ -t^2 & 2-t^2 & -2t \\ 2t & 2t & 2 \end{pmatrix} \end{aligned}$$

and acts transitively on the intersection of the semicone X_0^+ and the plane $x_1 + x_2 = 1$. We denote this parabola by γ_4.

Letting $\mathfrak{a} = \operatorname{Span}(e_3)$, by analogy, we obtain $\mathfrak{n} = \operatorname{Span}(e_{2,3} + e_2)$ and find its exponential image H_5 consisting of matrices

$$\begin{aligned} h_5(t) =& \operatorname{diag}(1,1,1) + t\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ &+ \frac{t^2}{2!}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} + \frac{t^3}{3!}\operatorname{diag}(0,0,0) = \frac{1}{2}\begin{pmatrix} 2+t^2 & 2t & -t^2 \\ 2t & 2 & -2t \\ t^2 & 2t & 2-t^2 \end{pmatrix}. \end{aligned}$$

We denote by γ_5 the intersection of the semicone X_0^+ and plane $x_1 + x_3 = 1$. This parabola is a homogeneous space of the subgroup H_5.

The infinitesimal operator $\mathfrak{c} = \mathfrak{d}_{2,3} + \mathfrak{d}_3$ associated with the generator $e_{2,3} + e_3$ within a one-dimensional subalgebra $\mathfrak{n}$ exhibits commutativity with infinitesimal operators associated with all vectors within $\mathfrak{n}$. Thus, it qualifies as a Casimir operator linked to the reduction $H_4 \subset G_0$.

In horospherical coordinates

$$x_1 = \frac{1+\alpha^2}{2}, \quad x_2 = \frac{1-\alpha^2}{2}, \quad x_3 = \alpha \tag{10}$$

on the parabola γ_4, where $\alpha \in \mathbb{R}$, we have $\mathfrak{c} = \mathbf{i}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}$.

Let us denote by $f_\downarrow$ an eigenfunction of the operator $\mathfrak{c}$ with the eigenvalue λ. Suppose that $f_\downarrow$ is the restriction of function $f \in \mathscr{L}$ to parabola γ_4. From the equation $\mathbf{i}\frac{\mathrm{d}f_\downarrow}{\mathrm{d}\alpha} = \lambda f_\downarrow$ we obtain $f_\downarrow \in \operatorname{Span}\big(\mathrm{e}^{-\mathbf{i}\lambda\alpha}\big)$, where $\lambda \in \mathbb{R}$. It gives the basis in the space of function restrictions from $\mathscr{L}$ to γ_4, which consists of functions $\mathrm{e}^{\mathbf{i}\lambda\alpha}$.

For any point $x \in X_0^+$ where $x_1 \neq -x_2$, considering σ-homogeneity and utilizing formula (5), we derive $x_1 + x_2 = r(1+\cos\alpha) \geqslant 0$. Consequently,

$$f(x) = (x_1 + x_2)^\sigma f_\downarrow\left(\frac{x_1}{x_1+x_2}, \frac{x_2}{x_1+x_2}, \frac{x_3}{x_1+x_2}\right).$$

Therefore, the functions

$$f_\lambda^{(4)}(x) = (x_1 + x_2)^\sigma\, \mathrm{e}^{\mathbf{i}\lambda\alpha} = (x_1 + x_2)^\sigma \exp\frac{\mathbf{i}\lambda x_3}{x_1+x_2},$$

where $\lambda \in \mathbb{R}$, form a basis B_4 in $\mathscr{L}$.

By analogy, finding eigenfunctions of the operator $\mathfrak{d}_{2,3} + \mathfrak{d}_2$, which corresponds to the reduction $H_5 \subset G_0$, we obtain a basis B_5 in $\mathscr{L}$, consisting of functions

$$f_\lambda^{(5)}(x) = (x_1 + x_3)^\sigma \, e^{\mathbf{i}\lambda\alpha} = (x_1 + x_3)^\sigma \, \exp \frac{\mathbf{i}\lambda x_2}{x_1 + x_3},$$

where $\lambda \in \mathbb{R}$.

