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Analyzing the Ricci Tensor for Slant Submanifolds in Locally Metallic Product Space Forms with a Semi-Symmetric Metric Connection

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Abstract: This article explores the Ricci tensor of slant submanifolds within locally metallic product space forms equipped with a semi-symmetric metric connection (SSMC). Our investigation includes the derivation of the Chen–Ricci inequality and an in-depth analysis of its equality case. More precisely, if the mean curvature vector at a point vanishes, then the equality case of this inequality is achieved by a unit tangent vector at the point if and only if the vector belongs to the normal space. Finally, we have shown that when a point is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at the point, and conversely.

Keywords: Chen–Ricci inequality; isotropic submanifolds; locally metallic product space forms

MSC: 53B50; 53C20; 53C40

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1. Introduction

The investigation of submanifolds immersed in Riemannian manifolds has captivated the attention of differential geometry scholars for numerous decades. Within this realm, a fundamental inquiry revolves around comprehending the geometric characteristics of submanifolds in relation to the curvature of the encompassing manifold.

A renowned inequality in differential geometry, known as the Chen–Ricci inequality, establishes a connection between the scalar curvature of a submanifold, its mean curvature, and the norm of its second fundamental form.

In 1996, Chen made a significant breakthrough by formulating an equation that links two fundamental geometric properties of a submanifold, denoted as $\mathcal{M}$, embedded within a space known as $\overline{\mathcal{M}}(c)$ with a constant curvature c. These properties are the Ricci curvature denoted by Ric and the squared mean curvature expressed as $||H||^2$. According to Chen's formula, for any unit vector χ positioned on the submanifold $\mathcal{M}$,

$$Ric(\chi) \leq (k-1)c + \frac{k^2}{2}||H||^2, \quad k = dim\mathcal{M}$$

Chen also established the aforementioned inequality for Lagrangian submanifolds [1]. Since then, this inequality has garnered the interest of geometers worldwide, leading to the proof of similar inequalities by various researchers for diverse types of submanifolds in various ambient manifolds [2–4]. Furthermore, there are some applications in geometric flow and tangent bundles. For example, the study of Harnack estimates [5], Li–Yau-type gradient estimates [6,7], Perelman-type differential Harnack inequalities and Li–Yau-type estimates [8], the new Harnack inequalities of a variety of geometric flows [9], etc. Recent works on differential Harnack inequalities can be found in [10–12]. In Ref. [13], Kumar, R.

et al. considered the problems of NSNMC in the tangent bundle. In Refs. [14,15], Kumar and R. et al. studied the tangent bundles with QSNMC in an LP-Sasakian manifold. The properties, theorems, and results of the curvature tensor and Ricci tensor relevant to QSMC on the tangent bundles were obtained in [16–18]. In the recent years, Li and Khan et al. conducted the research relevant to inequalities [19], solitons [20], submanifolds [21], and classical differential geometry [22] under the viewpoint of submanifold theory, soliton theory, and other related theories [23–25]. The results and methods of those papers motivate us to write the paper.

Simultaneously, a θ-slant submanifold represents a subtype of the submanifold in the domain of differential geometry that generalizes the concept of a slant submanifold. Similar to slant submanifolds, θ-slant submanifolds pertain to submanifolds of Riemannian manifolds that exhibit a tilted or slanted geometry concerning the ambient manifold. Nevertheless, unlike slant submanifolds, which are defined by the angle between the submanifold and a vector distribution in the ambient manifold [26–28], θ-slant submanifolds are defined by a more inclusive angle function θ, which can rely on the submanifold's position within the ambient manifold [29,30]. This broader definition allows for increased flexibility and generality in characterizing the submanifold.

Specifically, a θ-slant submanifold is determined by the prerequisite that its tangent space at each point be slanted in relation to a particular vector distribution in the ambient manifold, wherein the angle of slant is determined by evaluating the angle function θ at that specific point. This angle function enables the capture of various geometric properties of the submanifold, including its curvature or its embedding within the ambient manifold. The classification of minimal surfaces in Euclidean space has historically leveraged slant submanifolds, with Almgren's renowned theorem asserting that any complete, non-flat minimal surface in Euclidean space must be either a plane, a catenoid, or a helicoid. The proof of this theorem employs the theory of slant submanifolds to demonstrate the impossibility of certain types of minimal surfaces.

In the work by Mastsumoto [31], a bound for the Ricci tensor of slant submanifolds in complex space forms was obtained. They also demonstrated that a Kaehlerian slant submanifold satisfying the equality case identically is minimal. Kim et al. [32] derived the Ricci curvature for integral submanifolds of S-space forms and discussed the equality case of the inequality. Additionally, they obtained results for various subclasses, including almost semi-invariant submanifolds, θ-slant submanifolds, anti-invariant submanifolds, and invariant submanifolds.

