axioms

MDPI

Article

On Some New Dynamic Hilbert-Type Inequalities across Time Scales

Mohammed Zakarya [1], Ahmed I. Saied [2,3], Amirah Ayidh I Al-Thaqfan [4], Maha Ali [4] and Haytham M. Rezk [5,*]

1 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; mzibrahim@kku.edu.sa
2 Mathematical Institute, Slovak Academy of Sciences, Grekošákova 6, 04001 Košice, Slovakia; ahmed.abosaied@fsc.bu.edu.eg
3 Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt
4 Department of Mathematics, College of Arts and Sciences, King Khalid University, P.O. Box 64512, Abha 62529, Sarat Ubaidah, Saudi Arabia; amerhm@kku.edu.sa (A.A.I.A.-T.); mayoali@kku.edu.sa (M.A.)
5 Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt
* Correspondence: haythamrezk@azhar.edu.eg

Abstract: In this article, we present some novel dynamic Hilbert-type inequalities within the framework of time scales $\mathbb{T}$. We achieve this by utilizing Hölder's inequality, the chain rule, and the mean inequality. As specific instances of our findings (when $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), we obtain the discrete and continuous analogues of previously established inequalities. Additionally, we derive other inequalities for different time scales, such as $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, which, to the best of the authors' knowledge, is a largely novel conclusion.

Keywords: Hilbert-type inequalities; Hölder's inequality; chain rule; mean inequality; dynamic inequality; time scale delta calculus

MSC: 26D10; 26D15; 26E70

check for updates

Citation: Zakarya, M.; Saied, A.I.; Al-Thaqfan, A.A.I.; Ali, M.; Rezk, H.M. On Some New Dynamic Hilbert-Type Inequalities across Time Scales. *Axioms* **2024**, *13*, 475. https://doi.org/10.3390/axioms13070475

Academic Editors: Cheng-Cheng Zhu and Hari Mohan Srivastava

Received: 19 May 2024
Revised: 29 June 2024
Accepted: 12 July 2024
Published: 14 July 2024

1. Introduction

In the early 1900s, the renowned mathematician David Hilbert formulated his famous inequality, known as the double series Hilbert inequality (see [1]), wherein he established that if $\{G_m\}_{m=1}^{\infty}$, $\{Z_n\}_{n=1}^{\inf}$ are two real sequences, such that $0 < \sum_{m=1}^{\infty} G_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} Z_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{G_m Z_n}{m+n} \leq 2\pi\left(\sum_{m=1}^{\infty} G_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} Z_n^2\right)^{\frac{1}{2}}. \tag{1}$$

In 1911, Schur demonstrated in his paper [2] that the constant π in (1) is optimal. Furthermore, he established the integral analogue of (1), which is now recognized as the Hilbert integral inequality in the form

$$\int_0^{\infty}\int_0^{\infty}\frac{S(x)T(y)}{x+y}dxdy \leq \pi\left(\int_0^{\infty} S^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^{\infty} T^2(y)dy\right)^{\frac{1}{2}}, \tag{2}$$

where S and T are real functions such that $0 < \int_0^{\inf} S^2(x)dx < \infty$, $0 < \int_0^{\infty} T^2(y)dy < \infty$, and π in (2) is still the best possible constant factor.

The inequalities expressed as (1) and (2) are crucial in the theory and application of integral inequalities, especially in analyzing both the qualitative and quantitative aspects of solutions to differential and integral equations. Recently, there has been rapid development in fractal theory, which has found widespread use in science and engineering. Some researchers have utilized fractal theory and weight function methods to generalize classical inequalities effectively. For instance, Liu [3] established a Hilbert-type integral inequality and its equivalent form on a fractal set. Hilbert-type inequalities play a significant role in mathematics, particularly in complex and numerical analysis. Over the years, these inequalities have seen numerous refinements, generalizations, extensions, and applications in the literature (see [4–7]).

In 1925, Hardy [8] extended (1) by introducing a pair of conjugate exponents (η, λ), where $\eta, \lambda > 1$ and satisfying $1/\eta + 1/\lambda = 1$, as follows. If G_m, $Z_n \geq 0$, such that $0 < \sum_{m=1}^{\infty} G_m^{\eta} < \infty$, and $0 < \sum_{n=1}^{\infty} Z_n^{\lambda} < \infty$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{G_m Z_n}{m+n} \leq \frac{\pi}{\sin \frac{\pi}{\eta}} \left(\sum_{m=1}^{\infty} G_m^{\eta} \right)^{\frac{1}{\eta}} \left(\sum_{n=1}^{\infty} Z_n^{\lambda} \right)^{\frac{1}{\lambda}}. \tag{3}$$

In [9], the authors established the equivalent integral form of (3) as

$$\int_0^{\infty} \int_0^{\infty} \frac{S(x)T(y)}{x+y} dxdy \leq \frac{\pi}{\sin \frac{\pi}{\eta}} \left(\int_0^{\infty} S^{\eta}(x)dx \right)^{\frac{1}{\eta}} \left(\int_0^{\infty} T^{\lambda}(y)dy \right)^{\frac{1}{\lambda}}, \tag{4}$$

where S, $T \geq 0$, such that $0 < \int_0^{\infty} S^p(x)dx < \infty$ and $0 < \int_0^{\infty} T^{\lambda}(y)dy < \infty$. The constant factor $\pi/\sin(\pi/\eta)$ in (3) and (4) is optimal.

In 1998, Pachpatte [10] presented a new inequality akin to the Hilbert inequality as follows: let $G(s) : \{0, 1, 2, \ldots, p\} \subset \mathbb{N} \to \mathbb{R}$ and $Z(\vartheta) : \{0, 1, 2, \ldots, q\} \subset \mathbb{N} \to \mathbb{R}$ with $G(0) = Z(0) = 0$. Define the operators as $\nabla G_s = G_s - G_{s-1}$, $\nabla Z_{\vartheta} = Z_{\vartheta} - Z_{\vartheta-1}$. Then

$$\begin{aligned} \sum_{s=1}^{p} \sum_{\vartheta=1}^{q} \frac{|G_s||Z_{\vartheta}|}{s+\vartheta} \quad &\leq \quad \frac{1}{2}\sqrt{pq} \left(\sum_{s=1}^{p} (p-s+1)|\nabla G_s|^2 \right)^{\frac{1}{2}} \\ &\times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla Z_{\vartheta}|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{5}$$

In 2000, Pachpatte [11] generalized (5) by introducing one pair of conjugate exponents $(\eta, \mu) : \eta, \mu > 1$ with $1/\eta + 1/\mu = 1$ and proved that if $G(s) : \{0, 1, 2, \ldots, p\} \subset \mathbb{N} \to \mathbb{R}$ and $Z(\vartheta) : \{0, 1, 2, \ldots, q\} \subset \mathbb{N} \to \mathbb{R}$ with $G(0) = Z(0) = 0$, then

$$\begin{aligned} \sum_{s=1}^{p} \sum_{\vartheta=1}^{q} \frac{|G_s||Z_{\vartheta}|}{\mu s^{\eta-1} + \eta \vartheta^{\mu-1}} \quad &\leq \quad \frac{1}{\eta\mu} p^{\frac{\eta-1}{\eta}} q^{\frac{\mu-1}{\mu}} \left(\sum_{s=1}^{p} (p-s+1)|\nabla G_s|^{\eta} \right)^{\frac{1}{\eta}} \\ &\times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla Z_{\vartheta}|^{\mu} \right)^{\frac{1}{\mu}}. \end{aligned} \tag{6}$$

In 2002, Kim et al. [12] extended (6) and demonstrated that if $\eta, \mu > 1$, $G(s) : \{0, 1, 2, \ldots, p\} \subset \mathbb{N} \to \mathbb{R}$ and $Z(\vartheta) : \{0, 1, 2, \ldots, q\} \subset \mathbb{N} \to \mathbb{R}$ with $G(0) = Z(0) = 0$, then

$$\begin{aligned} \sum_{s=1}^{p} \sum_{\vartheta=1}^{q} \frac{|G_s||Z_{\vartheta}|}{\mu s^{\frac{(\eta-1)(\eta+\mu)}{\eta\mu}} + \eta \vartheta^{\frac{(\mu-1)(\eta+\mu)}{\eta\mu}}} \quad &\leq \quad \frac{1}{\eta+\mu} p^{\frac{\eta-1}{\eta}} q^{\frac{\mu-1}{\mu}} \left(\sum_{s=1}^{p} (p-s+1)|\nabla G_s|^{\eta} \right)^{\frac{1}{\eta}} \\ &\times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla Z_{\vartheta}|^{\mu} \right)^{\frac{1}{\mu}}. \end{aligned} \tag{7}$$

