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Nonlinear Contractions Employing Digraphs and Comparison Functions with an Application to Singular Fractional Differential Equations

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Abstract: After the initiation of Jachymski's contraction principle via digraph, the area of metric fixed point theory has attracted much attention. A number of outcomes on fixed points in the context of graph metric space employing various types of contractions have been investigated. The aim of this paper is to investigate some fixed point theorems for a class of nonlinear contractions in a metric space endued with a transitive digraph. The outcomes presented herewith improve, extend and enrich several existing results. Employing our findings, we describe the existence and uniqueness of a singular fractional boundary value problem.

Keywords: fixed points; digraphs; singular fractional differential equations

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1. Introduction

Fractional differential equations (abbreviated as FDEs) are generalisations of the ordinary differential equations to an arbitrary non-integer order. In the recent past, FDEs have been studied on account of their remarkable growth and relevance to the field of fractional calculus. For an extensive collection on the background of FDE, we refer the readers to consult [1–5] and the references therein. Various researchers (e.g., [6–11]) have discussed the existence theory of FDE employing the approaches of fixed point theory. Recall that a typical fractional BVP (abbreviation of 'boundary value problem') in a dependent variable ϑ and independent variable θ can be represented by

$$-D^{\iota}\vartheta(\theta) = \hbar\Big(\theta, \vartheta(\theta), D^{\alpha_1}\vartheta(\theta), D^{\alpha_2}\vartheta(\theta), \ldots, D^{\alpha_{r-1}}\vartheta(\theta)\Big)$$

$$\begin{cases} D^{\alpha_i}\vartheta(0) = 0, \quad 1 \le i \le r-1, \\ D^{\alpha_{r-1}+1}\vartheta(0) = 0, \\ D^{\alpha_{r-1}}\vartheta(1) = \sum\limits_{j=1}^{m-2} q_j D^{\alpha_{r-1}}\vartheta(\varpi_j) \end{cases} \tag{1}$$

where

- $r \in \mathbb{N},\ r \ge 3$ and $r-1 < \iota \le r$,
- $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{r-2} < \alpha_{r-1}$ and $r-3 < \alpha_{r-1} < \iota - 2$,
- D^{ι} is standard Riemann–Liouville derivative,
- $\hbar \in C\big([0,1] \times \mathbb{R}^r;\ [0,\infty)\big)$,
- $q_j \in \mathbb{R}$ and $0 < \varpi_1 < \varpi_2 < \cdots < \varpi_{m-1} < 1$ with $0 < \sum\limits_{j=1}^{m-2} q_j \varpi_j^{\iota-\alpha_{r-1}-1} < 1$.

Fixed point theory plays in metric space (in short, MS) a central role in nonlinear functional analysis. Throughout the foregoing century, BCP has been expanded and generalised by numerous authors. A common generalisation of this finding is to expand the standard contraction to φ-contraction by means of a proper auxiliary function $\varphi : [0, \infty) \to [0, \infty)$. A variety of generalisations has been developed through effectively modifying φ, resulting in a huge number of articles on this topic. Matkowski [12] invented a new class of φ-contraction that incorporated the concept of comparison functions, which has been further studied in ([13–17]) besides several others. Quite recently, Pant [17] established an interesting non-unique fixed point theorem enlarging the class of φ-contractions in a complete metric space.

In 2008, Jachymski [18] established a very interesting approach in fixed point theory in the setup of graph metric space. Graphs are algebraic structures that subsume the partial ordering. The chief feature of the graphic approach is that the contraction condition is required to hold for merely certain edges of the underlying graph. This approach gave rise to an emerging discipline of research in metric fixed point theory, which led to the appearance of numerous works, e.g., see [19–25]. In 2010, Bojor [19] extended the results of Jachymski [18] to (G, φ)-contraction in the sense of Matkowski [12].

The intent of this manuscript is to expand the outcomes of Bojor [19] adopting the idea of Pant [17] and to prove the fixed point theorems under the enlarged class of (G, φ)-contraction in the setup of graph metric space. Employing the findings proved herewith, we study the existence and uniqueness of positive solutions of a particular form of BVP (1), such that the FDE remains singular.

2. Graph Metric Space

The set of real numbers (resp. natural numbers) are indicated by $\mathbb{R}$ (resp. $\mathbb{N}$). By a graph G, we mean the pair $(V(G), E(G))$, whereas $V(G)$ (known as set of vertices) and a set $E(G)$ (known as set of edges) have a binary relation on $V(G)$.