Upon substituting σ with $-\sigma - 1$, the resultant is the linear space denoted as $\mathscr{L}^*$. Each $f \in \mathscr{L}$ has its counterpart in $\mathscr{L}^*$ as f^*. The bases B_1–B_5 are assumed to represent the equivalents of the bases B_1^*–B_5^*.

4. Composition of the Transformations $B_2 \longrightarrow B_5$ and $B_5 \longrightarrow B_4$

Considering the composition of the integral operators

$$f_\mu^{(4)} = \int_{-\infty}^{+\infty} \varepsilon_{\mu,\rho}^{5\to 4}(\sigma) \, f_\rho^{(5)} \, \mathrm{d}\rho$$

and

$$f_\rho^{(5)} = \int_{-\infty}^{+\infty} (\varepsilon_{\rho,\lambda,+}^{2\to 5}(\sigma) \, f_{\lambda,+}^{(2)} + \varepsilon_{\rho,\lambda,-}^{2\to 5}(\sigma) \, f_{\lambda,-}^{(2)}) \, \mathrm{d}\lambda,$$

we get

$$\begin{aligned} f_\mu^{(4)} = \int_{-\infty}^{+\infty} \Bigg[& \left(\int_{-\infty}^{+\infty} \varepsilon_{\mu,\rho}^{5\to 4}(\sigma) \, \varepsilon_{\rho,\lambda,+}^{2\to 5}(\sigma) \, \mathrm{d}\rho \right) f_{\lambda,+}^{(2)} \\ & + \left(\int_{-\infty}^{+\infty} \varepsilon_{\mu,\rho}^{5\to 4}(\sigma) \, \varepsilon_{\rho,\lambda,-}^{2\to 5}(\sigma) \, \mathrm{d}\rho \right) f_{\lambda,-}^{(2)} \Bigg] \, \mathrm{d}\lambda. \end{aligned} \tag{11}$$

Comparing

$$f_\mu^{(4)} = \int_{-\infty}^{+\infty} (\varepsilon_{\mu,\lambda,+}^{2\to 4}(\sigma) \, f_{\lambda,+}^{(2)} + \varepsilon_{\mu,\lambda,-}^{2\to 4} \, f_{\lambda,-}^{(2)}) \, \mathrm{d}\lambda \tag{12}$$

and (11), we derive

$$\varepsilon_{\mu,\lambda,\pm}^{2\to 4}(\sigma) = \int_{-\infty}^{+\infty} \varepsilon_{\mu,\rho}^{5\to 4}(\sigma) \, \varepsilon_{\rho,\lambda,\pm}^{2\to 5}(\sigma) \, \mathrm{d}\rho. \tag{13}$$

In view of equality $f_{\lambda,+}^{*(2)}(\cosh\alpha, \sinh\alpha, -1) = f_{\lambda,-}^{*(2)}(\cosh\alpha, \sinh\alpha, 1) = 0$, we derive from (12) that

$$\begin{aligned} & \int_{-\infty}^{+\infty} f_{\nu,+}^{*(2)}(\cosh\alpha, \sinh\alpha, 1) \, f_\mu^{(4)}(\cosh\alpha, \sinh\alpha, 1) \, \mathrm{d}\alpha \\ & = \int_{-\infty}^{+\infty} \varepsilon_{\mu,\lambda,+}^{2\to 4}(\sigma) \, \mathrm{d}\lambda \int_{-\infty}^{+\infty} e^{\mathbf{i}(\lambda+\nu)\alpha} \, \mathrm{d}\alpha = 2\pi \int_{-\infty}^{+\infty} \varepsilon_{\mu,\lambda,+}^{2\to 4}(\sigma) \, \delta(\lambda + \nu) \, \mathrm{d}\lambda = 2\pi \, \varepsilon_{\mu,-\nu,+}^{2\to 4}(\sigma), \end{aligned}$$

where δ is the Dirac function. Therefore,

$$\varepsilon_{\mu,\lambda,+}^{2\to 4}(\sigma) = (2\pi)^{-1} \int_{-\infty}^{+\infty} f_{-\lambda,+}^{*(2)}(\cosh\alpha, \sinh\alpha, 1) \, f_\mu^{(4)}(\cosh\alpha, \sinh\alpha, 1) \, \mathrm{d}\alpha.$$