In 2010, Mihai and Ozgur [33] established an inequality for submanifolds of real space forms with a semi-symmetric connection. They also considered the equality case for this inequality. Mihai and Radulescu improved the inequality for Kaehlerian slant submanifolds in complex space forms [34]. Deng [35] enhanced the Chen–Ricci inequality for Lagrangian submanifolds in complex space forms by utilizing an optimization technique. Mihai [36] improved the Chen–Ricci inequalities for Lagrangian submanifolds of dimension n (where $n \geq 2$) in a $2n$-dimensional complex space form with a semi-symmetric metric connection, as well as for Legendrian submanifolds in a Sasakian space with a semi-symmetric metric connection.

In their work [37], Khan and Ozel established a relationship between the Ricci curvature and the squared norm of the second fundamental form for contact CR-warped product submanifolds in generalized Sasakian space forms admitting a trans-Sasakian structure. Recently, Lee et al. [38] derived Chen–Ricci inequalities for Riemannian maps with different ambient spaces and discussed numerous applications, for which we can refer to [39–41].

Motivated by the above studies, this article centers its focus on θ-slant submanifolds within locally metallic product space forms and explores the Chen–Ricci inequality as it applies to these submanifolds.

Our principal outcome involves the formulation of the Chen–Ricci inequality for θ-slant submanifolds within locally metallic product space forms, along with deriving the conditions under which equality to the inequality is established.

Furthermore, we delve into several applications of our findings, showcasing how our inequality facilitates the derivation of significant geometric properties specific to θ-slant submanifolds.

2. Fundamental Results

In the subsequent section, we present the relevant mathematical formulas and concepts necessary to grasp the Chen–Ricci inequality concerning isotropic submanifolds in locally metallic product space forms.

Let $\overline{\mathcal{M}}$ denote a Riemannian manifold equipped with the linear connection $\overline{\nabla}$. A connection is classified as semi-symmetric if its torsion tensor T satisfies the expression:

$$T(\chi_1, \chi_2) = \pi(\chi_2)\chi_1 - \pi(\chi_1)\chi_2$$

where π is a 1-form. Consequently, $\overline{\nabla}$ is referred to as a semi-symmetric connection. Assuming a Riemannian metric g on $\overline{\mathcal{M}}$, if $\overline{\nabla} g = 0$, then $\overline{\nabla}$ qualifies as a semi-symmetric metric connection on $\overline{\mathcal{M}}$. The mathematical form of this connection is given by:

$$\overline{\nabla}_{\chi_1}\chi_2 = \tilde{\overline{\nabla}}_{\chi_1}\chi_2 + \pi(\chi_2)\chi_1 - g(\chi_1, \chi_2)\Gamma \tag{1}$$

where χ_1 and χ_2 are arbitrary vectors in $\overline{\mathcal{M}}$, $\tilde{\overline{\nabla}}$ represents the Levi–Civita connection with respect to the Riemannian metric g, and Γ is a vector field.

Suppose $\mathcal{M}$ is an m-dimensional submanifold within the Riemannian manifold $\overline{\mathcal{M}}$. Let ∇ denote the semi-symmetric metric connection induced on $\mathcal{M}$, and let $\tilde{\nabla}$ denote the Levi–Civita connection. In this case, the Gauss formulas can be expressed as follows:

$$\overline{\nabla}_{\chi_1}\chi_2 = \nabla_{\chi_1}\chi_2 + \zeta(\chi_1, \chi_2), \quad \chi_1, \chi_2 \in \Gamma(T\mathcal{M}), \tag{2}$$

$$\tilde{\overline{\nabla}}_{\chi_1}\chi_2 = \tilde{\nabla}_{\chi_1}\chi_2 + \tilde{\zeta}(\chi_1, \chi_2), \quad \chi_1, \chi_2 \in \Gamma(T\mathcal{M}), \tag{3}$$

Let $\tilde{\zeta}$ represent the second fundamental form. Additionally, let $\overline{R}$ and $\tilde{\overline{R}}$ denote the curvature tensors of $\overline{\mathcal{M}}$ and $\mathcal{M}$ respectively, with respect to the connections $\overline{\nabla}$ and $\tilde{\overline{\nabla}}$. Similarly, R and $\tilde{R}$ denote the curvature tensors of $\overline{\mathcal{M}}$ and $\mathcal{M}$ respectively, with respect to the connections ∇ and $\tilde{\nabla}$. Given these definitions, we can express the following relations:

$$\begin{aligned}\tilde{\overline{R}}(\chi_1, \chi_2, \chi_3, \chi_4) &= \tilde{R}(\chi_1, \chi_2, \chi_3, \chi_4) \\ &\quad + g(\zeta(\chi_1, \chi_3), \zeta(\chi_2, \chi_4)) - g(\zeta(\chi_1, \chi_4), \zeta(\chi_2, \chi_3)),\end{aligned} \tag{4}$$

for $\chi_1, \chi_2, \chi_3, \chi_4 \in T\overline{\mathcal{M}}$. Let us introduce the $(0, 2)$ tensors:

$$\beta(\chi_1, \chi_2) = (\tilde{\overline{\nabla}}_{\chi_1}\pi)(\chi_2) - \pi(\chi_1)\pi(\chi_2) + \frac{1}{2}g(\chi_1, \chi_2)\pi(\Gamma).$$