Definition 1 ([26])**.** *A graph is named as a digraph (or, directed graph) if every edge remains an ordered pair of vertices.*

Definition 2 ([26])**.** *The transpose of a graph G, is a graph denoted by G^{-1}, described as*

$$V(G^{-1}) = V(G) \quad \text{and} \quad E(G^{-1}) = \{(v, u) \in V(G)^2 : (u, v) \in E(G)\}.$$

Definition 3 ([26])**.** *Each digraph $G = (V(G), E(G))$ induces an undirected graph $\tilde{G}$, defined by*

$$V(\tilde{G}) = V(G) \quad \text{and} \quad E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 4 ([26])**.** *For any two vertices v and u in the graph G, a finite sequence $\{v_0, v_1, v_2, \dots v_p\}$ of vertices is said to form a path in G from v to u of length p if $v_0 = v$, $v_p = u$ and $(v_{r-1}, v_r) \in E(G)$, $\forall\, r \in \{1, 2, \dots p\}$.*

Definition 5 ([26])**.** *A graph G is known as connected if any two vertices of G enjoy a path. If $\tilde{G}$ is connected then G is referred as weakly connected.*

Definition 6 ([18])**.** *Let (V, ϱ) be a MS and $G := (V(G), E(G))$ a digraph. Then the triplet (V, ϱ, G) called a graph MS if*

- *$V(G) = V$;*
- *$E(G)$ contains all loops;*
- *G admits no parallel edge.*

Definition 7 ([20])**.** *Given a graph MS (V, ϱ, G), G is referred as a (C)-graph if for every sequence $\{v_n\} \subset V$ having the properties: $v_n \to v$ and $(v_n, v_{n+1}) \in E(G)$, for every $n \in \mathbb{N}$, $\exists$ a subsequence $\{v_{n_r}\}$ with $(v_{n_r}, v) \in E(G)$, $\forall\, r \in \mathbb{N}$.*

Definition 8 ([23])**.** *Given a graph MS (V, ϱ, G), a map $R : V \to V$ is named as G-edge preserving if*

$$(v, u) \in E(G) \Longrightarrow (Rv, Ru) \in E(G).$$

Definition 9 ([24])**.** *A digraph G is referred as transitive if for all $v, u, w \in V(G)$ with*

$$(v, u) \in E(G) \quad \text{and} \quad (u, w) \in E(G) \Longrightarrow (v, w) \in E(G).$$

Definition 10 ([27])**.** *An increasing function $\varphi : [0, \infty) \to [0, \infty)$ is named as comparison function if $\lim_{n\to\infty} \varphi^n(t) = 0, \forall\, t > 0$.*

For further discussions on comparison functions, we refer the monographs of Rus [27] and Berinde [28].

Proposition 1 ([27,28])**.** *Every comparison function φ verifies that $\varphi(t) < t, \forall\, t > 0$ and $\varphi(0) = 0$.*

Definition 11 ([29])**.** *A self-map R defined on a MS (V, ϱ) is referred as*

- *PM (Picard mapping) if $\mathrm{Fix}(R) = \{v^*\}$ (a singleton set) and $R^n(v) \to v^*, \forall\, v \in V$;*
- *WPM (weakly Picard mapping) if $\mathrm{Fix}(R) \neq \emptyset$ and the sequence $\{R^n v\}$ converges to a fixed point of R, $\forall\, v \in V$.*

3. Main Results

Given a digraph $G := (V(G), E(G))$, a self-map R on V and $v \in V(G)$, we adopt the succeeding notations:

$$[v]_G = \{u \in V(G) : \exists \text{ a path in } G \text{ from } v \text{ to } u\};$$

$$V_R = \{v \in V : (v, Rv) \in E(G)\};$$

and

$$\mathrm{Fix}(R) = \{v \in V : R(v) = v\}.$$

We are now going to demonstrate the following fpt in a graph MS over a class of (G, φ)-contractivity condition.

Theorem 1. *Let (V, ϱ, G) be a graph MS whereas (V, ϱ) is a complete MS and G is a transitive. Let $R : V \to V$ be a G-edge preserving map and $V_R \neq \emptyset$. Also, assume that either, R is orbitally G-continuous, or, G is a (C)-graph. If there exists a comparison function φ such that*

$$\varrho(Rv, Ru) \leq \varphi(\varrho(v, u)) \quad \forall\, (v, u) \in E(G) \text{ with } [v \neq R(v) \text{ or } u \neq R(u)], \tag{2}$$

then R is a WPM.

Proof. Take $v_0 \in V_R$ so that $(v_0, Rv_0) \in E(G)$. Construct a sequence $\{v_n\}$ in the following way:

$$v_{n+1} = R^n(v_0) = R(v_n), \quad \forall\, n \in \mathbb{N}_0. \tag{3}$$

Since $(v_0, Rv_0) \in E(G)$ and R is a G-edge preserving, by easy induction, we have

$$(R^n v_0, R^{n+1} v_0) \in E(G)$$

which through (3) simplifies to

$$(v_n, v_{n+1}) \in E(G) \quad \forall\, n \in \mathbb{N}_0. \tag{4}$$

Define $\varrho_n := \varrho(v_n, v_{n+1})$. If there is some $n_0 \in \mathbb{N}_0$ with $\varrho_{n_0} = 0$, then by (3), we find $v_{n_0} = v_{n_0+1} = R(v_{n_0})$; so $v_{n_0} \in \mathrm{Fix}(R)$, unless, we have $\varrho_n > 0$ for every $n \in \mathbb{N}_0$. Then, we have $v_n \neq v_{n+1} = R(v_n)$. On implementing (4) and the contractivity condition (2), we find

$$\varrho_n = \varrho(v_n, v_{n+1}) = \varrho(Rv_{n-1}, Rv_n) \leq \varphi(\varrho(v_{n-1}, v_n)),$$

or,

$$\varrho_n \leq \varphi(\varrho_{n-1}) \quad \forall\, n \in \mathbb{N}_0. \tag{5}$$