Also, the authors [12] established the continuous analogue of (7) as follows: if η, $\mu > 1$ and $S(\theta)$, and $T(\vartheta)$ are real continuous functions on $(0,x)$, $(0,y)$, respectively, with $S(0) = T(0) = 0$, then for $x, y \in (0,\infty)$, we have

$$\int_0^x \int_0^y \frac{|S(\theta)||T(\vartheta)|}{\mu\theta^{\frac{(\eta-1)(\eta+\mu)}{\eta\mu}} + \eta\vartheta^{\frac{(\mu-1)(\eta+\mu)}{\eta\mu}}} d\theta d\vartheta$$
$$\leq \frac{1}{\eta+\mu} x^{\frac{\eta-1}{\eta}} y^{\frac{\mu-1}{\mu}} \left(\int_0^x (x-\theta)\left|S'(\theta)\right|^\eta d\theta\right)^{\frac{1}{\eta}} \left(\int_0^y (y-\vartheta)\left|T'(\vartheta)\right|^\mu d\vartheta\right)^{\frac{1}{\mu}}.$$

In 2011, Chang-Jian et al. [13] generalized (5) and demonstrated that if $\lambda_i > 1$, such that $1/\lambda_i + 1/q_i = 1$, $G_i(s_i)$ is a real sequence defined for $s_i = 0, 1, 2, \ldots, m_i$, where m_i is a natural number and $G_i(0) = 0$, $i = 1, 2, \ldots, n$.. Define the operator ∇ by $\nabla G_i(s_i) = G_i(s_i) - G_i(s_i - 1)$ for any function $G_i(s_i)$, $i = 1, 2, \ldots, n$. Then

$$\sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} \frac{\prod_{i=1}^n |G_i(s_i)|}{\left(\sum\limits_{i=1}^n s_i/q_i\right)^{\sum\limits_{i=1}^n 1/q_i}}$$
$$\leq M \prod_{i=1}^n \left(\sum_{s_i=1}^{m_i} (m_i - s_i + 1) |\nabla G_i(s_i)|^{\lambda_i}\right)^{\frac{1}{\lambda_i}}, \tag{8}$$

where

$$M = \left(n - \sum_{i=1}^n \frac{1}{\lambda_i}\right)^{\sum_{i=1}^n 1/\lambda_i - n} . \prod_{i=1}^n m_i^{1/q_i}.$$

Also, the authors of [13] proved that if $h_i \geq 1$ and $\lambda_i > 1$ are constants such that $1/\lambda_i + 1/q_i = 1$, $T_i(s_i)$ is a real valued differentiable function defined on $[0, x_i)$, where $x_i \in (0,\infty)$. Assume $T_i(0) = 0$ for $i = 1, 2, \ldots, n$. Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left|T_i^{h_i}(s_i)\right|}{\left(\sum\limits_{i=1}^n s_i/q_i\right)^{\sum\limits_{i=1}^n 1/q_i}} ds_n \ldots ds_1$$
$$\leq K \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) \left|T_i^{h_i-1}(s_i).T_i'(s_i)\right|^{\lambda_i} ds_i\right)^{\frac{1}{\lambda_i}}, \tag{9}$$

where

$$K = \left(n - \sum_{i=1}^n \frac{1}{\lambda_i}\right)^{\sum_{i=1}^n 1/\lambda_i - n} . \prod_{i=1}^n h_i x_i^{1/q_i}.$$

Also, they demonstrated that if $\lambda_i, q_i > 1$, such that $1/\lambda_i + 1/q_i = 1$, $G_i(s_i, t_i)$ is a real sequence defined for (s_i, t_i), where $s_i = 0, 1, 2, \ldots, m_i$, $t_i = 0, 1, 2, \ldots, n_i$ and m_i, n_i $(i = 1, 2, \ldots, n)$ are natural numbers and assuming that $G_i(0, t_i) = G_i(s_i, 0) = 0$ for all $i = 1, 2, \ldots, n$. Define the operator ∇_1 and ∇_2 as

$$H\nabla_1 G_i(s_i, t_i) = G_i(s_i, t_i) - G_i(s_i - 1, t_i),$$

$$\nabla_2 G_i(s_i, t_i) = G_i(s_i, t_i) - G_i(s_i, t_i - 1).$$

Then

$$\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\cdots\sum_{s_n=1}^{m_n}\sum_{t_n=1}^{n_n}\frac{\prod_{i=1}^{n}|G_i(s_i,t_i)|}{\left(\sum\limits_{i=1}^{n}s_it_i/q_i\right)^{\sum\limits_{i=1}^{n}1/q_i}}$$
$$\leq \quad L\prod_{i=1}^{n}\left(\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}(m_i-s_i+1)(n_i-t_i+1)|\nabla_2\nabla_1G_i(s_i,t_i)|^{\lambda_i}\right)^{\frac{1}{\lambda_i}}, \tag{10}$$

where

$$L=\left(n-\sum_{i=1}^{n}\frac{1}{\lambda_i}\right)^{\sum_{i=1}^{n}1/\lambda_i-n}.\prod_{i=1}^{n}(m_in_i)^{1/q_i}.$$

In the last few decades, much attention has been devoted to establishing discrete analogues of the corresponding continuous results in various fields of analysis. This appears along with establishing a dynamic inequality in this paper by using a general domain called a time scale $\mathbb{T}$. A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. For more details about dynamic inequalities and applications on time scales, see [14–19].

The aim of this paper is to prove similar analogues of the inequalities (8), (9) on time scales, and we can also generalize (10) on time scale delta calculus for an increasing function by establishing some new dynamic Hilbert-type inequalities on time scale delta calculus.

The remainder of this paper is organized as follows. In Section 2, we show some basics of the time scale theory and some lemmas on time scales needed in Section 3, where we prove our results. These results as special cases when $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=\mathbb{R}$ give the inequalities ((8) and (10)), (9), respectively. Also, we can obtain other inequalities on different time scales, like $\mathbb{T}=q^{\mathbb{N}}$ for $q>1$.

2. Preliminaries and Basic Lemmas

In 2001, Bohner and Peterson [20] defined the forward jump operator by $\sigma(\tau):=\inf\{s\in\mathbb{T}:s>\tau\}$. For any function $S:\mathbb{T}\to\mathbb{R}$, the notation $S^{\sigma}(\tau)$ denotes $S(\sigma(\tau))$. We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$.

In the following, we state the definition of $rd-$continuous and $\Delta-$derivative function.

Definition 1 ([20]). *A function $S:\mathbb{T}\to\mathbb{R}$ is called $rd-$continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of $rd-$continuous functions $S:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.*

Definition 2 ([20]). *Assume that $S:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}$. We define $S^{\Delta}(t)$ to be the number, provided it exists, as follows: for any $\varepsilon>0$, there is a neighborhood U of t, $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$, such that*

$$|S(\sigma(t))-S(\gamma)-S^{\Delta}(t)(\sigma(t)-\gamma)|\leq\varepsilon|\sigma(t)-\gamma| \quad \textit{for all} \quad \gamma\in U,\ \gamma\neq\sigma(t).$$

In this case, we say $S^{\Delta}(t)$ is the delta or Hilger derivative of S at t.

In the following, we state several values of Δ—differentiable function at a point $t\in\mathbb{T}$.

Theorem 1 ([20]). *Assume $S:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}^k$. Then we have the following.*

1. *If S is differentiable at t, then f is continuous at t.*
2. *If S is continuous at t and t is right-scattered, then S is differentiable at t with*

$$S^{\Delta}(t)=\frac{S(\sigma(t))-S(t)}{\sigma(t)-t}.$$

3. *If t is right-dense, then S is differentiable if the limit*

$$\lim_{\gamma \to t} \frac{S(t) - S(\gamma)}{t - \gamma},$$

exists as a finite number. In this case,

$$S^{\Delta}(t) = \lim_{\gamma \to t} \frac{S(t) - S(\gamma)}{t - \gamma}.$$

Example 1. *1. If $\mathbb{T} = \mathbb{R}$, then for $S : \mathbb{R} \to \mathbb{R}$, we obtain*

$$S^{\Delta}(t) = \lim_{\gamma \to t} \frac{S(t) - S(\gamma)}{t - \gamma} = S^{'}(t),$$

where $S^{'}$ is the usual derivative.

2. If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t + 1$, and for $S : \mathbb{N} \to \mathbb{R}$, we have

$$S^{\Delta}(t) = \frac{S(\sigma(t)) - S(t)}{\sigma(t) - t} = \frac{S(t+1) - S(t)}{1} = \Delta S(t),$$

where Δ is the usual forward difference operator.

3. If $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, then we have $\sigma(t) = qt$ and

$$S^{\Delta}(t) = \Delta_q S(t) = \frac{S(q\,t) - S(t)}{(q-1)\,t}.$$

The following theorem is about the chain rule formula on time scales.

Theorem 2 (Chain Rule [20] Theorem 1.87)**.** *Assume $T : \mathbb{R} \to \mathbb{R}$ is continuous, $T : \mathbb{T} \to \mathbb{R}$ is delta-differentiable on $\mathbb{T}$ and $S : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then γ exists in the real interval $[t, \sigma(t)]$ with*

$$(S \circ T)^{\Delta}(t) = S^{'}(T(\gamma))T^{\Delta}(t). \tag{11}$$

Definition 3 ([20])**.** *A function $S : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $s : \mathbb{T} \to \mathbb{R}$, provided that*

$$S^{\Delta}(t) = s(t) \quad \text{holds for all} \quad t \in \mathbb{T}^k.$$

In this case, the Cauchy integral of s is defined by

$$\int_r^{\alpha} s(t)\Delta t = S(\alpha) - S(r), \quad \text{for all } \; r, \alpha \in \mathbb{T}.$$

Theorem 3 ([20])**.** *Every rd–continuous function $S : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then*

$$\left(\int_{t_0}^{t} S(\tau)\Delta\tau \right)^{\Delta} = S(t), \quad \text{for} \quad t \in \mathbb{T}.$$

In the following, we present the properties of integration on time scales.