Using the substitution $\cosh\alpha = \frac{1+(t-1)^2}{2}$, we have

$$\varepsilon^{2\to4}_{\mu,\lambda,+}(\sigma) = 2^{\sigma}\,\pi^{-1}\,\mathrm{e}^{-\mathbf{i}\mu}\int\limits_{0}^{2} t^{-\mathbf{i}\lambda-\sigma-1}\,(2-t)^{\mathbf{i}\lambda-\sigma-1}\,\mathrm{e}^{\mathbf{i}\mu t}\,\mathrm{d}t. \tag{14}$$

Theorem 1. *Let $\Re(\sigma) < 0$. Then,*

$$\varepsilon^{2\to4}_{\mu,\lambda,+}(\sigma) = 2^{-\sigma-1}\,\pi^{-1}\,\mathrm{e}^{-\mathbf{i}\mu}\,\mathrm{B}(-\mathbf{i}\lambda-\sigma,\mathbf{i}\lambda-\sigma)\,{}_1F_1(-\mathbf{i}\lambda-\sigma;-2\sigma;-2\mathbf{i}\mu), \tag{15}$$

where B *is the Beta function.*

Proof. Utilizing the equation [10] (Entry 2.3.6.1) on (14) leads us to derive the intended expression, resulting in Formula (15). □

Likewise, we obtain the subsequent equalities through a similar derivation process:

$$\varepsilon^{2\to5}_{\mu,\lambda,+}(\sigma) = (2\pi)^{-1}\left[\int\limits_{-\infty}^{+\infty} f^{*(2)}_{-\lambda,+}(\cosh\alpha,\sinh\alpha,1)\,f^{(5)}_{\mu}(\cosh\alpha,\sinh\alpha,1)\,\mathrm{d}\alpha \right.$$
$$\left. + \int\limits_{-\infty}^{+\infty} f^{*(2)}_{-\lambda,+}(\cosh\alpha,\sinh\alpha,1)\,f^{(5)}_{\mu}(\cosh\alpha,\sinh\alpha,1)\,\mathrm{d}\alpha\right]$$

and

$$\varepsilon^{4\to5}_{\mu,\lambda}(\sigma) = (2\pi)^{-1}\int\limits_{-\infty}^{+\infty} f^{*(4)}_{-\lambda}\left(\frac{1+\alpha^2}{2},\frac{1-\alpha^2}{2},\alpha\right) f^{(5)}_{\mu}\left(\frac{1+\alpha^2}{2},\frac{1-\alpha^2}{2},\alpha\right)\mathrm{d}\alpha.$$

Theorem 2. *Let $-1 < \Re(\sigma) < 0$. Then,*

$$\varepsilon^{2\to5}_{\mu,\lambda,+}(\sigma) = 2^{\sigma}\,\pi^{-1}\,(-\mathbf{i}\mu)^{\sigma-\mathbf{i}\lambda}\,\Gamma(\mathbf{i}\lambda-\sigma), \tag{16}$$

where Γ is Gamma function.

Proof. By substituting a new variable t such that $\cosh\alpha = \frac{1+t^2}{2}$, we obtain

$$\varepsilon^{2\to5}_{\mu,\lambda,+}(\sigma) = 2^{\sigma}\,\pi^{-1}\int\limits_{-\infty}^{+\infty}(t)_+^{\mathbf{i}\lambda-\sigma-1}\,\mathrm{e}^{\mathbf{i}\mu t}\,\mathrm{d}t = 2^{\sigma}\,\pi^{-1}\int\limits_{0}^{+\infty} t^{\mathbf{i}\lambda-\sigma-1}\,\mathrm{e}^{\mathbf{i}\mu t}\,\mathrm{d}t.$$

Then, use of the Laplace transform to the last integral gives the desired result (16). □

Theorem 3. *Let $-1 < \Re(\sigma) < 0$. For $\operatorname{sign}\lambda = \operatorname{sign}\mu \neq 0$, we have*

$$\varepsilon^{4\to5}_{\mu,\lambda}(\sigma) = -\frac{2\sqrt{2}}{\pi}\,\mathrm{e}^{\mathbf{i}(\lambda-\mu)}\left(\frac{\mu}{\lambda}\right)^{\sigma+1/2}\sin\sigma\pi\,K_{2\sigma+1}\Big(2\sqrt{2\mu\lambda}\Big). \tag{17}$$