Using monotonicity of φ in (5), we have

$$\varrho_n \leq \varphi(\varrho_{n-1}) \leq \varphi^2(\varrho_{n-2}) \leq \cdots \leq \varphi^n(\varrho_0),$$

or,

$$\varrho_n \leq \varphi^n(\varrho_0), \quad \forall\, n \in \mathbb{N}. \tag{6}$$

With $n \to \infty$ in (6) and employing the definition of φ, we find

$$\lim_{n\to\infty} \varrho_n = 0. \tag{7}$$

Choose $\varepsilon > 0$. Then, owing to (7), we can find $n \in \mathbb{N}_0$ allows for

$$\varrho_n < \varepsilon - \varphi(\varepsilon). \tag{8}$$

Now, we seek to verify that $\{v_n\}$ is Cauchy. Implementing the monotonicity of φ, (5) and (8), we find

$$\begin{aligned} \varrho(v_n, v_{n+2}) &\leq \varrho(v_n, v_{n+1}) + \varrho(v_{n+1}, v_{n+2}) = \varrho_n + \varrho_{n+1} \\ &\leq \varrho_n + \varphi(\varrho_n) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi[\varepsilon - \varphi(\varepsilon)] \leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

Implementing the monotonicity of φ, transitivity of G, (4), (8), and the contractivity condition (2), we find

$$\begin{aligned} \varrho(v_n, v_{n+3}) &\leq \varrho(v_n, v_{n+1}) + \varrho(v_{n+1}, v_{n+3}) \\ &= \varrho_n + \varrho(Rv_n, Rv_{n+2}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varrho(v_n, v_{n+2})) \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

By easy induction, one finds

$$\varrho(v_n, v_{n+p}) < \varepsilon, \quad \forall\, p \in \mathbb{N}.$$

It turns out that $\{v_n\}$ continues to be Cauchy. Through the completeness of (V, ϱ), there exists $v \in V$ whereby $v_n \xrightarrow{\varrho} v$.

Suppose that R is orbitally G-continuous. Then, one finds

$$v_{n+1} = R(v_n) \xrightarrow{\varrho} R(v),$$

leading to, in turn, $R(v) = v$. Therefore, v is a fixed point of R. Otherwise, if G is a (C)-graph, then, a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ can be determined that satisfies $(v_{n_k}, v) \in E(G)$ for every $k \in \mathbb{N}_0$. By contractivity condition (2), we have

$$\varrho(v_{n_k+1}, Rv) = \varrho(Rv_{n_k}, Rv) \leq \varphi(\varrho(v_{n_k}, v)), \quad \forall\, k \in \mathbb{N}_0.$$

Using Proposition 1 (whether $\varrho(v_{n_k}, v)$ is zero or non-zero), the above inequality becomes

$$\varrho(v_{n_k+1}, Rv) \leq \varrho(v_{n_k}, v).$$

Taking $k \to \infty$ in the above inequality and using $v_{n_k} \xrightarrow{\varrho} v$, we get

$$v_{n_k+1} \xrightarrow{\varrho} R(v),$$

leading to, in turn, $R(v) = v$. Hence, v is a fixed point of R. □

Next, we present the uniqueness theorem corresponding to Theorem 1.

Theorem 2. *Let (V, ϱ, G) be a graph MS whereas (V, ϱ) is a complete MS and G is a transitive and weakly connected. Let $R : V \to V$ be a G-edge preserving map and $V_R \neq \emptyset$. Also, assume that either, R is orbitally G-continuous, or, G is a (C)-graph. If there exists a comparison function φ such that*

$$\varrho(Rv, Ru) \leq \varphi(\varrho(v, u)) \quad \forall\, (v, u) \in E(G),$$

then R is a PM.