Theorem 4 ([20])**.** *If $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $S, T \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$, then*

1. $\int_a^b [\alpha S(t) + \beta T(t)]\Delta t = \alpha \int_a^b S(t)\Delta t + \beta \int_a^b T(t)\Delta t.$
2. $\int_a^b S(t)\Delta t = -\int_b^a S(t)\Delta t.$
3. $\int_a^b S(t)\Delta t = \int_a^c S(t)\Delta t + \int_c^b S(t)\Delta t.$
4. $\int_a^a S(t)\Delta t = 0.$
5. $\left| \int_a^b S(t)\Delta t \right| \leq \int_a^b |S(t)|\Delta t.$

6. *If $S(t) \geq 0$ for all $t \in [a,b]_{\mathbb{T}}$, then $\int_a^b S(t)\Delta t \geq 0$.*

Theorem 5 ([21]). *Let $a, b \in \mathbb{T}$ and $S \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then, the following properties hold:*
(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b S(t)\Delta t = \int_a^b S(t)dt.$$

(ii) If $\mathbb{T} = \mathbb{N}_0 \cup \{0\}$, then

$$\int_a^b S(t)\Delta t = \sum_{t=a}^{b-1} S(t).$$

(iii) If $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0, q > 1\}$, then

$$\int_{t_0}^{\infty} S(t)\Delta t = \sum_{k=n_0}^{\infty} S(q^k)\mu(q^k).$$

In the following, we present some auxiliary lemmas that we need to prove our results.

Lemma 1 (Integration by Parts [21]). *If $a, b \in \mathbb{T}$ and $S, T \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, then*

$$\int_a^b S(t)T^{\Delta}(t)\Delta t = [S(t)T(t)]_a^b - \int_a^b S^{\Delta}(t)T^{\sigma}(t)\Delta t. \tag{12}$$

Lemma 2 (lHölder's Inequality [21,22]). *If $a, b \in \mathbb{T}$ and $S, T \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, then*

$$\int_a^b |S(t)T(t)|\Delta t \leq \left[\int_a^b |S(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |T(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{13}$$

where $\gamma > 1$ and $1/\gamma + 1/\nu = 1$. The inequality (13) is reversed for $0 < \gamma < 1$ or $\gamma < 0$.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales. Assume that CC_{rd} denotes the set of functions $S(t_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$, where S is $rd-$ continuous in t_1 and t_2. Let CC'_{rd} denote the set of all functions CC_{rd} for which both the Δ_1 partial derivative with respect to t_1 and the Δ_2 partial derivative with respect to t_2 exist and are in CC_{rd}.

Lemma 3 (Two dimensional Hölder's inequality [23] Theorem 3.3). *Assume that $a, b \in \mathbb{T}$ with $a < b$, $S, T \in CC_{rd}([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \mathbb{R})$ and $\gamma, \nu > 1$ such that $1/\gamma + 1/\nu = 1$. Then*

$$\int_a^b \int_a^b |S(\tau,\xi)T(\tau,\xi)|\Delta_1\tau\Delta_2\xi \leq \left[\int_a^b \int_a^b |S(\tau,\xi)|^{\gamma}\Delta_1\tau\Delta_2\xi\right]^{\frac{1}{\gamma}} \left[\int_a^b \int_a^b |T(\tau,\xi)|^{\nu}\Delta_1\tau\Delta_2\xi\right]^{\frac{1}{\nu}}. \tag{14}$$

Lemma 4 (Fubini's theorem [24]). *If $a, b, c, d \in \mathbb{T}$ and $S \in CC_{rd}([a,b]_{\mathbb{T}} \times [c,d]_{\mathbb{T}}, \mathbb{R})$ is $\Delta-$integrable, then*

$$\int_a^b \left(\int_c^d S(x,y)\Delta_2 y\right)\Delta_1 x = \int_c^d \left(\int_a^b S(x,y)\Delta_1 x\right)\Delta_2 y.$$

Lemma 5 (Mean inequality [9]). *If $\alpha_i, \beta_i > 0$ for $i = 1, 2, \ldots, n$, then*

$$\prod_{i=1}^{n} \alpha_i^{\beta_i} \leq \frac{(\sum_{i=1}^{n} \alpha_i\beta_i)^{\sum_{i=1}^{n}\beta_i}}{(\sum_{i=1}^{n}\beta_i)^{\sum_{i=1}^{n}\beta_i}}. \tag{15}$$

Lemma 6. *Let $p_i, r_i > 1$ with $1/p_i + 1/r_i = 1$ and $s_i > 0$, where $i = 1, 2, \ldots, n$. Then*

$$\prod_{i=1}^{n} s_i^{1/r_i} \leq \frac{(\sum_{i=1}^{n} s_i/r_i)^{\sum_{i=1}^{n} 1/r_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{\left(n - \sum_{i=1}^{n} 1/p_i\right)}}. \tag{16}$$

Proof. Applying Lemma 5 with $\alpha_i = s_i$ and $\beta_i = 1/r_i$, we observe that

$$\prod_{i=1}^{n} s_i^{1/r_i} \le \frac{(\sum_{i=1}^{n} s_i/r_i)^{\sum_{i=1}^{n} 1/r_i}}{(\sum_{i=1}^{n} 1/r_i)^{\sum_{i=1}^{n} 1/r_i}}. \tag{17}$$

Since $1/r_i = 1 - 1/p_i$, we can obtain that

$$\sum_{i=1}^{n} 1/r_i = \sum_{i=1}^{n} (1 - 1/p_i) = n - \sum_{i=1}^{n} 1/p_i,$$

and then the inequality (17) becomes

$$\prod_{i=1}^{n} s_i^{1/r_i} \le \frac{(\sum_{i=1}^{n} s_i/r_i)^{\sum_{i=1}^{n} 1/r_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{n - \sum_{i=1}^{n} 1/p_i}},$$

which is (16). □

3. Main Results

In this section, we present the key results of our study. Firstly, we establish the time scale version of (8).

Theorem 6. *Let $a_i, \varepsilon_i \in \mathbb{T}$, $p_i, r_i > 1$, such that $1/p_i + 1/r_i = 1$ and let $\lambda_i \in C_{rd}([a_i, \varepsilon_i]_{\mathbb{T}}, \mathbb{R})$ be a delta-differentiable function with $\lambda_i(a_i) = 0$; $i = 1, 2, \ldots, n$. Then*

$$\begin{aligned} &\int_{a_n}^{\varepsilon_n} \cdots \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n} |\lambda_i(\xi_i)|}{(\sum_{i=1}^{n} (\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}} \Delta\xi_1 \ldots \Delta\xi_n \\ \le\ & M \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} [\varepsilon_i - \sigma(\xi_i)] \left| \lambda_i^{\Delta}(\xi_i) \right|^{p_i} \Delta\xi_i \right)^{\frac{1}{p_i}}, \end{aligned} \tag{18}$$

where

$$M = \left(n - \sum_{i=1}^{n} 1/p_i \right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} (\varepsilon_i - a_i)^{\frac{1}{q_i}}. \tag{19}$$

Proof. Applying the property (5) of Theorem 4 and the hypothesis $\lambda_i(a_i) = 0$, we obtain

$$\begin{aligned} \int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta}(t_i) \right| \Delta t_i &\ge \left| \int_{a_i}^{\xi_i} \lambda_i^{\Delta}(t_i) \Delta t_i \right| \\ &= |\lambda_i(\xi_i) - \lambda_i(a_i)| = |\lambda_i(\xi_i)|, \end{aligned}$$

and then

$$\prod_{i=1}^{n} |\lambda_i(\xi_i)| \le \prod_{i=1}^{n} \int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta}(t_i) \right| \Delta t_i. \tag{20}$$

Applying (13) on $\int_{a_i}^{\xi_i} |\lambda_i^{\Delta}(t_i)| \Delta t_i$ with $f(t_i) = |\lambda_i^{\Delta}(t_i)|$ and $g(t_i) = 1$, we observe that

$$\begin{aligned} \int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta}(t_i) \right| \Delta t_i &\le \left(\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta}(t_i) \right|^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \left(\int_{a_i}^{\xi_i} \Delta t_i \right)^{\frac{1}{r_i}} \\ &= (\xi_i - a_i)^{\frac{1}{r_i}} \left(\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta}(t_i) \right|^{p_i} \Delta t_i \right)^{\frac{1}{p_i}}, \end{aligned}$$

and then

$$\begin{aligned}\prod_{i=1}^{n}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|\Delta t_i &\leq \prod_{i=1}^{n}(\xi_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\\ &= \prod_{i=1}^{n}(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}. \end{aligned}\tag{21}$$