According to Wang [42], the expression for the curvature tensor $\overline{R}$ of the manifold $\overline{\mathcal{M}}$ is as follows:

$$\begin{aligned}\overline{R}(\chi_1, \chi_2, \chi_3, \chi_4) \quad &= \quad \tilde{\overline{R}}(\chi_1, \chi_2, \chi_3, \chi_4) + \beta(\chi_1, \chi_3)g(\chi_2, \chi_4) \\ &- \quad \beta(\chi_2, \chi_3)g(\chi_1, \chi_4) + \beta(\chi_2, \chi_4)g(\chi_1, \chi_3) - \beta(\chi_1, \chi_4)g(\chi_2, \chi_3).\end{aligned} \tag{5}$$

Let us define λ as the trace of β.

Let $\mathcal{M}$ be a Riemannian manifold, and let $\pi \subset T_x\mathcal{M}$ be a plane section at a point $x \in \mathcal{M}$. The sectional curvature of π is denoted by $K(\pi)$. For any $x \in \mathcal{M}$, if $\{\varrho_1, \ldots, \varrho_n\}$

and $\{\varrho_{n+1}, \ldots, \varrho_m\}$ are orthonormal bases of $T_x\mathcal{M}$ and $T_x^{\perp}\mathcal{M}$, respectively, then the scalar curvature τ can be expressed as follows:

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(\varrho_i \wedge \varrho_j). \tag{6}$$

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^{n} g(\zeta(\varrho_i, \varrho_i)).$$

The orthonormal frames $\{\varrho_1, \ldots, \varrho_n\}$ and $\{\varrho_{n+1}, \ldots, \varrho_m\}$ represent the tangent and normal spaces, respectively, on the Riemannian manifold $\mathcal{M}$.

The relative null space of the Riemannian manifold at a point x in $\mathcal{M}$ is defined as:

$$\mathcal{N}_x = \{\chi_1 \in T_x\mathcal{M} | \zeta(\chi_1, \chi_2) = 0 \quad \forall \quad \chi_2 \in T_x\mathcal{M}\}. \tag{7}$$

This refers to the subset of the tangent space at point x where the second fundamental form is constantly zero. It is also referred to as the normal space of $\mathcal{M}$ at x.

In the context of minimal submanifolds, it is stated that the mean curvature vector $\mathcal{H}$ is always zero.

On a m-dimensional Riemannian manifold $(\overline{\mathcal{M}}, g)$ with real numbers $a_1, \ldots, a_n$, a polynomial structure is defined as a tensor field ϑ of type (1, 1) that satisfies the following equation:

$$\mathcal{B}(\vartheta) \equiv \vartheta^n + a_{n-1}\vartheta^{n-1} + \ldots + a_2\vartheta + a_1\mathcal{I} = 0,$$

where $\mathcal{I}$ denotes the identity transformation. The following remark presents a few notable instances of polynomial structures.

Remark 1. *1. ϑ is an almost complex structure if $\mathcal{B}(\vartheta) = \vartheta^2 + \mathcal{I}$.*
2. ϑ is an almost product structure if $\mathcal{B}(\vartheta) = \vartheta^2 - \mathcal{I}$.
3. ϑ is a metallic structure if $\mathcal{B}(\vartheta) = \vartheta^2 - p\vartheta - q\mathcal{I}$,
where p and q are two integers.

If for all $\chi_1, \chi_2 \in \Gamma(T\overline{\mathcal{M}})$

$$g(\vartheta\chi_1, \chi_2) \quad = \quad g(\chi_1, \vartheta\chi_2), \tag{8}$$

then in such a case, the Riemannian metric g is referred to as being compatible with the polynomial structure ϑ.

In the context of Riemannian manifolds, a metallic structure refers to a tensor field ϑ that satisfies two conditions: it is ϑ-compatible with the metric g, and the manifold $(\overline{\mathcal{M}}, g)$ itself is a metallic Riemannian manifold.