Proof. In regard to Theorem 1, if $v, u \in \mathrm{Fix}(R)$, then, for every $n \in \mathbb{N}_0$, we find

$$R^n(v) = v, \quad R^n(u) = u.$$

By the weak connectedness of G, there is a path $\{w_0, w_1, w_2, \ldots w_p\}$ between v and u, i.e.,

$$w_0 = v,\ w_p = u \text{ and } (w_{r-1}, w_r) \in E(G), \quad \forall\, r \in \{1, 2, \ldots p\}.$$

As R is G-edge preserving, we find for each $0 \leq r \leq p-1$ that

$$(R^n w_r, R^n w_{r+1}) \in E(\tilde{G}), \quad \forall\, n \in \mathbb{N}_0. \tag{9}$$

The application of the triangle inequality reveals that

$$\varrho(v, u) = \varrho(R^n w_0, R^n w_p) \leq \sum_{r=0}^{p-1} \varrho(R^n w_r, R^n w_{r+1}). \tag{10}$$

For every $r (0 \leq r \leq p-1)$, δ_n^r denotes $\varrho(R^n w_r, R^n w_{r+1})$, where $n \in \mathbb{N}_0$. Now, it is claimed that

$$\lim_{n\to\infty} \delta_n^r = 0.$$

To substantiate this, on fixing r, assuming first that $\delta_{n_0}^r = 0$ for some $n_0 \in \mathbb{N}_0$, then, $R^{n_0+1}(w_r) = R^{n_0+1}(w_{r+1})$. Thus, we find $\delta_{n_0+1}^r = \varrho(R^{n_0+1} w_r, R^{n_0+1} w_{r+1}) = 0$; so inductively, we find $\delta_n^r = 0$ for every $n \geq n_0$, so that $\lim_{n\to\infty} \delta_n^r = 0$. In contrast, if $\delta_n^r > 0$ for every $n \in \mathbb{N}_0$, then, by (9) and the contractivity condition (2), we get

$$\begin{aligned} \delta_{n+1}^r &= \varrho(R^{n+1} w_r, R^{n+1} w_{r+1}) \\ &\leq \varphi(\varrho(R^n w_r, R^n w_{r+1})) \\ &= \varphi(\delta_n^r). \end{aligned}$$

Using the monotonicity of φ in (11), we get

$$\delta_n^r \leq \varphi(\delta_{n-1}^r) \leq \varphi^2(\delta_{n-2}^r) \leq \cdots \leq \varphi^n(\delta_0^r)$$

so that

$$\delta_n^r \leq \varphi^n(\delta_0^r). \tag{11}$$

If $\delta_0 = 0$, then by Proposition 1, one gets $\delta_n^r = 0$ yielding thereby $\lim_{n\to\infty} \delta_n = 0$. Otherwise, in case $\delta_0 > 0$, using the limit in (11) and the property of φ, one gets

$$\lim_{n\to\infty} \delta_n^r \leq \lim_{n\to\infty} \varphi^n(\delta_0) = 0.$$

Thus in each case, one has

$$\lim_{n\to\infty} \delta_n^r = 0. \tag{12}$$

Further, (10) can be written as

$$\begin{aligned} \varrho(v,u) = \varrho(R^n w_0, R^n w_p) &\leq \sum_{r=0}^{p-1} \varrho(R^n w_r, R^n w_{r+1}) \\ &\leq \delta_n^0 + \delta_n^1 + \cdots + \delta_n^{p-1} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which yields that $v = u$, so R has a unique fixed point. □

4. Applications to Fractional BVP

Consider the following fractional BVP:

$$\begin{cases} D_{0+}^{\iota}\vartheta(\theta) + \hbar(\theta, \vartheta(\theta)) = 0, & \forall\, \theta \in (0,1), \\ \vartheta(0) = \vartheta'(0) = \vartheta''(0) = 0, & \vartheta''(1) = \eta\vartheta''(\ell), \end{cases} \tag{13}$$

along with the following assumptions:

- $3 < \iota \leq 4$,
- $0 < \ell < 1$,
- $0 < \eta\ell^{\iota-3} < 1$,
- $\hbar : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous,
- $\hbar$ remains singular at $\theta = 0$, which means $\lim_{\theta\to 0+} \hbar(\theta,\cdot) = \infty$.

Obviously, the BVP (13) is identical to an integral equation given as under

$$\vartheta(\theta) = \int_0^1 G(\theta,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma + \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma \tag{14}$$

where the Green function is

$$G(\theta,\sigma) = \begin{cases} \frac{\theta^{\iota-1}(1-\sigma)^{\iota-3} - (\theta-\sigma)^{\iota-1}}{\Gamma(\iota)}, & 0 \leq \sigma \leq \theta \leq 1, \\ \frac{\theta^{\iota-1}(1-\sigma)^{\iota-3}}{\Gamma(\iota)}, & 0 \leq \theta \leq \sigma \leq 1 \end{cases}$$

and the function $H(\theta,\sigma) := \dfrac{\partial^2 G(\theta,\sigma)}{\partial\theta^2}$ becomes

$$H(\theta,\sigma) = \begin{cases} \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\left[\theta^{\iota-3}(1-\sigma)^{\iota-3} - (\theta-\sigma)^{\iota-3}\right], & 0 \leq \sigma \leq \theta \leq 1, \\ \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\theta^{\iota-3}(1-\sigma)^{\iota-3}, & 0 \leq \theta \leq \sigma \leq 1. \end{cases}$$

As usual, $\Gamma(\cdot)$ and $\beta(\cdot,\cdot)$ will denote the special functions: gamma function and beta function, respectively. Motivated by [8,9], we will determine the unique positive solution of (13).