Substituting (21) into (20), we obtain

$$\prod_{i=1}^{n}|\lambda_i(\xi_i)| \leq \prod_{i=1}^{n}(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}.\tag{22}$$

Applying (16) with $s_i=\xi_i-a_i$, we have

$$\prod_{i=1}^{n}(\xi_i-a_i)^{1/r_i} \leq \frac{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}{\left(n-\sum_{i=1}^{n}1/p_i\right)^{n-\sum_{i=1}^{n}1/p_i}}.\tag{23}$$

Substituting (23) into (22), we obtain

$$\prod_{i=1}^{n}|\lambda_i(\xi_i)| \leq \frac{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}{\left(n-\sum_{i=1}^{n}1/p_i\right)^{n-\sum_{i=1}^{n}1/p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}.\tag{24}$$

Dividing (24) by $\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}$ and integrating over ξ_i from a_i to ε_i, $i=1,2,\ldots,n$, we observe that

$$\begin{aligned}&\int_{a_n}^{\varepsilon_n}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}|\lambda_i(\xi_i)|}{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta\xi_1\ldots\Delta\xi_n\\ &\leq \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\int_{a_n}^{\varepsilon_n}\cdots\int_{a_1}^{\varepsilon_1}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\Delta\xi_1\ldots\Delta\xi_n\\ &= \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\Delta\xi_i. \end{aligned}\tag{25}$$

Applying (13) on $\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}|\lambda_i^{\Delta}(t_i)|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\Delta\xi_i$ with $f(\xi_i)=\left(\int_{a_i}^{\xi_i}|\lambda_i^{\Delta}(t_i)|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}$ and $g(\xi_i)=1$, we have

$$\begin{aligned}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\Delta\xi_i &\leq \left(\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\Delta\xi_i\right)^{\frac{1}{p_i}}\left(\int_{a_i}^{\varepsilon_i}\Delta\xi_i\right)^{\frac{1}{r_i}}\\ &= (\varepsilon_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\Delta\xi_i\right)^{\frac{1}{p_i}}, \end{aligned}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\Delta\xi_i\\ &\leq \prod_{i=1}^{n}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\Delta\xi_i\right)^{\frac{1}{p_i}}\\ &= \prod_{i=1}^{n}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta}(t_i)\right|^{p_i}\Delta t_i\Delta\xi_i\right)^{\frac{1}{p_i}}. \end{aligned}\tag{26}$$

Substituting (26) into (25), we have

$$\int_{a_n}^{\varepsilon_n} \dots \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n} |\lambda_i(\xi_i)|}{\left(\sum_{i=1}^{n}(\xi_i - a_i)/r_i\right)^{\sum_{i=1}^{n} 1/r_i}} \Delta\xi_1 \dots \Delta\xi_n$$
$$\le \left(n - \sum_{i=1}^{n} 1/p_i\right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} (\varepsilon_i - a_i)^{\frac{1}{r_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\xi_i} \left|\lambda_i^{\Delta}(t_i)\right|^{p_i} \Delta t_i \Delta\xi_i\right)^{\frac{1}{p_i}}. \tag{27}$$

Applying (12) on $\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} |\lambda_i^{\Delta}(t_i)|^{p_i} \Delta t_i\right) \Delta\xi_i$ with $f(\xi_i) = \int_{a_i}^{\xi_i} |\lambda_i^{\Delta}(t_i)|^{p_i} \Delta t_i$ and $g^{\Delta}(\xi_i) = 1$, we obtain

$$\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left|\lambda_i^{\Delta}(t_i)\right|^{p_i} \Delta t_i\right) \Delta\xi_i$$
$$= \left(\int_{a_i}^{\xi_i} \left|\lambda_i^{\Delta}(t_i)\right|^{p_i} \Delta t_i\right) g(\xi_i)\Big|_{a_i}^{\varepsilon_i} - \int_{a_i}^{\varepsilon_i} \left|\lambda_i^{\Delta}(\xi_i)\right|^{p_i} g^{\sigma}(\xi_i) \Delta\xi_i, \tag{28}$$

where $g(\xi_i) = \xi_i - \varepsilon_i$. Since $g(\varepsilon_i) = 0$, we can find from (28) that

$$\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left|\lambda_i^{\Delta}(t_i)\right|^{p_i} \Delta t_i\right) \Delta\xi_i = \int_{a_i}^{\varepsilon_i} \left|\lambda_i^{\Delta}(\xi_i)\right|^{p_i} [\varepsilon_i - \sigma(\xi_i)] \Delta\xi_i. \tag{29}$$

Substituting (29) into (27), we obtain

$$\int_{a_n}^{\varepsilon_n} \dots \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n} |\lambda_i(\xi_i)|}{\left(\sum_{i=1}^{n}(\xi_i - a_i)/r_i\right)^{\sum_{i=1}^{n} 1/r_i}} \Delta\xi_1 \dots \Delta\xi_n$$
$$\le \left(n - \sum_{i=1}^{n} 1/p_i\right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} (\varepsilon_i - a_i)^{\frac{1}{r_i}}$$
$$\times \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} [\varepsilon_i - \sigma(\xi_i)] \left|\lambda_i^{\Delta}(\xi_i)\right|^{p_i} \Delta\xi_i\right)^{\frac{1}{p_i}}. \tag{30}$$

Substituting (19) into (30), we obtain

$$\int_{a_n}^{\varepsilon_n} \dots \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n} |\lambda_i(\xi_i)|}{\left(\sum_{i=1}^{n}(\xi_i - a_i)/r_i\right)^{\sum_{i=1}^{n} 1/r_i}} \Delta\xi_1 \dots \Delta\xi_n$$
$$\le M \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} [\varepsilon_i - \sigma(\xi_i)] \left|\lambda_i^{\Delta}(\xi_i)\right|^{p_i} \Delta\xi_i\right)^{\frac{1}{p_i}},$$

which is (18). □

Corollary 1. *If we put $\mathbb{T} = \mathbb{N}_0$ and $a_i = 0$ for $i = 1, 2, \dots, n$, into Theorem 6, then $\sigma(\xi_i) = \xi_i + 1$, and we obtain the analogue of inequality (8) as follows*

$$\sum_{\xi_1=0}^{\varepsilon_1 - 1} \dots \sum_{\xi_n=0}^{\varepsilon_n - 1} \frac{\prod_{i=1}^{n} |\lambda_i(\xi_i)|}{\left(\sum_{i=1}^{n} \xi_i / r_i\right)^{\sum_{i=1}^{n} 1/r_i}} \le M \prod_{i=1}^{n} \left(\sum_{\xi_i=0}^{\varepsilon_i - 1} [\varepsilon_i - \xi_i - 1] |\Delta\lambda_i(\xi_i)|^{p_i}\right)^{\frac{1}{p_i}}, \tag{31}$$

where

$$M = \left(n - \sum_{i=1}^{n} 1/p_i\right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} \varepsilon_i^{\frac{1}{r_i}}.$$

In the following, we present some special cases in (the continuous and quantum) calculie, i.e., when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$. These cases are new and interesting for the reader.

Corollary 2. *In Theorem 6, if* $\mathbb{T} = \mathbb{R}$, $a_i = 0$, $p_i, r_i > 1$, *such that* $1/p_i + 1/r_i = 1$ *and* $\lambda_i \in C([0, \varepsilon_i], \mathbb{R})$ *is a differentiable function with* $\lambda_i(0) = 0$; $i = 1, 2, \ldots, n$, *then* $\sigma(\xi_i) = \xi_i$ *and we obtain*

$$\int_0^{\varepsilon_n} \ldots \int_0^{\varepsilon_1} \frac{\prod_{i=1}^n |\lambda_i(\xi_i)|}{(\sum_{i=1}^n \xi_i/r_i)^{\sum_{i=1}^n 1/r_i}} d\xi_1 \ldots d\xi_n \leq M \prod_{i=1}^n \left(\int_0^{\varepsilon_i} [\varepsilon_i - \xi_i] |\lambda_i'(\xi_i)|^{p_i} d\xi_i \right)^{\frac{1}{p_i}},$$

where

$$M = \left(n - \sum_{i=1}^n 1/p_i \right)^{\sum_{i=1}^n 1/p_i - n} \prod_{i=1}^n \varepsilon_i^{\frac{1}{q_i}}.$$

Corollary 3. *In Theorem 6, if* $\mathbb{T} = q^{\mathbb{N}_0}$ *for* $q > 1$, $p_i, r_i > 1$, *such that* $1/p_i + 1/r_i = 1$ *and* $\lambda_i \in C([a_i, \varepsilon_i]_{\mathbb{T}}, \mathbb{R})$ *with* $\lambda_i(a_i) = 0$; $i = 1, 2, \ldots, n$, *then* $\sigma(\xi_i) = q\xi_i$ *and we obtain*

$$\sum_{\xi_1 = a_1}^{\varepsilon_1/q} \ldots \sum_{\xi_n = a_n}^{\varepsilon_n/q} \frac{\prod_{i=1}^n (q-1)\xi_i |\lambda_i(\xi_i)|}{(\sum_{i=1}^n (\xi_i - a_i)/r_i)^{\sum_{i=1}^n 1/r_i}} \leq M \prod_{i=1}^n \left(\sum_{\xi_i = a_i}^{\varepsilon_i/q} (q-1)\xi_i [\varepsilon_i - q\xi_i] |\Delta_q \lambda_i(\xi_i)|^{p_i} \right)^{\frac{1}{p_i}},$$

where

$$\Delta_q \lambda_i(\xi_i) = \frac{\lambda_i(q\xi_i) - \lambda_i(\xi_i)}{(q-1)\xi_i},$$

and

$$M = \left(n - \sum_{i=1}^n 1/p_i \right)^{\sum_{i=1}^n 1/p_i - n} \prod_{i=1}^n (\varepsilon_i - a_i)^{\frac{1}{r_i}}.$$

In the following theorem, we generalize the previous results for two variables.