By utilizing Equation (8), we derive

$$g(\vartheta\chi_1, \vartheta\chi_2) = g(\vartheta^2\chi_1, \chi_2) = pg(\chi_1, \vartheta\chi_2) + qg(\chi_1, \chi_2).$$

An almost product structure $\mathcal{F}$ defined on an m-dimensional (Riemannian) manifold $(\overline{\mathcal{M}}, g)$ is characterized by being a (1,1)-tensor field that satisfies $\mathcal{F}^2 = I$ and $\mathcal{F} \neq \pm\mathcal{I}$. When $\mathcal{F}$ additionally fulfills the condition $g(\mathcal{F}\chi_1, \chi_2) = g(\chi_1, \mathcal{F}\chi_2)$ for all $\chi_1, \chi_2 \in \Gamma(T\overline{\mathcal{M}})$, the manifold $(\overline{\mathcal{M}}, g)$ is said to be an almost product Riemannian manifold [43].

There exist two almost product structures, denoted as $\mathcal{F}_1$ and $\mathcal{F}_2$, induced by a metallic structure ϕ on $\overline{\mathcal{M}}$ [44]. These structures can be expressed using the following equation:

$$\begin{cases} \mathcal{F}_1 = \frac{2}{2\sigma_{p,q} - p}\phi - \frac{p}{2\sigma_{p,q} - p}\mathcal{I}, \\ \\ \mathcal{F}_2 = \frac{2}{2\sigma_{p,q} - p}\phi + \frac{p}{2\sigma_{p,q} - p}\mathcal{I}, \end{cases}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ are the members of the metallic means family or the metallic proportions.

Likewise, for any given almost product structure $\mathcal{F}$ on $\overline{\mathcal{N}}$, two corresponding metallic structures ϕ_1 and ϕ_2 are induced, and they can be defined as follows:

$$\begin{cases} \phi_1 = \frac{p}{2}\mathcal{I} + \frac{2\sigma_{p,q}-p}{2}\mathcal{F}, \\ \phi_2 = \frac{p}{2}\mathcal{I} - \frac{2\sigma_{p,q}-p}{2}\mathcal{F}. \end{cases}$$

Definition 1 ([45]). *Consider a metallic structure ϕ on $\overline{\mathcal{M}}$ and a linear connection $\overline{\nabla}$ such that $\nabla\phi = 0$. In this case, $\overline{\nabla}$ is referred to as a ϕ connection. A locally metallic Riemannian manifold is defined as a metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$, wherein the Levi–Civita connection $\overline{\nabla}$ associated with the metric g serves as a ϕ connection.*

Here, we recall the following.

Remark 2. *It is essential to bear in mind that the metallic family includes various members, which are categorized as follows [44]:*

1. *The golden structure, when $p = q = 1$.*
2. *The copper structure, when $p = 1$ and $q = 2$.*
3. *The nickel structure, when $p = 1$ and $q = 3$.*
4. *The silver structure, when $p = 2$ and $q = 1$.*
5. *The bronze structure, when $p = 3$ and $q = 1$.*
6. *The subtle structure, when $p = 4$ and $q = 1$, and so on.*

Suppose we have an m-dimensional metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$ and an n-dimensional submanifold $(\mathcal{M}, g)$ that is isometrically immersed into $\overline{\mathcal{M}}$ with the induced metric g. For any $x \in \mathcal{M}$, the tangent space $T_x\overline{\mathcal{M}}$ of $\overline{\mathcal{M}}$ at x can be expressed as the direct sum of $T_x\mathcal{M}$ and $T_x^{\perp}\mathcal{M}$, where $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x, and $T_x^{\perp}\mathcal{M}$ is the orthogonal complement of $T_x\mathcal{M}$ in $T_x\overline{\mathcal{M}}$.

In an almost Hermitian manifold $\overline{\mathcal{M}}$, a submanifold $\mathcal{M}$ is considered to be a slant submanifold if the angle between $J\mathcal{M}$ and $T_x\mathcal{M}$ remains constant for any $x \in \mathcal{M}$ and a non-zero vector $X \in T_x\mathcal{M}$. The slant angle of $\mathcal{M}$ in $\overline{\mathcal{M}}$ is denoted by θ and takes values in the interval $[0, \frac{\pi}{2}]$.

Moreover, if $\mathcal{M}$ is a slant submanifold of a metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$ with a slant angle θ, then according to [45]:

$$g(T\chi_1, T\chi_2) = \cos^2\theta[pg(\chi_1, T\chi_2) + qg(\chi_1, \chi_2)]$$

and

$$g(N\chi_1, N\chi_2) = \sin^2\theta[pg(\chi_1, T\chi_2) + qg(\chi_1, \chi_2)],$$

$\forall \chi_1, \chi_2 \in \Gamma(T\mathcal{M})$.

Additionally,

$$T^2 = \cos^2\theta(pT + q\mathcal{I}),$$

Here, $\mathcal{I}$ denotes the identity operator acting on $\Gamma(T\mathcal{M})$, the space of smooth sections of the tangent bundle of $\mathcal{M}$, and

$$\nabla T^2 = p\cos^2\theta\nabla T.$$

Consider $\mathcal{M}_1$, a Riemannian manifold with constant sectional curvature c_1, and $\mathcal{M}_2$, a Riemannian manifold with constant sectional curvature c_2. In this case, the Riemannian curvature tensor $\overline{R}$ of the locally Riemannian product manifold can be expressed as follows.