Proposition 2 ([9]). *The functions G and H enjoy the following properties:*

- *G and H both are continuous;*
- *$G(\theta,\sigma) \geq 0$ and $H(\theta,\sigma) \geq 0$;*
- *$G(\theta,1) = 0$;*
- *$\sup\limits_{0\leq\theta\leq 1} \int_0^1 G(\theta,\sigma)d\sigma = \frac{2}{(\iota-2)\Gamma(\iota+1)}$;*
- *$\int_0^1 H(\ell,\sigma)d\sigma = \frac{\ell^{\iota-3}(\iota-1)(1-\ell)}{\Gamma(\iota)}$.*

Lemma 1. *If $0 < \rho < 1$, then,*

$$\sup_{0\leq\theta\leq 1} \int_0^1 G(\theta,\sigma)\sigma^{-\rho}d\sigma = \frac{1}{\Gamma(\iota)}(\beta(1-\rho,\iota-2) - \beta(1-\rho,\iota)).$$

Proof. Making use of definition of G, we get

$$\begin{aligned}
\int_0^1 G(\theta,\sigma)\sigma^{-\rho}d\sigma &= \int_0^\theta G(\theta,\sigma)\sigma^{-\rho}d\sigma + \int_\theta^1 G(\theta,\sigma)\sigma^{-\rho}d\sigma \\
&= \int_0^\theta \frac{\theta^{\iota-1}(1-\sigma)^{\iota-3} - (\theta-\sigma)^{\iota-1}}{\Gamma(\iota)}\sigma^{-\rho}d\sigma + \int_\theta^1 \frac{\theta^{\iota-1}(1-\sigma)^{\iota-3}}{\Gamma(\iota)}\sigma^{-\rho}d\sigma \\
&= \int_0^1 \frac{\theta^{\iota-1}(1-\sigma)^{\iota-3}}{\Gamma(\iota)}\sigma^{-\rho}d\sigma - \int_0^\theta \frac{(\theta-\sigma)^{\iota-1}}{\Gamma(\iota)}\sigma^{-\iota}d\sigma \\
&= \frac{\theta^{\iota-1}}{\Gamma(\iota)}\int_0^1 (1-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma - \frac{1}{\Gamma(\iota)}\int_0^\theta (\theta-\sigma)^{\iota-1}\sigma^{-\rho}d\sigma \\
&= \frac{\theta^{\iota-1}}{\Gamma(\iota)}\beta(1-\rho,\iota-2) - \frac{1}{\Gamma(\iota)}I, \qquad (15)
\end{aligned}$$

where

$$I = \int_0^\theta (\theta-\sigma)^{\iota-1}\sigma^{-\rho}d\sigma = \int_0^\theta \left(1-\frac{\sigma}{\theta}\right)^{\iota-1}\theta^{\iota-1}\sigma^{-\rho}d\sigma = \theta^{\theta-\rho}\int_0^\theta \left(1-\frac{\sigma}{\theta}\right)^{\iota-1}\left(\frac{\sigma}{\theta}\right)^{-\rho}\theta d\sigma.$$

Applying the change of variables $v = \sigma/\theta$ so that $\theta dv = d\sigma$ in the above integral, we find

$$I = \theta^{\theta-\rho}\int_0^\theta (1-v)^{\iota-1}v^{-\rho}dv = \theta^{1-\rho}\beta(1-\rho,\iota). \qquad (16)$$

By (15) and (16), we obtain

$$\int_0^1 G(\theta,\sigma)\sigma^{-\rho}d\sigma = \frac{\theta^{\iota-1}}{\Gamma(\iota)}\beta(1-\rho,\iota-2) - \frac{\theta^{\iota-\rho}}{\Gamma(\iota)}\beta(1-\rho,\iota).$$

Defining

$$\phi(\theta) := \frac{\beta(1-\rho,\iota-2)}{\Gamma(\iota)}\theta^{\iota-1} - \frac{\beta(1-\rho,\iota)}{\Gamma(\iota)}\theta^{\iota-\rho}$$

Naturally, the function $\phi(\theta)$ remains increasing on $[0,1]$. Hence, we conclude

$$\sup_{0\leq\theta\leq 1}\int_0^1 G(\theta,\sigma)\sigma^{-\rho}d\sigma = \sup_{0\leq\theta\leq 1}\phi(\theta) = \phi(1) = \frac{1}{\Gamma(\iota)}[\beta(1-\rho,\iota-2) - \beta(1-\rho,\iota)].$$