For $\operatorname{sign}\lambda = -\operatorname{sign}\mu \neq 0$, we have

$$\varepsilon^{4\to5}_{\mu,\lambda}(\sigma) = \frac{\mathrm{e}^{\mathbf{i}(\lambda-\mu)}}{\sqrt{2}\,\pi}\left(\frac{-\mu}{\lambda}\right)^{\sigma+1/2}\cos\sigma\pi\left[J_{-2\sigma-1}\Big(2\sqrt{-2\mu\lambda}\Big) - J_{2\sigma+1}\Big(2\sqrt{-2\mu\lambda}\Big)\right]. \tag{18}$$

Proof. Introducing a new variable $t = \alpha + 1$, we obtain

$$\varepsilon^{4\to5}_{\mu,\lambda}(\sigma) = 2^{-\sigma}\,\pi^{-1}\,\mathrm{e}^{\mathbf{i}(\lambda-\mu)}\int\limits_{0}^{+\infty} t^{2\sigma}\cos\left(\lambda t - \frac{2\mu}{t}\right)\mathrm{d}t. \tag{19}$$

Utilizing the formulas specified as [10] (Entries 2.5.24.4, 2.5.24.7), and applying them to the integral presented in (19), results in the derivation of both (17) and (18). □

Theorem 4. *Let $\kappa > 0$ and $-1 < \Re(\sigma) < 0$. Then,*

$$\begin{aligned} &4\,(-\mathbf{i})^{\sigma-\mathbf{i}\lambda}\,\sin(\pi\sigma)\int_0^{+\infty} t^{-\mathbf{i}\lambda}\,\mathrm{e}^{-\mathbf{i}t^2/4}\,K_{2\sigma+1}(\kappa t)\,\mathrm{d}t \\ &-\mathbf{i}^{\sigma-\mathbf{i}\lambda}\,\cos(\pi\sigma)\int_0^{+\infty} t^{-\mathbf{i}\lambda}\,\mathrm{e}^{-\mathbf{i}t^2/4}\,[J_{2\sigma+1}(\kappa t) - J_{-2\sigma-1}(\kappa t)]\,\mathrm{d}t \\ &\quad = \frac{\pi\,\Gamma(-\mathbf{i}\lambda-\sigma)}{2^{\sigma}\,\kappa^{2\sigma+1}\,\mathrm{e}^{\mathbf{i}\kappa^2}\,\Gamma(-2\sigma)}\,{}_1F_1(-\mathbf{i}\lambda;-2\sigma;-\mathbf{i}\kappa^2). \end{aligned}$$

Proof. Let $\mu > 0$. Taking the relationship $\varepsilon^{5\to4}_{\mu,\rho}(\sigma) = \varepsilon^{4\to5}_{-\rho,-\mu}(-\sigma-1)$, we obtain from (13) that

$$\varepsilon^{2\to4}_{\mu,\lambda,+}(\sigma) = M_+(\lambda) + M_-(\lambda),$$

where

$$M_\pm(\lambda) = \pm\int_0^{\pm\infty} \varepsilon^{2\to5}_{\rho,\lambda,+}(\sigma)\,\varepsilon^{4\to5}_{-\rho,-\mu}(-\sigma-1)\,\mathrm{d}\rho.$$

Choosing new variables $t = 2\sqrt{\pm\rho}$ in $M_\pm(\lambda)$ and supposing that $\kappa = \sqrt{2\mu}$, we complete the proof. □

Remark 1. *The result in Theorem 4 can be rewritten in terms of the Coulomb wave function $F_\mu(\rho;x) \sim x^{\mu+1}\,\mathrm{e}^{-\mathbf{i}x}\,{}_1F_1(1+\mu-\mathbf{i}\rho;2\mu+2;2\mathbf{i}x)$.*