Theorem 7. *Assume that* $a_i, \varepsilon_i, \epsilon_i \in \mathbb{T}$, $p_i, r_i > 1$, *such that* $1/p_i + 1/r_i = 1$ *and* $\lambda_i \in CC_{rd}'([a_i, \varepsilon_i]_{\mathbb{T}} \times [a_i, \epsilon_i]_{\mathbb{T}}, \mathbb{R})$ *with* $\lambda_i(a_i, \xi_i) = \lambda_i(\tau_i, a_i) = 0$ *for* $\xi_i \in [a_i, \varepsilon_i]_{\mathbb{T}}$ *and* $\tau_i \in [a_i, \epsilon_i]_{\mathbb{T}}$, $i = 1, 2, \ldots, n$. *Then*

$$\begin{aligned} &\int_{a_n}^{\epsilon_n} \int_{a_1}^{\epsilon_1} \ldots \int_{a_n}^{\varepsilon_n} \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^n |\lambda_i(\tau_i, \xi_i)|}{(\sum_{i=1}^n (\tau_i - a_i)(\xi_i - a_i)/r_i)^{\sum_{i=1}^n 1/r_i}} \Delta_2\xi_1 \ldots \Delta_2\xi_n \Delta_1\tau_1 \ldots \Delta_1\tau_n \\ &\leq N \prod_{i=1}^n \left(\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} (\epsilon_i - \sigma(\tau_i))(\varepsilon_i - \sigma(\xi_i)) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_1\tau_i \Delta_2\xi_i \right)^{\frac{1}{p_i}}, \end{aligned} \tag{32}$$

where

$$N = \left(n - \sum_{i=1}^n 1/p_i \right)^{\sum_{i=1}^n 1/p_i - n} \prod_{i=1}^n (\epsilon_i - a_i)^{\frac{1}{r_i}} (\varepsilon_i - a_i)^{\frac{1}{r_i}}. \tag{33}$$

Here, the Δ_1*—derivative of the function* $\lambda(\tau, \xi)$ *is the* Δ*—derivative with respect to the first variable* τ *and the* Δ_2*—derivative of the function* $\lambda(\tau, \xi)$ *is the* Δ*—derivative with respect to the second variable* ξ.

Proof. Applying the property (5) of Theorem 4, Fubini's theorem and the hypothesis $\lambda_i(a_i, \xi_i) = \lambda_i(\tau_i, a_i) = 0$, we obtain

$$\begin{aligned} &\int_{a_i}^{\tau_i} \int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(t_i, \vartheta_i) \right| \Delta_2\vartheta_i \Delta_1 t_i \\ \geq\ & \left| \int_{a_i}^{\tau_i} \int_{a_i}^{\xi_i} \lambda_i^{\Delta_2\Delta_1}(t_i, \vartheta_i) \Delta_2\vartheta_i \Delta_1 t_i \right| = \left| \int_{a_i}^{\xi_i} \left(\int_{a_i}^{\tau_i} \left[\lambda_i^{\Delta_2}(t_i, \vartheta_i) \right]^{\Delta_1} \Delta_1 t_i \right) \Delta_2\vartheta_i \right| \\ =\ & |\lambda_i(\tau_i, \xi_i) - \lambda_i(\tau_i, a_i) - \lambda_i(a_i, \xi_i) + \lambda_i(a_i, a_i)| = |\lambda_i(\tau_i, \xi_i)|, \end{aligned}$$

and then

$$\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)| \leq \prod_{i=1}^{n}\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|\Delta_2\vartheta_i\Delta_1 t_i. \tag{34}$$

Applying (14) on $\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|\Delta_2\vartheta_i\Delta_1 t_i$ with $h(t_i,\vartheta_i) = 1$, $f(t_i,\vartheta_i) = 1$ and $g(t_i,\vartheta_i) = \left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|$, we observe

$$\begin{aligned}&\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|\Delta_2\vartheta_i\Delta_1 t_i \\ \leq\ & (\tau_i-a_i)^{\frac{1}{r_i}}(\xi_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}},\end{aligned}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n}\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|\Delta_2\vartheta_i\Delta_1 t_i \\ \leq\ & \prod_{i=1}^{n}(\tau_i-a_i)^{\frac{1}{r_i}}(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}.\end{aligned} \tag{35}$$

Substituting (35) into (34), we observe that

$$\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)| \leq \prod_{i=1}^{n}(\tau_i-a_i)^{\frac{1}{r_i}}(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}. \tag{36}$$

Applying (16) with $s_i = (\tau_i - a_i)(\xi_i - a_i)$, we have

$$\prod_{i=1}^{n}(\tau_i-a_i)^{\frac{1}{r_i}}(\xi_i-a_i)^{\frac{1}{r_i}} \leq \frac{\left(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}{\left(n-\sum_{i=1}^{n}1/p_i\right)^{n-\sum_{i=1}^{n}1/p_i}}. \tag{37}$$

Substituting (37) into (36), we obtain

$$\begin{aligned}\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)| \leq\ & \frac{\left(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}{\left(n-\sum_{i=1}^{n}1/p_i\right)^{n-\sum_{i=1}^{n}1/p_i}} \\ & \times\prod_{i=1}^{n}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}.\end{aligned} \tag{38}$$

Dividing (38) by $\left(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}$ and integrating over ξ_i and τ_i from a_i to ε_i and ϵ_i for $i = 1,2,\ldots,n$, respectively, we observe that

$$\begin{aligned}&\int_{a_n}^{\epsilon_n}\int_{a_1}^{\epsilon_1}\cdots\int_{a_n}^{\varepsilon_n}\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)|}{\left(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta_2\xi_1\ldots\Delta_2\xi_n\Delta_1\tau_1\ldots\Delta_1\tau_n \\ &\leq \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n} \\ &\times\int_{a_n}^{\epsilon_n}\int_{a_1}^{\epsilon_1}\cdots\int_{a_n}^{\varepsilon_n}\int_{a_1}^{\varepsilon_1}\prod_{i=1}^{n}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}\Delta_2\xi_1\ldots\Delta_2\xi_n\Delta_1\tau_1\ldots\Delta_1\tau_n \\ &= \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n} \\ &\times\prod_{i=1}^{n}\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}\Delta_2\xi_i\Delta_1\tau_i.\end{aligned} \tag{39}$$

Applying (14) on $\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}\Delta_2\xi_i\Delta_1\tau_i$ with $h(\xi_i,\tau_i)=1$, $f(\xi_i,\tau_i)=1$ and

$$g(\xi_i,\tau_i)=\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}},$$

we have

$$\begin{aligned}&\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}\Delta_2\xi_i\Delta_1\tau_i\\&\leq(\epsilon_i-a_i)^{\frac{1}{r_i}}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\Delta_2\xi_i\Delta_1\tau_i\right)^{\frac{1}{p_i}},\end{aligned}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n}\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right)^{\frac{1}{p_i}}\Delta_2\xi_i\Delta_1\tau_i\\&\leq\prod_{i=1}^{n}(\epsilon_i-a_i)^{\frac{1}{r_i}}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\Delta_2\xi_i\Delta_1\tau_i\right)^{\frac{1}{p_i}}.\end{aligned}\tag{40}$$

Substituting (40) into (39) and applying the Fubini theorem, we obtain

$$\begin{aligned}&\int_{a_n}^{\epsilon_n}\int_{a_1}^{\epsilon_1}\cdots\int_{a_n}^{\varepsilon_n}\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)|}{\left(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta_2\xi_1\ldots\Delta_2\xi_n\Delta_1\tau_1\ldots\Delta_1\tau_n\\&\leq\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}(\epsilon_i-a_i)^{\frac{1}{r_i}}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\\&\times\prod_{i=1}^{n}\left(\int_{a_i}^{\epsilon_i}\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\Delta_2\xi_i\Delta_1\tau_i\right)^{\frac{1}{p_i}}\\&=\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}(\epsilon_i-a_i)^{\frac{1}{r_i}}(\varepsilon_i-a_i)^{\frac{1}{r_i}}\\&\times\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\epsilon_i}\left[\int_{a_i}^{\tau_i}\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\Delta_1 t_i\right]\Delta_1\tau_i\right)\Delta_2\xi_i\right)^{\frac{1}{p_i}}.\end{aligned}\tag{41}$$