□

Lemma 2. *If* $0 < \rho < 1$, *then,*

$$\int_0^1 H(\ell,\sigma)\sigma^{-\rho}d\sigma = \frac{(\iota - 1(\iota-2)}{\Gamma(\iota)}\left(\ell^{\iota-3} - \ell^{\iota-\rho-2}\beta(1-\rho,\iota-2)\right),$$

Proof. We have

$$\begin{aligned}
&\int_0^1 H(\ell,\sigma)\sigma^{-\rho}d\sigma = \int_0^\ell H(\ell,\sigma)\sigma^{-\rho}d\sigma + \int_\ell^1 H(\ell,\sigma)\sigma^{-\rho}d\sigma \\
&= \int_0^\ell \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\Big[\ell^{\iota-3}(1-\sigma)^{\iota-3} - (\ell-\sigma)^{\iota-3}\Big]\sigma^{-\rho}d\sigma + \int_\ell^1 \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-3}(1-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma \\
&= \int_0^1 \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-3}(1-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma - \int_0^\ell \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}(\ell-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma \\
&= \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-3}\int_0^1 (1-\sigma)^{\iota-1}\sigma^{-\rho}d\sigma \\
&- \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\int_0^\ell (\ell-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma \\
&= \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-3}\beta(1-\rho,\iota-2) - \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\int_0^\ell (\ell-\sigma)^{\iota-3}\sigma^{-\rho}d\sigma
\end{aligned}$$

In keeping with the argument of the proof of Lemma 1, we conclude

$$\begin{aligned}
\int_0^1 H(\ell,\sigma)\sigma^{-\rho}d\sigma &= \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-3}\beta(1-\rho,\iota-2) \\
&- \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\ell^{\iota-\rho-2}\beta(1-\rho,\iota-2) \\
&= \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)}\left(\ell^{\iota-3} - \ell^{\iota-\rho-2}\right)\beta(1-\rho,\iota-2).
\end{aligned}$$

□

Remark 1. *Denote*

$$\lambda := \frac{1}{\Gamma(\iota)}\left[\left(1 + \frac{\beta(\ell^{\iota-3} - \ell^{\iota-\rho-2})}{1-\beta\ell^{\iota-3}}\right)\beta(1-\rho,\iota-2) - \beta(1-\rho,\iota)\right].$$

Finally, we present the main results.

Theorem 3. *Let the BVP* (13) *satisfy the above standard assumptions. Also, assume that* $0 < \rho < 1$ *and that* $\theta^\rho\hbar(\theta,\sigma)$ *is continuous. If* $\mu \in (0, 1/\lambda]$ *and* φ *remains a comparison function with*

$$\sigma_1 \geq \sigma_2 \geq 0 \text{ and } 0 \leq \theta \leq 1 \Longrightarrow 0 \leq \theta^\rho[\hbar(\theta,\sigma_1) - \hbar(\theta,\sigma_2)] \leq \mu\varphi(\sigma_1 - \sigma_2), \tag{17}$$

then, BVP (13) *possesses a unique solution.*

Proof. Endow the following metric on $C[0,1]$:

$$\varrho(\vartheta,\mu) = \sup_{0\leq\theta\leq 1}|\vartheta(\theta) - \mu(\theta)|.$$

Defining

$$V = \{\vartheta \in C[0,1] : \vartheta(\theta) \geq 0\}.$$

Then, (V,ϱ) forms a complete MS. On V, consider the relation

$$E(G) = \{(\vartheta,\mu) \in V^2 : \vartheta(\theta) \leq \mu(\theta), \text{ for each } \theta \in [0,1]\}.$$

Clearly, G is transitive, and (V, ϱ, G) forms a graph MS. Now, choose $\vartheta, \mu \in V$. Define $\omega := \max\{\vartheta, \mu\} \in V$. Then, $\{\vartheta, \omega, \mu\}$ admits a path in $\tilde{G}$ from ϑ to μ. Thus, G remains weakly connected.

We will verify that G is a (C)-graph. Assuming $\{\vartheta_n\} \subset V$ verifying $\vartheta_\mathsf{n} \to \vartheta$ and $(\vartheta_\mathsf{n}, \vartheta_{\mathsf{n}+1}) \in E(G)$, $\forall$ $\mathsf{n} \in \mathbb{N}$. Then, $\forall$ $\theta \in [0,1]$, $\{\vartheta_n(\theta)\}$ is an increasing sequence in $\mathbb{R}$ that converges to $\vartheta(\theta)$. Hence, $\forall$ $n \in \mathbb{N}$ and $\forall$ $\theta \in [0,1]$, we find $\vartheta_n(\theta) \leq \vartheta(\theta)$ so that $(\vartheta_n, \vartheta) \in E(G)$, $\forall$ $n \in \mathbb{N}$.