$\overline{\mathcal{M}} = \mathcal{M}_1 \times \mathcal{M}_2$ is given by [43]

$$
\begin{aligned}
\tilde{\overline{R}}(\chi_1, \chi_2)\chi_3 \quad = \quad & \frac{1}{4}(c_1 + c_2)[g(\chi_2, \chi_3)\chi_1 - g(\chi_1, \chi_3)\chi_2] \\
& + \frac{1}{4}(c_1 + c_2)\Big\{ \frac{4}{(2\sigma_{p,q} - p)^2}[g(\phi\chi_2, \chi_3)\phi\chi_1 - g(\phi\chi_1, \chi_3)\phi\chi_2] \\
& + \frac{p^2}{(2\sigma_{p,q} - p)^2}[g(\chi_2, \chi_3)X - g(\chi_1, \chi_3)\chi_2] \\
& + \frac{2p}{(2\sigma_{p,q} - p)^2}[g(\phi\chi_1, \chi_3)\chi_2 + g(\chi_1, \chi_3)\phi\chi_2 \\
& - g(\phi\chi_2, \chi_3)\chi_1 - g(\chi_1, \chi_3)\phi\chi_1] \Big\} \\
& \pm \frac{1}{2}(c_1 - c_2)\Big\{ \frac{1}{(2\sigma_{p,q} - p)}[g(\chi_2, \chi_3)\phi\chi_1 - g(\chi_1, \chi_3)\phi\chi_2] \\
& + \frac{1}{(2\sigma_{p,q} - p)}[g(\phi\chi_2, \chi_3)\chi_1 - g(\phi\chi_1, \chi_3)\chi_2] \\
& + \frac{p}{(2\sigma_{p,q} - p)}[g(\chi_1, \chi_3)\chi_2 - g(\chi_2, \chi_3)\chi_1] \Big\}.
\end{aligned} \tag{9}
$$

From (5) and (9), we have

$$
\begin{aligned}
\overline{R}(\chi_1, \chi_2, \chi_3, \chi_4) = & \frac{1}{4}(c_1 + c_2)[g(\chi_2, \chi_3)g(\chi_1, \chi_4) - g(\chi_1, \chi_3)g(\chi_2, \chi_4)] \\
& + \frac{1}{4}(c_1 + c_2)\Big\{ \frac{4}{(2\sigma_{p,q} - p)^2}[g(\phi\chi_2, \chi_3)g(\phi\chi_1, \chi_4) - g(\phi\chi_1, \chi_3)g(\phi\chi_2, \chi_4)] \\
& + \frac{p^2}{(2\sigma_{p,q} - p)^2}[g(\chi_2, \chi_3)g(\chi_1, \chi_4) - g(\chi_1, \chi_3)g(\chi_2, \chi_4)] \\
& + \frac{2p}{(2\sigma_{p,q} - p)^2}[g(\phi\chi_1, \chi_3)g(\chi_2, \chi_4) + g(\chi_1, \chi_3)g(\phi\chi_2, \chi_4) \\
& - g(\phi\chi_2, \chi_3)g(\chi_1, \chi_4) - g(\chi_2, \chi_3)g(\phi\chi_1, \chi_4)] \Big\} \\
& \pm \frac{1}{2}(c_1 - c_2)\Big\{ \frac{1}{(2\sigma_{p,q} - p)}[g(\chi_2, \chi_3)g(\phi\chi_1, \chi_4) - g(\chi_1, \chi_3)g(\phi\chi_2, \chi_4)] \\
& + \frac{1}{(2\sigma_{p,q} - p)}[g(\phi\chi_2, \chi_3)g(\chi_1, \chi_4) - g(\phi\chi_1, \chi_3)g(\chi_2, \chi_4)] \\
& + \frac{p}{(2\sigma_{p,q} - p)}[g(\chi_1, \chi_3)g(\chi_2, \chi_4) - g(\chi_2, \chi_3)g(\chi_1, \chi_4)] \Big\} \\
& + \beta(\chi_1, \chi_3)g(\chi_2, \chi_4) - \beta(\chi_2, \chi_3)g(\chi_1, \chi_4) \\
& + \beta(\chi_2, \chi_4)g(\chi_1, \chi_3) - \beta(\chi_1, \chi_4)g(\chi_2, \chi_3).
\end{aligned} \tag{10}
$$

3. Ricci Tensor Analysis with Semi-Symmetric Metric Connection

The primary objective of this section is to introduce and analyze the principal outcome.