Applying (12) on $\int_{a_i}^{\epsilon_i}\left(\int_{a_i}^{\tau_i}\left[\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\right]\Delta_1 t_i\right)\Delta_1\tau_i$, with

$$f(\tau_i)=\int_{a_i}^{\tau_i}\left[\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\right]\Delta_1 t_i\ \text{ and } g^{\Delta}(\tau_i)=1,$$

we obtain

$$\begin{aligned}&\int_{a_i}^{\epsilon_i}\left(\int_{a_i}^{\tau_i}\left[\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\right]\Delta_1 t_i\right)\Delta_1\tau_i\\=\;&g(\tau_i)\int_{a_i}^{\tau_i}\left[\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(t_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\right]\Delta_1 t_i\Big|_{a_i}^{\epsilon_i}\\&-\int_{a_i}^{\epsilon_i}g^{\sigma}(\tau_i)\left[\int_{a_i}^{\xi_i}\left|\lambda_i^{\Delta_2\Delta_1}(\tau_i,\vartheta_i)\right|^{p_i}\Delta_2\vartheta_i\right]\Delta_1\tau_i,\end{aligned}\tag{42}$$

where $g(\tau_i) = \tau_i - \epsilon_i$. Since $g(\epsilon_i) = 0$, we know from (42) that

$$\begin{aligned} &\int_{a_i}^{\epsilon_i} \left(\int_{a_i}^{\tau_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(t_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_1 t_i \right) \Delta_1 \tau_i \\ = &\int_{a_i}^{\epsilon_i} [\epsilon_i - \sigma(\tau_i)] \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_1 \tau_i. \end{aligned} \tag{43}$$

Integrating (43) over ξ_i from a_i to ε_i and then applying Fubini's theorem, we obtain

$$\begin{aligned} &\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} \left(\int_{a_i}^{\tau_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(t_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_1 t_i \right) \Delta_1 \tau_i \Delta_2 \xi_i \\ = &\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} [\epsilon_i - \sigma(\tau_i)] \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_1 \tau_i \Delta_2 \xi_i \\ = &\int_{a_i}^{\epsilon_i} \int_{a_i}^{\varepsilon_i} [\epsilon_i - \sigma(\tau_i)] \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_2 \xi_i \Delta_1 \tau_i \\ = &\int_{a_i}^{\epsilon_i} [\epsilon_i - \sigma(\tau_i)] \left(\int_{a_i}^{\varepsilon_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_2 \xi_i \right) \Delta_1 \tau_i. \end{aligned} \tag{44}$$

Applying (12) on $\int_{a_i}^{\varepsilon_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_2 \xi_i$ with $f(\xi_i) = \int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i$ and $g^{\Delta}(\xi_i) = 1$, we see

$$\begin{aligned} &\int_{a_i}^{\varepsilon_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_2 \xi_i \\ = &\ g(\xi_i) \left(\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right) \Bigg|_{a_i}^{\varepsilon_i} \\ &- \int_{a_i}^{\varepsilon_i} g^{\sigma}(\xi_i) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_2 \xi_i, \end{aligned} \tag{45}$$

where $g(\xi_i) = \xi_i - \varepsilon_i$. Since $g(\varepsilon_i) = 0$, we know from (45) that

$$\int_{a_i}^{\varepsilon_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_2 \xi_i = \int_{a_i}^{\varepsilon_i} (\varepsilon_i - \sigma(\xi_i)) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_2 \xi_i. \tag{46}$$

Substituting (46) into (44) and applying Fubini's theorem, we obtain

$$\begin{aligned} &\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} \left(\int_{a_i}^{\tau_i} \left[\int_{a_i}^{\xi_i} \left| \lambda_i^{\Delta_2\Delta_1}(t_i, \vartheta_i) \right|^{p_i} \Delta_2\vartheta_i \right] \Delta_1 t_i \right) \Delta_1 \tau_i \Delta_2 \xi_i \\ = &\int_{a_i}^{\epsilon_i} (\epsilon_i - \sigma(\tau_i)) \left(\int_{a_i}^{\varepsilon_i} (\varepsilon_i - \sigma(\xi_i)) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_2 \xi_i \right) \Delta_1 \tau_i \\ = &\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} (\epsilon_i - \sigma(\tau_i))(\varepsilon_i - \sigma(\xi_i)) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_1 \tau_i \Delta_2 \xi_i. \end{aligned} \tag{47}$$

Substituting (47) into (41), we see that

$$\begin{aligned} &\int_{a_n}^{\epsilon_n} \int_{a_1}^{\epsilon_1} \cdots \int_{a_n}^{\varepsilon_n} \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n} |\lambda_i(\tau_i, \xi_i)|}{(\sum_{i=1}^{n} (\tau_i - a_i)(\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}} \Delta_2 \xi_1 \ldots \Delta_2 \xi_n \Delta_1 \tau_1 \ldots \Delta_1 \tau_n \\ \leq &\left(n - \sum_{i=1}^{n} 1/p_i \right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} (\epsilon_i - a_i)^{\frac{1}{r_i}} (\varepsilon_i - a_i)^{\frac{1}{r_i}} \\ &\times \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} \int_{a_i}^{\epsilon_i} (\epsilon_i - \sigma(\tau_i))(\varepsilon_i - \sigma(\xi_i)) \left| \lambda_i^{\Delta_2\Delta_1}(\tau_i, \xi_i) \right|^{p_i} \Delta_1 \tau_i \Delta_2 \xi_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{48}$$

Substituting (33) into (48), we have

$$\int_{a_n}^{\epsilon_n}\int_{a_1}^{\epsilon_1}\cdots\int_{a_n}^{\varepsilon_n}\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)|}{(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i)^{\sum_{i=1}^{n}1/r_i}}\Delta_2\xi_1\ldots\Delta_2\xi_n\Delta_1\tau_1\ldots\Delta_1\tau_n$$
$$\leq N\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}\int_{a_i}^{\epsilon_i}(\epsilon_i-\sigma(\tau_i))(\varepsilon_i-\sigma(\xi_i))\left|\lambda_i^{\Delta_2\Delta_1}(\tau_i,\xi_i)\right|^{p_i}\Delta_1\tau_i\Delta_2\xi_i\right)^{\frac{1}{p_i}},$$

which is (32). □

Corollary 4. *If $\mathbb{T}=\mathbb{N}_0$ and $a_i=0$, then $\sigma(\xi_i)=\xi_i+1$ and we obtain the analogue of inequality (10) as follows:*

$$\sum_{\tau_1=0}^{\epsilon_1-1}\sum_{\tau_n=0}^{\epsilon_n-1}\cdots\sum_{\xi_1=0}^{\varepsilon_1-1}\sum_{\xi_n=0}^{\varepsilon_n-1}\frac{\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)|}{(\sum_{i=1}^{n}\tau_i\xi_i/r_i)^{\sum_{i=1}^{n}1/r_i}}$$
$$\leq N\prod_{i=1}^{n}\left(\sum_{\xi_i=0}^{\varepsilon_i-1}\sum_{\tau_i=0}^{\epsilon_i-1}(\epsilon_i-\tau_i-1)(\varepsilon_i-\xi_i-1)|\Delta_2\Delta_1\lambda_i(\tau_i,\xi_i)|^{p_i}\right)^{\frac{1}{p_i}},$$

where $\Delta_1\lambda_i(\tau_i,\xi_i)=\lambda_i(\tau_i+1,\xi_i)-\lambda_i(\tau_i,\xi_i)$, $\Delta_2\lambda_i(\tau_i,\xi_i)=\lambda_i(\tau_i,\xi_i+1)-\lambda_i(\tau_i,\xi_i)$ and

$$N=\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}(\varepsilon_i\epsilon_i)^{\frac{1}{r_i}}.$$

In the following corollaries, we show some particular cases in (the continuous and quantum) calculie, i.e., when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=q^{\mathbb{N}_0}$ for $q>1$, which are original.