Now, define the map $R : V \to V$ by

$$(R\vartheta)(\theta) = \int_0^1 G(\theta,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma + \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma. \tag{18}$$

Let $\mathbf{0} \in V$ be zero function. Then, for every $\theta \in [0,1]$, we find $\mathbf{0}(\theta) \leq (R\mathbf{0})(\theta)$, thereby yielding $(\mathbf{0}, R\mathbf{0}) \in E(G)$. Thus, $\mathbf{0} \in V_R$ i.e., $V_R \neq \emptyset$.

Take $(\vartheta, \mu) \in E(G)$, thereby implying $\vartheta(\theta) \leq \mu(\theta)$, for each $\theta \in [0,1]$. Consequently, we find

$$\begin{aligned}
(R\vartheta)(\theta) &= \int_0^1 G(\theta,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma + \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\hbar(\sigma,\vartheta(\sigma))d\sigma. \\
&= \int_0^1 G(\theta,\sigma)\sigma^{-\rho}\sigma^{\rho}\hbar(x,\vartheta(\sigma))d\sigma \\
&+ \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\sigma^{-\rho}\sigma^{\rho}\hbar(\sigma,\vartheta(\sigma))d\sigma \\
&\leq \int_0^1 G(\theta,\sigma)\sigma^{-\rho}\sigma^{\rho}\hbar(\sigma,\mu(\sigma))d\sigma \\
&+ \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\sigma^{-\rho}\sigma^{\rho}\hbar(\sigma,\mu(\sigma))d\sigma \\
&= \int_0^1 G(\theta,\sigma)\hbar(\sigma,\mu(\sigma))d\sigma + \frac{\eta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\hbar(\sigma,\mu(\sigma))d\sigma \\
&= (R\mu)(\theta)
\end{aligned}$$

yielding $(R\vartheta, R\mu) \in E(G)$. Hence, R is G-edge preserving.

On the other hand, for $(\vartheta, \mu) \in E(G)$, we also have

$$\begin{aligned}
\varrho(R\vartheta, R\mu) &= \sup_{0\leq\theta\leq1} |(R\vartheta)(\theta) - (R\mu)(\theta)| = \sup_{0\leq\theta\leq1} [(R\mu)(\theta) - (R\vartheta)(\theta)] \\
&= \sup_{0\leq\theta\leq1} \left[\int_0^1 G(\theta,\sigma)(\hbar(\sigma,\mu(\sigma)) - \hbar(\sigma,\vartheta(\sigma)))\,d\sigma\right. \\
&\left.+ \frac{\eta\theta^{\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)(\hbar(\sigma,\mu(\sigma)) - \hbar(\sigma,v)(\sigma))d\sigma\right] \\
&\leq \sup_{0\leq\theta\leq1}\int_0^1 G(\theta,\sigma)\sigma^{-\rho}\sigma^{\rho}[\hbar(\sigma,\mu(\sigma)) - \hbar(\sigma,\vartheta(\sigma))]d\sigma \\
&+ \frac{\eta}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\sigma^{-\rho}\sigma^{\rho}[\hbar(\sigma,\mu(\sigma)) - \hbar(\sigma,\vartheta)(\sigma)]d\sigma \\
&\leq \sup\int_0^1 G(\theta,\sigma)\sigma^{-\rho}\mu\varphi(\mu(\sigma) - \vartheta(\sigma))d\sigma \\
&+ \frac{\eta}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})}\int_0^1 H(\ell,\sigma)\sigma^{-\rho}\mu\varphi(\mu(\sigma)) - \vartheta(\sigma)d\sigma.
\end{aligned}$$

Using the monotonicity of φ, the above relation reduces to

$$\begin{aligned}\varrho(R\vartheta, R\mu) \leq \mu\varphi(\varrho(\vartheta,\mu)) \sup_{0\leq\theta\leq 0} \int_0^1 G(\theta,\sigma)\sigma^{-\rho} d\sigma \\ + \frac{\eta}{(\iota-1)(\iota-2)(1-\eta\ell\iota-3)} \mu\varphi(\varrho(\mu,v)) \int_0^1 H(\ell,\sigma)\sigma^{\rho} d\sigma \\ = \mu\varphi(\varrho(\vartheta,\mu)) \Bigg[\sup_{0\leq\theta\leq 0} \int_0^1 G(\theta,\sigma)\sigma^{-\rho} d\sigma \\ + \frac{\eta}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})} \int_0^1 H(\ell,\sigma)\sigma^{-\rho} d\sigma \Bigg].\end{aligned} \tag{19}$$