Theorem 1. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of θ in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$ with SSMC.*

Given a point p on $\mathcal{M}$ and a unit vector X in the tangent space $T_p\mathcal{M}$, the following inequality holds:

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2}\frac{c_1 - c_2}{\sqrt{p^2+4q}}\big[2\,\mathrm{tr}\,\phi - p(n-1)\big] - (n-2)\beta(X,X) - \lambda$$
$$+ \frac{1}{2}\frac{c_1+c_2}{p^2+4q}(n-1)\big[p^2 + 2q - \frac{1}{n-1}(p\,\mathrm{tr}\,\phi + q\cos^2\theta)\big]. \tag{11}$$

Moreover, if $H(p) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at p, and conversely.

Proof. Let $\{\varrho_1, \ldots, \varrho_n\}$ be an orthonormal tangent frame and $\{\varrho_{n+1}, \ldots, \varrho_m\}$ be an orthonormal frame of $T_x\mathcal{M}$ and $T_x^{\perp}\mathcal{M}$, respectively at any point $x \in \mathcal{M}$. Substituting $\chi_1 = \chi_4 = \varrho_i$, $\chi_2 = \chi_3 = \varrho_j$ in (10) with the Equation (4) and take $i \neq j$, we obtain

$$\begin{aligned}
R(\varrho_i,\varrho_j,\varrho_j,\varrho_i) &= \frac{1}{4}(c_1+c_2)\big[g(\varrho_j,\varrho_j)g(\varrho_i,\varrho_i) - g(\varrho_i,\varrho_j)g(\varrho_j,\varrho_i)\big] \\
&+ \frac{1}{4}(c_1+c_2)\bigg\{\frac{4}{(2\sigma_{p,q}-p)^2}\big[g(\phi\varrho_j,\varrho_j)g(\phi\varrho_i,\varrho_i) \\
&- g(\phi\varrho_i,\varrho_j)g(\phi\varrho_j,\varrho_i)\big] \\
&+ \frac{p^2}{(2\sigma_{p,q}-p)^2}\big[g(\varrho_j,\varrho_j)g(\varrho_i,\varrho_i) - g(\varrho_i,\varrho_j)g(\varrho_j,\varrho_i)\big] \\
&+ \frac{2p}{(2\sigma_{p,q}-p)^2}\big[g(\phi\varrho_i,\varrho_j)g(\varrho_j,\varrho_i) + g(\varrho_i,\varrho_j)g(\phi\varrho_j,\varrho_i) \\
&- g(\phi\varrho_j,\varrho_j)g(\varrho_i,\varrho_i) - g(\varrho_j,\varrho_j)g(\phi\varrho_i,\varrho_i)\big]\bigg\} \\
&\pm \frac{1}{2}(c_1-c_2)\bigg\{\frac{1}{(2\sigma_{p,q}-p)}\big[g(\varrho_j,\varrho_j)g(\phi\varrho_i,\varrho_i) \\
&- g(\varrho_i,\varrho_j)g(\phi\varrho_j,\varrho_i)\big] \\
&+ \frac{1}{(2\sigma_{p,q}-p)}\big[g(\phi\varrho_j,\varrho_j)g(\varrho_i,\varrho_i) - g(\phi\varrho_i,\varrho_j)g(\varrho_j,\varrho_i)\big] \\
&+ \frac{p}{(2\sigma_{p,q}-p)}\big[g(\varrho_i,\varrho_j)g(\varrho_j,\varrho_i) - g(\varrho_j,\varrho_j)g(\varrho_i,\varrho_i)\big]\bigg\} \\
&+ \beta(\varrho_i,\varrho_j)g(\varrho_j,\varrho_i) - \beta(\varrho_j,\varrho_j)g(\varrho_i,\varrho_i) \\
&+ \beta(\varrho_j,\varrho_i)g(\varrho_i,\varrho_j) - \beta(\varrho_i,\varrho_i)g(\varrho_j,\varrho_j) \\
&+ g(\zeta(\varrho_i,\varrho_i),\zeta(\varrho_j,\varrho_j)) - g(\zeta(\varrho_i,\varrho_j),\zeta(\varrho_j,\varrho_i)).
\end{aligned} \tag{12}$$

Using $1 \leq i,j \leq n$ in (12), we find

$$2\tau(x) = \frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\bigg\{2p^2 + 4q + \frac{4}{n(n-1)}\big[tr^2\phi - \cos^2\theta(p.trT + nq)\big]$$
$$- \frac{4p}{n}tr\phi\bigg\} - 2(n-1)\lambda$$
$$\pm \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) + n^2||H||^2 - ||\zeta||^2. \tag{13}$$

Now, we consider

$$\delta = 2\tau - \frac{n^2}{2}||\mathcal{H}||^2 \mp \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) + 2(n-1)\lambda$$
$$-\frac{1}{4}(c_1-c_2)\frac{n(n-1)}{p^2+4q}\left\{2p^2+4q+\frac{4}{n(n-1)}\left[tr^2\phi - \cos^2\theta(ptrT+nq)\right] - \frac{4p}{n}tr\phi\right\}. \quad (14)$$