The particular case $\lambda = 0$ in (4) can be expressed in terms of Bessel and Hankel functions. Indeed, using [11] (Entry 2.6.15.2), we have

$$\begin{aligned} M_+(0) &= -\sqrt{2}\,\pi\,(-\mathbf{i})^{\sigma}\,\kappa^{2\sigma+1}\,\mathrm{e}^{\mathbf{i}(\sigma+1)/2}\,\Gamma(-\sigma)\,\big(J_{\sigma+1/2}(\mu) + \mathbf{i}\,Y_{\sigma+1/2}(\mu)\big) \\ &= -\sqrt{2}\,\pi\,(-\mathbf{i})^{\sigma}\,\kappa^{2\sigma+1}\,\mathrm{e}^{\mathbf{i}(\sigma+1)/2}\,\Gamma(-\sigma)\,H^{(1)}_{\sigma+1/2}(\mu), \end{aligned}$$

where $Y_{\sigma+1/2}$ is the Bessel function of the second kind. Also, employing [11] (Entry 2.12.18.2), we obtain

$$\begin{aligned} M_-(0) &= 2^{-1/2}\,\pi^{-2}\,\mathbf{i}^{\sigma}\,\kappa^{2\sigma+1}\,\cos(\sigma\pi)\,\Gamma(-\sigma)\Big[\mathrm{e}^{-\mathbf{i}\sigma\pi/2}\,J_{-\sigma-1/2}(\mu) - \mathbf{i}\,\mathrm{e}^{\mathbf{i}\sigma\pi/2}\,J_{\sigma+1/2}(\mu)\Big] \\ &= -2^{-1/2}\,\pi^{-2}\,\mathbf{i}^{\sigma+1}\,\kappa^{2\sigma+1}\,\cos^2(\sigma\pi)\,\mathrm{e}^{-\mathbf{i}\sigma\pi/2}\,\Gamma(-\sigma)\,H^{(2)}_{\sigma+1/2}(\mu). \end{aligned}$$

Moreover, the integral in (14) with $\lambda = 0$ can be evaluated using the formula [10] (Entry 2.3.6.2) as follows:

$$\int_0^a t^{\nu-1}\,(a-t)^{\nu-1}\,\mathrm{e}^{-pt}\,\mathrm{d}t = \sqrt{\pi}\,\Gamma(\nu)\left(\frac{a}{p}\right)^{\nu-1/2}\mathrm{e}^{-ap/2}\,I_{\nu-1/2}\Big(\frac{ap}{2}\Big),$$

which is valid for $\Re(\nu) > 0$. Note $J_\nu(\mathbf{i}z) = \mathrm{e}^{\mathbf{i}\nu\pi/2}\,I_\nu(z)$.

5. Composition of the Transformations $B_1 \longrightarrow B_4$ and $B_4 \longrightarrow B_5$

Considering the operator

$$f^{(5)}_\lambda = \sum_{n=-\infty}^{\infty} \varepsilon^{1\to5}_{\lambda,n}(\sigma)\,f^{(1)}_n$$

as a composition of the operators

$$f_\lambda^{(5)} = \int_{-\infty}^{+\infty} \varepsilon_{\lambda,\rho}^{4\to 5}(\sigma)\, f_\rho^{(4)}\, \mathrm{d}\rho$$

and

$$f_\rho^{(4)} = \sum_{n=-\infty}^{\infty} \varepsilon_{\rho,n}^{1\to 4}(\sigma)\, f_n^{(1)},$$

we obtain the equality

$$\varepsilon_{\lambda,n}^{1\to 5}(\sigma) = \int_{-\infty}^{+\infty} \varepsilon_{\lambda,\rho}^{4\to 5}(\sigma)\, \varepsilon_{\rho,n}^{1\to 4}(\sigma)\, \mathrm{d}\rho. \tag{20}$$