Using Lemmas 1 and 2, (19) reduces to

$$\begin{aligned}\varrho(R\vartheta, R\mu) \leq & \mu\varphi(\varrho(\vartheta,\mu)) \Bigg[\frac{1}{\Gamma(\iota)} (\beta(1-\rho,\iota-2) - \beta(1-\rho\iota)) + \frac{\eta}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})} \\ & \times \frac{(\iota-1)(\iota-2)}{\Gamma(\iota)} \left(\ell^{\iota-3} - \ell^{\iota-\rho-2}\right) \Bigg] \\ & = \mu\varphi(\varrho(\vartheta,\mu)) \Bigg[\frac{1}{\Gamma(\iota)} (\beta(1-\rho,\iota-2) - \beta(1-\rho,\iota)) \\ & + \frac{\eta(\ell^{\iota-3} - \ell^{\iota-\rho-2})}{(1-\eta\ell^{\iota-3})\Gamma(\iota)} \beta(1-\rho,\iota-2) \Bigg] \\ & = \mu\varphi(\varrho(\vartheta,\mu)) \Bigg[\frac{1}{\Gamma(\iota)} \Bigg[\left(1 + \frac{\eta(\ell^{\iota-3} - \ell^{\iota-\rho-2})}{1-\eta-\eta\ell^{\iota-3}}\right) \beta(1-\rho,\iota-2) - \beta(1-\rho,\iota) \Bigg] \Bigg] \\ & = \mu\varphi(\varrho(\vartheta,\mu))\lambda.\end{aligned}$$

As $0 < \mu \leq 1/\lambda$, the last inequality becomes

$$\varrho(R\vartheta, R\mu) \leq \mu\varphi(\varrho(\vartheta,\mu))\lambda \leq \varphi(\varrho(\vartheta,\mu)). \tag{20}$$

Thus, R verifies the contraction condition mentioned in Theorem 2. Therefore, by Theorem 2, R is a PM. Thus, in view of (14) and (18), the unique fixed point of R will form the unique solution of BVP (13). □

Theorem 4. *Along with the assertions of Theorem 3, BVP (13) owns a unique positive solution.*

Proof. By Theorem 3, let $\varpi \in V$ be the unique solution of BVP (13). Owing to the fact $\varpi \in V$, we have $\varpi(\theta) \geq 0, \forall\, \theta \in [0,1]$. This means that ϖ is a unique nonnegative solution of given BVP. By contradiction method, we will verify that ϖ remains a unique positive solution of the BVP, i.e., $\bar{p}(\mathrm{x}) > 0$, for all $\mathrm{x} \in (0,1)$. If $\exists\, 0 < \theta^* < 1$ verifying $\varpi(\theta^*) = 0$, then by (14), we observe that

$$\varpi(\theta^*) = \int_0^1 G(\theta^*,\sigma)\hbar(\sigma,\varpi(\sigma))d\sigma + \frac{\eta\theta^{*\iota-1}}{(\iota-1)(\iota-2)(1-\eta\ell^{\iota-3})} \int_0^1 H(\ell,\sigma)\hbar(\sigma,x(\sigma))d\sigma = 0.$$

By the definition, $\hbar$ is nonnegative. Thus in view of Proposition 2, both summands in RHS are nonnegative. Consequently, we find

$$\begin{aligned}\int_0^1 G(\theta^*,\sigma)\hbar(\sigma,\varpi(\sigma))d\sigma = 0, \\ \int_0^1 H(\ell,\sigma)\hbar(\sigma,\sigma(\sigma))d\sigma = 0\end{aligned}$$

thereby implying

$$\begin{cases} G(\theta^*,\sigma)\hbar(\sigma,\varpi(\sigma)) = 0, & a.e.\ (\sigma), \\ H(\ell,\sigma)\hbar(\sigma,\varpi(\sigma)) = 0, & a.e\ (\sigma). \end{cases} \tag{21}$$

Take an arbitrary $\kappa > 0$. By the singular property of $\hbar$, we can find an $\epsilon > 0$ with $\hbar(\sigma, 0) > \kappa$, $\forall\, \sigma \in [0, 1] \cap (0, \epsilon)$. Note that

$$[0,1] \cap (0,\epsilon) \subset \{\sigma \in [0,1] : \hbar(\sigma,\varpi(\sigma)) > \kappa\}$$

and

$$\aleph([0,1] \cap (0,\epsilon)) > 0,$$

where $\aleph$ denotes the Lebesque measure. Hence, (21) yields that

$$\begin{cases} G(\theta^*,\sigma) = 0, & a.e.\ (\sigma), \\ H(\ell,\sigma) = 0, & a.e.\ (\sigma) \end{cases}$$

which contradicts the fact that $G(\theta^*, \cdot)$ and $H(\ell, \cdot)$ are rational functions. This completes the proof. □

5. Discussions

This article is devoted to prove some outcomes on fixed points under an expanded class of (G, φ)-contraction in the setup of graph metric space. The results presented in this article give new insights into graph metric spaces. Our findings extend, enrich, unify, sharpen and improve a few fixed point theorems, especially due to Matkowski [12], Pant [17], Jachymski [18] and Bojor [19]. Applying our findings, we describe the existence of the unique positive solution of a BVP involving singular fractional differential equations. We can prove the analogues of our results under Boyd–Wong contractions, weak contractions, (ψ, ϕ)-contractions, F-contractions, $\mathcal{Z}$-contractions, and similar others.

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