Combining (13) and (14), we obtain

$$n^2||H||^2 = 2(\delta + ||\zeta||^2). \quad (15)$$

Consequently, when employing the orthonormal frame $\{\varrho_1, \ldots, \varrho_n\}$, Equation (15) takes on the subsequent expression:

$$\left(\sum_{i=1}^{n}\zeta_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(\zeta_{ii}^{n+1})^2 + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2\right\}. \quad (16)$$

If we substitute $d_1 = \zeta_{11}^{n+1}$, $d_2 = \sum_{i=2}^{n-1}\zeta_{ii}^{n+1}$ and $d_3 = \zeta_{nn}^{n+1}$, then (16) reduces to

$$\left(\sum_{i=1}^{3}d_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3}d_i^2 + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2 - \sum_{2\leq j\neq k_\leq n-1}\zeta_{jj}^{n+1}\zeta_{kk}^{n+1}\right\}. \quad (17)$$

As a result, d_1, d_2, d_3 fulfill Chen's Lemma [41], that is,

$$\left(\sum_{i=1}^{3}d_i\right)^2 = 2\left(v + \sum_{i=1}^{3}d_i^2\right).$$

Clearly, $2d_1d_2 \geq v$ with equality holds if $d_1 + d_2 = d_3$ and vice versa. This signifies

$$\sum_{1\leq j\neq k_\leq n-1}\zeta_{jj}^{n+1}\zeta_{kk}^{n+1} \geq \delta + 2\sum_{i<j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2. \quad (18)$$

It is possible to write (18) as

$$\frac{n^2}{2}||\mathcal{H}||^2 \pm \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) - 2(n-1)\lambda$$
$$+\frac{1}{4}(c_1-c_2)\frac{n(n-1)}{p^2+4q}\left\{2p^2+4q+\frac{4}{n(n-1)}\left[tr^2\phi - \cos^2\theta(ptrT+nq)\right] - \frac{4p}{n}tr\phi\right\}$$
$$\geq 2\tau - \sum_{1\leq j\neq k_\leq n-1}\zeta_{jj}^{n+1}\zeta_{kk}^{n+1} + 2\sum_{i<j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2. \quad (19)$$

Invoking the Gauss equation once again and making use of (19), we obtain

$$Ric(X) \leq \frac{n^2}{4}||H||^2 \pm \frac{1}{2}\frac{(c_1-c_2)}{\sqrt{p^2+4q}}\left[2tr\phi - p(n-1)\right] - (n-2)\beta(X,X) - \lambda$$
$$+\frac{1}{2}\frac{(c_1+c_2)}{p^2+4q}(n-1)\left[p^2+2q-\frac{1}{n-1}(ptr\phi + q\cos^2\theta)\right]. \quad (20)$$

Hence, we have obtained the required inequality (1).

Further, assume that $H(p) = 0$. Equality holds in (1) if and only if

$$\begin{cases} \zeta^r_{1n} = \cdots = \zeta^r_{n-1n} = 0 \\ \zeta^r_{nn} = \sum_{i=1}^{n-1} \zeta^r_{ii}, \quad r \in \{n+1, \dots, m\}. \end{cases} \tag{21}$$

Then,

$$\zeta^r_{in} = 0,$$

for all $i \in \{1, \dots, n\}$, and $r \in \{n+1, \dots, m\}$, i.e., $X \in \mathcal{N}_p$.

In conclusion, the equality condition of (1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} \zeta^r_{ij} = 0, i \neq j, r \in \{n+1, \dots, m\} \\ \zeta^r_{11} + \cdots + \zeta^r_{nn} - 2\zeta^r_{ii} = 0, \quad i \in \{1, \dots, n\} \quad r \in \{n+1, \dots, m\}. \end{cases} \tag{22}$$

From here, we separate the two situations:

(i) p is a totally geodesic point if $n \neq 2$;
(ii) It is evident that p is a totally umbilical point if $n = 2$.

It goes without saying that the converse applies. □

4. Some Applications

We can have two different approaches to see the various applications: either by considering particular classes of locally metallic product space forms, or by considering particular classes of θ-slant submanifolds.

4.1. Application by Considering Particular Classes of θ-Slant Submanifolds

Two specific classes of θ-slant submanifolds, namely, invariant and anti-invariant submanifolds, were introduced in [45] for metallic Riemannian manifolds. With the help of the definitions of these submanifolds in Theorem 1, we obtain the following results.