Corollary 5. *If $\mathbb{T}=\mathbb{R}$, $a_i=0$, $p_i,r_i>1$, such that $1/p_i+1/r_i=1$, $\lambda_i\in C([0,\varepsilon_i]\times[0,\epsilon_i],\mathbb{R})$ with $\lambda_i(0,\xi_i)=\lambda_i(\tau_i,0)=0$ for $\xi_i\in[0,\varepsilon_i]$ and $\tau_i\in[0,\epsilon_i]$, $i=1,2,\ldots,n$, then $\sigma(\xi_i)=\xi_i$ and we obtain*

$$\int_0^{\epsilon_n}\int_0^{\epsilon_1}\cdots\int_0^{\varepsilon_n}\int_0^{\varepsilon_1}\frac{\prod_{i=1}^{n}|\lambda_i(\tau_i,\xi_i)|}{(\sum_{i=1}^{n}\tau_i\xi_i/r_i)^{\sum_{i=1}^{n}1/r_i}}d\xi_1\ldots d\xi_n d\tau_1\ldots d\tau_n$$
$$\leq N\prod_{i=1}^{n}\left(\int_0^{\varepsilon_i}\int_0^{\epsilon_i}(\epsilon_i-\tau_i)(\varepsilon_i-\xi_i)\left|\frac{\partial^2}{\partial\xi_i\partial\tau_i}\lambda_i(\tau_i,\xi_i)\right|^{p_i}d\tau_i d\xi_i\right)^{\frac{1}{p_i}},$$

where

$$N=\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}(\epsilon_i\varepsilon_i)^{\frac{1}{r_i}}.$$

Corollary 6. *If $\mathbb{T}=q^{\mathbb{N}_0}$ for $q>1$, $p_i,r_i>1$ such that $1/p_i+1/r_i=1$ and $\lambda_i:[a_i,\varepsilon_i]_{\mathbb{T}}\times[a_i,\epsilon_i]_{\mathbb{T}}\to\mathbb{R}$ with $\lambda_i(a_i,\xi_i)=\lambda_i(\tau_i,a_i)=0$ for $\xi_i\in[a_i,\varepsilon_i]_{\mathbb{T}}$ and $\tau_i\in[a_i,\epsilon_i]_{\mathbb{T}}$, $i=1,2,\ldots,n$, then $\sigma(t)=qt$ and we obtain*

$$\sum_{\tau_1=a_1}^{\epsilon_1/q}\sum_{\tau_n=a_n}^{\epsilon_n/q}\cdots\sum_{\xi_1=a_1}^{\varepsilon_1/q}\sum_{\xi_n=a_n}^{\epsilon_n/q}\frac{\prod_{i=1}^{n}(q-1)^2\tau_i\xi_i|\lambda_i(\tau_i,\xi_i)|}{(\sum_{i=1}^{n}(\tau_i-a_i)(\xi_i-a_i)/r_i)^{\sum_{i=1}^{n}1/r_i}}$$
$$\leq N\prod_{i=1}^{n}\left(\sum_{\xi_i=a_i}^{\varepsilon_i/q}\sum_{\tau_i=a_i}^{\epsilon_i/q}(\epsilon_i-q\tau_i)(\varepsilon_i-q\xi_i)\left|\lambda_i^{\Delta_2\Delta_1}(\tau_i,\xi_i)\right|^{p_i}\right)^{\frac{1}{p_i}},$$

where

$$N=\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}(\epsilon_i-a_i)^{\frac{1}{q_i}}(\varepsilon_i-a_i)^{\frac{1}{q_i}}.$$

Theorem 8. *Let $a_i,\varepsilon_i\in\mathbb{T}$, $h_i\geq 1$, $p_i,r_i>1$ such that $1/p_i+1/r_i=1$ and $\lambda_i\in C_{rd}([a_i,\varepsilon_i]_{\mathbb{T}},\mathbb{R}^+\cup\{0\})$ is a delta-differentiable function and an increasing function with $\lambda_i(a_i)=0$; $i=1,2,\dots,n$. Then*

$$\begin{aligned}&\int_{a_1}^{\varepsilon_1}\cdots\int_{a_n}^{\varepsilon_n}\frac{\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta\xi_1\dots\Delta\xi_n\\ \leq\ & Q\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}(\varepsilon_i-\sigma(\xi_i))\left([\lambda_i^{\sigma}(\xi_i)]^{h_i-1}\lambda_i^{\Delta}(\xi_i)\right)^{p_i}\Delta\xi_i\right)^{\frac{1}{p_i}},\end{aligned}\tag{49}$$

where

$$Q=\left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}h_i(\varepsilon_i-a_i)^{\frac{1}{r_i}}.\tag{50}$$

Proof. Applying the chain rule formula (11) on the term $\lambda_i^{h_i}(t_i)$, $h_i\geq 1$, we obtain

$$\left[\lambda_i^{h_i}(t_i)\right]^{\Delta}=h_i\lambda_i^{h_i-1}(\zeta_i)\lambda_i^{\Delta}(t_i),\tag{51}$$

where $\zeta_i\in[t_i,\sigma(t_i)]$. Since λ_i is an increasing function, $h_i\geq 1$ and $\zeta_i\leq\sigma(t_i)$, we know from (51) that

$$\left[\lambda_i^{h_i}(t_i)\right]^{\Delta}\leq h_i[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i),$$

and then (where $\lambda_i(a_i)=0$), we observe that

$$h_i\int_{a_i}^{\xi_i}[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\Delta t_i\geq\int_{a_i}^{\xi_i}\left[\lambda_i^{h_i}(t_i)\right]^{\Delta}\Delta t_i=\lambda_i^{h_i}(\xi_i)-\lambda_i^{h_i}(a_i)=\lambda_i^{h_i}(\xi_i).$$

Thus,

$$\prod_{i=1}^{n}h_i\int_{a_i}^{\xi_i}[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\Delta t_i\geq\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i).\tag{52}$$

Applying (13) on $\int_{a_i}^{\xi_i}[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\Delta t_i$ with $f(t_i)=[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)$ and $g(t_i)=1$, we have

$$\begin{aligned}\int_{a_i}^{\xi_i}[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\Delta t_i &\leq \left(\int_{a_i}^{\xi_i}\Delta t_i\right)^{\frac{1}{r_i}}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}\\ &= (\xi_i-a_i)^{\frac{1}{r_i}}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)^{\frac{1}{p_i}},\end{aligned}$$

and then

$$\prod_{i=1}^{n}\int_{a_i}^{\xi_i}[\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\Delta t_i\leq\prod_{i=1}^{n}(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}.\tag{53}$$

Substituting (53) into (52), we get

$$\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)\leq\prod_{i=1}^{n}h_i(\xi_i-a_i)^{\frac{1}{r_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)^{\frac{1}{p_i}}.\tag{54}$$

Applying (16) with $s_i = \xi_i - a_i$, we have

$$\prod_{i=1}^{n}(\xi_i - a_i)^{1/r_i} \le \frac{(\sum_{i=1}^{n}(\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{n - \sum_{i=1}^{n} 1/p_i}}. \tag{55}$$

Substituting (55) into (54), we obtain

$$\begin{aligned}\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i) \quad &\le \quad \frac{(\sum_{i=1}^{n}(\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}}{(n - \sum_{i=1}^{n} 1/p_i)^{n - \sum_{i=1}^{n} 1/p_i}} \\ &\quad \times \prod_{i=1}^{n} h_i \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{56}$$

Dividing (56) by $(\sum_{i=1}^{n}(\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}$ and integrating over ξ_i from a_i to ε_i, $i = 1, 2, \ldots, n$, we observe that

$$\begin{aligned}&\int_{a_n}^{\varepsilon_n} \ldots \int_{a_1}^{\varepsilon_1} \frac{\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)}{(\sum_{i=1}^{n}(\xi_i - a_i)/r_i)^{\sum_{i=1}^{n} 1/r_i}} \Delta\xi_1 \ldots \Delta\xi_n \\ &\le \left(n - \sum_{i=1}^{n} 1/p_i \right)^{\sum_{i=1}^{n} 1/p_i - n} \\ &\times \int_{a_n}^{\varepsilon_n} \ldots \int_{a_1}^{\varepsilon_1} \prod_{i=1}^{n} h_i \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \Delta\xi_1 \ldots \Delta\xi_n \\ &= \left(n - \sum_{i=1}^{n} 1/p_i \right)^{\sum_{i=1}^{n} 1/p_i - n} \\ &\times \prod_{i=1}^{n} h_i \int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \Delta\xi_i. \end{aligned} \tag{57}$$

Applying (13) on $\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \Delta\xi_i$ with $f(\xi_i) = 1$ and

$$g(\xi_i) = \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}},$$

we have

$$\begin{aligned}&\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \Delta\xi_i \\ \le \quad & (\varepsilon_i - a_i)^{\frac{1}{r_i}} \left(\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right) \Delta\xi_i \right)^{\frac{1}{p_i}}, \end{aligned}$$

and then

$$\begin{aligned}&\prod_{i=1}^{n} \int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right)^{\frac{1}{p_i}} \Delta\xi_i \\ &\le \prod_{i=1}^{n} (\varepsilon_i - a_i)^{\frac{1}{r_i}} \prod_{i=1}^{n} \left(\int_{a_i}^{\varepsilon_i} \left(\int_{a_i}^{\xi_i} \left([\lambda_i^{\sigma}(t_i)]^{h_i - 1}\lambda_i^{\Delta}(t_i) \right)^{p_i} \Delta t_i \right) \Delta\xi_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{58}$$

Substituting (58) into (57), we have

$$\begin{aligned} &\int_{a_n}^{\varepsilon_n}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta\xi_1\ldots\Delta\xi_n \\ \leq\ & \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}h_i(\varepsilon_i-a_i)^{\frac{1}{q_i}} \\ &\times\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)\Delta\xi_i\right)^{\frac{1}{p_i}}. \end{aligned} \tag{59}$$