Theorem 5. *Let $k \in \mathbb{N}$, $\kappa > 0$ and $0 < \Re(\nu) < \frac{1}{2}$. Then,*

$$\begin{aligned} W_{k,\nu}\left(\kappa^2/2\right) =& \left[\pi\, \mathbf{i}^k\, \Gamma(\nu + k + \frac{1}{2})\right]^{-1} \\ & \times \left[\frac{\sin(\nu\pi)}{2^{1/2}\,\Gamma(\nu - k - \frac{3}{2})} \int_0^{+\infty} t\, \mathrm{e}^{-\mathbf{i}t^2/2}\, W_{k,\nu}\left(t^2\right) K_{2\nu}(\kappa t)\, \mathrm{d}t \right. \\ & \left. + \frac{2^{3/2}\,\cos(\nu\pi)}{\Gamma(\nu - k + \frac{1}{2})} \int_0^{+\infty} t\, \mathrm{e}^{\mathbf{i}t^2/2}\, W_{-k,\nu}\left(t^2\right) \left(J_{-2\nu}(\kappa t) - J_{2\nu}(\kappa t)\right) \mathrm{d}t\right]. \end{aligned} \tag{21}$$

In particular,

$$\begin{aligned} K_\nu\left(\kappa^2/4\right) =& \left[\pi\, \Gamma(\nu + \frac{1}{2})\right]^{-1} \\ & \times \left[\frac{\sin(\nu\pi)}{2^{1/2}\,\Gamma(\nu - \frac{3}{2})} \int_0^{+\infty} t^{3/2}\, \mathrm{e}^{-\mathbf{i}t^2/2}\, K_\nu\left(t^2/2\right) K_{2\nu}(\kappa t)\, \mathrm{d}t \right. \\ & \left. + \frac{2^{3/2}\,\cos(\nu\pi)}{\Gamma(\nu + \frac{1}{2})} \int_0^{+\infty} t^{3/2}\, \mathrm{e}^{\mathbf{i}t^2/2}\, K_\nu\left(t^2/2\right) \left(J_{-2\nu}(\kappa t) - J_{2\nu}(\kappa t)\right) \mathrm{d}t\right]. \end{aligned} \tag{22}$$

Proof. Since

$$\int_{-\pi}^{\pi} f_k^{*(1)}(1, \cos\alpha, \sin\alpha)\, f_\lambda^{(4)}(1, \cos\alpha, \sin\alpha)\, \mathrm{d}\alpha = \sum_{n=-\infty}^{\infty} \varepsilon_{\lambda,n}^{1\to 4}(\sigma) \int_{-\pi}^{\pi} \mathrm{e}^{\mathbf{i}(k+n)\alpha}\, \mathrm{d}\alpha,$$

we have

$$\varepsilon_{\lambda,n}^{1\to 4}(\sigma) = (2\pi)^{-1} \int_{-\pi}^{\pi} f_{-n}^{*(1)}(1, \cos\alpha, \sin\alpha)\, f_\lambda^{(4)}(1, \cos\alpha, \sin\alpha)\, \mathrm{d}\alpha.$$

Using the substitution $\sin\alpha = \frac{2t}{1+t^2}$, we obtain

$$\varepsilon_{\lambda,n}^{1\to 4}(\sigma) = 2^\sigma\, \pi^{-1} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathbf{i}\lambda t}\, (1 + \mathbf{i}t)^{-\sigma-n-1}\, (1 - \mathbf{i}t)^{n-\sigma-1}\, \mathrm{d}t.$$

The value of this integral can be evaluated in terms of the Whittaker function [12] (Entry 3.384.9). Moreover, $\varepsilon_{\lambda,n}^{1\to 5}(\sigma) = \mathrm{i}^{-n}\, \varepsilon_{\lambda,-n}^{1\to 4}(\sigma)$. Considering Theorem 3 and Formula (20), introducing new variables $t = \sqrt{2\rho}$ for $\rho > 0$ and $t = 2\sqrt{-\rho}$ for $\rho < 0$ and setting $k = -n$, $\nu = \sigma + \frac{1}{2}$, and $\kappa = 2\sqrt{\lambda}$, we get (21).

Setting $k = 0$ in (21) yields the Formula (22). $\square$

6. Concluding Remarks

We crafted a variety of bases featuring eigenfunctions of Casimir infinitesimal operators, intricately linked to a reduction in the group of unimodular $\mathrm{diag}(1, -1, -1)$-matrices to specific subgroups. Through diligent exploration, we uncovered the kernels of base transformation integral operators and delved into the fusion of these transformations. By virtue of composition being a closed operation, we unveiled integral relationships for some select special functions, elegantly encapsulated in Theorems 4 and 5.

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