Corollary 1. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is invariant in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$ with SSMC.*

For any unit vector X in the tangent space $T_p\mathcal{M}$ at a point p on $\mathcal{M}$, the following inequality holds:

$$\begin{aligned} Ric(X) \leq{}& \frac{n^2}{4}||H||^2 \pm \frac{1}{2}\frac{(c_1 - c_2)}{\sqrt{p^2 + 4q}}[2tr\phi - p(n-1)] - (n-2)\beta(X, X) - \lambda \\ &+ \frac{1}{2}\frac{(c_1 + c_2)}{p^2 + 4q}(n-1)\left[p^2 + 2q - \frac{1}{n-1}(ptr\phi + q)\right]. \end{aligned} \tag{23}$$

Moreover, if $H(p) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at p, and vice versa.

Proof. The result is directly obtained by taking $\theta = 0$ in Theorem 1. □

Corollary 2. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is anti-invariant in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$ with SSMC.*

For any unit vector X in the tangent space $T_p\mathcal{M}$ at a point p on $\mathcal{M}$, the following inequality holds:

$$Ric(X) \leq \frac{n^2}{4}||H||^2 \pm \frac{1}{2}\frac{(c_1-c_2)}{\sqrt{p^2+4q}}\big[2tr\phi - p(n-1)\big] - (n-2)\beta(X,X) - \lambda \\ + \frac{1}{2}\frac{(c_1+c_2)}{p^2+4q}(n-1)\big[p^2+2q-\frac{1}{n-1}(ptr\phi)\big]. \tag{24}$$

Moreover, if $H(p)=0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n=2$, the equality case of this inequality holds true for all unit tangent vectors at p, and vice versa.

Proof. The result is directly obtained by taking $\theta = \frac{\pi}{2}$ in Theorem 1. □

4.2. Application by Considering Particular Classes of Locally Metallic Product Space Forms

As a consequence of Theorem 1 and together with Remark 2 (1), we obtained the following results.

Corollary 3. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of θ in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1)\times\mathcal{M}_2(c_2)$ with SSMC.*

For any unit vector X in the tangent space $T_p\mathcal{M}$ at a point p on $\mathcal{M}$, the following inequality holds:

$$Ric(X) \leq \frac{n^2}{4}||H||^2 \pm \frac{1}{2\sqrt{5}}(c_1-c_2)\big[2.tr\phi - (n-1)\big] - (n-2)\beta(X,X) - \lambda \\ + \frac{1}{10}(c_1+c_2)(n-1)\big[3-\frac{1}{n-1}(tr\phi + \cos^2\theta)\big]. \tag{25}$$

Moreover, if $H(p)=0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n=2$, the equality case of this inequality holds true for all unit tangent vectors at p, and vice versa.

Proof. The result is directly obtained by taking $p=q=1$ in Theorem 1. □

Corollary 4. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is invariant in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1)\times\mathcal{M}_2(c_2)$.*

For any unit vector X in the tangent space $T_p\mathcal{M}$ at a point p on $\mathcal{M}$, the following inequality holds:

$$Ric(X) \leq \frac{n^2}{4}||H||^2 \pm \frac{1}{2\sqrt{5}}(c_1-c_2)\big[2tr\phi - (n-1)\big] \\ + \frac{1}{10}(c_1+c_2)\big[3n-4-tr\phi\big] - (n-2)\beta(X,X) - \lambda. \tag{26}$$

Moreover, if $H(p)=0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n=2$, the equality case of this inequality holds true for all unit tangent vectors at p, and vice versa.

Proof. The result is directly obtained by taking $\theta=0$ and $p=q=1$ in Theorem 1. □

Corollary 5. *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is anti-invariant in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1)\times\mathcal{M}_2(c_2)$ with SSMC.*

For any unit vector X in the tangent space $T_p\mathcal{M}$ at a point p on $\mathcal{M}$, the following inequality holds:

$$Ric(X) \leq \frac{n^2}{4}||H||^2 \pm \frac{1}{2\sqrt{5}}(c_1 - c_2)\left[2tr\phi - (n-1)\right] + \frac{1}{10}(c_1 + c_2)\left[3n - 3 - tr\phi\right] - (n-2)\beta(X,X) - \lambda. \quad (27)$$

Moreover, if $H(p) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at p if and only if X belongs to the normal space $\mathcal{N}_p$. Finally, when p is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at p, and vice versa.

Proof. The result is directly obtained by taking $\theta = \frac{\pi}{2}$ and $p = q = 1$ in Theorem 1. □

Remark 3. *Similar results can also be obtained for other particular classes such as copper, silver, nickel, bronze, etc., by providing different particular values to p and q with the help of Remark 2.*

5. Conclusions

This article has not only explored the Ricci tensor of slant submanifolds within locally metallic product space forms equipped with a semi-symmetric metric connection but has also contributed to our understanding of these mathematical constructs. The derivation of the Chen–Ricci inequality, the analysis of its equality case, and the applications arising from our findings collectively demonstrate the significance and relevance of this research.

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