Applying (12) on $\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)\Delta\xi_i$ with $f(\xi_i)=\int_{a_i}^{\xi_i}([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i))^{p_i}\Delta t_i$ and $g^{\Delta}(\xi_i)=1$, we obtain

$$\begin{aligned} &\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)\Delta\xi_i \\ =\ & g(\xi_i)\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)\Bigg|_{a_i}^{\varepsilon_i} \\ &-\int_{a_i}^{\varepsilon_i}g^{\sigma}(\xi_i)\left([\lambda_i^{\sigma}(\xi_i)]^{h_i-1}\lambda_i^{\Delta}(\xi_i)\right)^{p_i}\Delta\xi_i, \end{aligned} \tag{60}$$

where $g(\xi_i)=\xi_i-\varepsilon_i$. Since $g(\varepsilon_i)=0$, we know from (60) that

$$\int_{a_i}^{\varepsilon_i}\left(\int_{a_i}^{\xi_i}\left([\lambda_i^{\sigma}(t_i)]^{h_i-1}\lambda_i^{\Delta}(t_i)\right)^{p_i}\Delta t_i\right)\Delta\xi_i=\int_{a_i}^{\varepsilon_i}(\varepsilon_i-\sigma(\xi_i))\left([\lambda_i^{\sigma}(\xi_i)]^{h_i-1}\lambda_i^{\Delta}(\xi_i)\right)^{p_i}\Delta\xi_i, \tag{61}$$

Substituting (61) into (59), we observe that

$$\begin{aligned} &\int_{a_n}^{\varepsilon_n}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta\xi_1\ldots\Delta\xi_n \\ \leq\ & \left(n-\sum_{i=1}^{n}1/p_i\right)^{\sum_{i=1}^{n}1/p_i-n}\prod_{i=1}^{n}h_i(\varepsilon_i-a_i)^{\frac{1}{q_i}} \\ &\times\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}(\varepsilon_i-\sigma(\xi_i))\left([\lambda_i^{\sigma}(\xi_i)]^{h_i-1}\lambda_i^{\Delta}(\xi_i)\right)^{p_i}\Delta\xi_i\right)^{\frac{1}{p_i}}. \end{aligned} \tag{62}$$

From (50), the inequality (62) becomes

$$\begin{aligned} &\int_{a_n}^{\varepsilon_n}\cdots\int_{a_1}^{\varepsilon_1}\frac{\prod_{i=1}^{n}\lambda_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}(\xi_i-a_i)/r_i\right)^{\sum_{i=1}^{n}1/r_i}}\Delta\xi_1\ldots\Delta\xi_n \\ \leq\ & Q\prod_{i=1}^{n}\left(\int_{a_i}^{\varepsilon_i}(\varepsilon_i-\sigma(\xi_i))\left([\lambda_i^{\sigma}(\xi_i)]^{h_i-1}\lambda_i^{\Delta}(\xi_i)\right)^{p_i}\Delta\xi_i\right)^{\frac{1}{p_i}}, \end{aligned}$$

which is (49). □

Remark 1. *If $\mathbb{T}=\mathbb{R}$ and $a_i=0$ for $i=1,2,\ldots,n$, then we obtain the inequality (9) for the non-negative increasing function λ with $\lambda_i(0)=0$, $i=1,2,\ldots,n$.*

In the following remark, we present the discrete analogue of (9), i.e., when $\mathbb{T}=\mathbb{N}$, which is new and interesting for the reader.

Corollary 7. *If $\mathbb{T} = \mathbb{N}$, $h_i \geq 1$, $p_i, r_i > 1$ such that $1/p_i + 1/r_i = 1$ and λ_i is a non-negative and increasing sequence with $\lambda_i(a_i) = 0$; $i = 1, 2, \ldots, n$, then*

$$\sum_{\xi_1=a_1}^{\varepsilon_1-1} \cdots \sum_{\xi_n=a_n}^{\varepsilon_n-1} \frac{\prod_{i=1}^{n} \lambda_i^{h_i}(\xi_i)}{\left(\sum_{i=1}^{n}(\xi_i - a_i)/r_i\right)^{\sum_{i=1}^{n} 1/r_i}}$$
$$\leq Q \prod_{i=1}^{n} \left(\sum_{\xi_i=a_i}^{\varepsilon_i-1} (\varepsilon_i - \xi_i - 1) \left([\lambda_i(\xi_i+1)]^{h_i-1} \Delta\lambda_i(\xi_i) \right)^{p_i} \right)^{\frac{1}{p_i}},$$

where

$$Q = \left(n - \sum_{i=1}^{n} 1/p_i\right)^{\sum_{i=1}^{n} 1/p_i - n} \prod_{i=1}^{n} h_i(\varepsilon_i - a_i)^{\frac{1}{r_i}}.$$

4. Conclusions and Future Work

In this paper, we establish some new dynamic Hilbert-type inequalities on time scale delta calculus by applying Hölder's inequality, the chain rule and the mean inequality. In the future, we will prove Hilbert-type inequalities on diamond—α calculus and fractional conformable calculus.

Author Contributions: Software and writing—original draft, H.M.R. and A.I.S.; writing—review and editing, M.Z., A.A.I.A.-T. and M.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through large Research Project under grant number RGP 2/190/45.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through large Research Project under grant number RGP 2/190/45.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hilbert, D. *Grundzüge Einer Allgemeinen Theorie der Linearen Intergraleichungen*; Springer: Göttingen, Germany, 1906; pp. 157–227.
2. Schur, I. Bernerkungen sur theorie der beschrankten Bilinearformen mit unendlich vielen veranderlichen. *J. Math.* **1911**, *140*, 1–28.
3. Liu, Q.; Sun, W. A Hilbert-type fractal integral inequality and its applications. *J. Inequal. Appl.* **2017**, *2017*, 83. [CrossRef] [PubMed]
4. Debnath, L.; Yang, B. Recent developments of Hilbert-type discrete and integral inequalities with applications. *Int. J. Math. Math. Sci.* **2012**, *2012*, 871845. [CrossRef]
5. Guariglia, E. Riemann zeta fractional derivative-functional equation and link with primes. *Adv. Differ. Equ.* **2019**, *2019*, 261. [CrossRef]
6. Hardy, G.H.; Littlewood, J.E.; Polya, G. The maximum of a certain bilinear form. *Proc. Lond. Math. Soc.* **1926**, *25*, 265–282. [CrossRef]
7. Hardy, G.H. Notes on a theorem of Hilbert. *Math. Z.* **1920**, *6*, 314–317. [CrossRef]
8. Hardy, G.H. Note on a theorem of Hilbert concerning series of positive term. *Proc. Lond. Math. Soc.* **1925**, *23*, 45–46.
9. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1934.
10. Pachpatte, B.G. A note on Hilbert type inequality. *Tamkang J. Math.* **1998**, *29*, 293–298. [CrossRef]
11. Pachpatte, B.G. Inequalities Similar to Certain Extensions of Hilbert's Inequality. *J. Math. Anal. Appl.* **2000**, *243*, 217–227. [CrossRef]
12. Kim, Y.H.; Kim, B.I. An Analogue of Hilbert's inequality and its extensions. *Bull. Korean Math. Soc.* **2002**, *39*, 377–388. [CrossRef]
13. Zhao, C.-J.; Chen, L.-Y.; Cheung, W.S. On some new Hilbert-type inequalities. *Math. Slovaca* **2011**, *61*, 15–28.
14. Řehak, P. Hardy inequality on time scales and its application to half-linear dynamic equations. *J. Inequal. Appl.* **2005**, *2005*, 495–507. [CrossRef]
15. Saker, S.H.; Ahmed, A.M.; Rezk, H.M.; O'Regan, D.; Agarwal, R.P. New Hilbert dynamic inequalities on time scales. *Math. Inequal. Appl.* **2017**, *20*, 1017–1039. [CrossRef]

16. Ahmed, A.M.; AlNemer, G.; Zakarya, M.; Rezk, H.M. Some Dynamic Inequalities of Hilbert's Type. *J. Funct. Spaces* **2020**, *2020*, 4976050. [CrossRef]
17. Gulsen, T.; Jadlovská, I.; Yilmaz, E. On the number of eigenvalues for parameter-dependent diffusion problem on time scales. *Math. Metho. Appl. Sci.* **2021**, *44*, 985–992. [CrossRef]
18. AlNemer, G.; Zakarya, M.; Abd El-Hamid, H.A.; Agarwal, P.; Rezk, H.M. Some Dynamic Hilbert-Type Inequalities on Time Scales. *Symmetry* **2020**, *12*, 1410. [CrossRef]
19. El-Hamid, H.A.A.; Rezk, H.M.; Ahmed, A.M.; AlNemer, G.; Zakarya, M.; ELsaify, H.A. Some Dynamic Hilbert-Type Inequalities for Two Variables on Time Scales. *J. Inequal. Appl.* **2021**, *2021*, 31. [CrossRef]
20. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
21. Agarwal, R.P.; O'Regan, D.; Saker, S.H. *Dynamic Inequalities on Time Scales*; Springer: Cham, Switzerland, 2014.
22. Agarwal, R.P.; O'Regan, D.; Saker, S.H. *Hardy Type Inequalities on Time Scales*; Springer: Cham, Switzerland, 2016.
23. Tuna, A.; Kutukcu, S. Some integral inequalities on time scales. *Appl. Math. Mech.* **2008**, *29*, 23–29. [CrossRef]
24. Bohner, M.; Guseinov, G.S. Multiple integration on time scales. *Dyn. Syst. Appl.* **2005**, *14*, 579–606.