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On a Neumann Problem with an Intrinsic Operator

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Abstract: This paper investigates the existence and location of solutions for a Neumann problem driven by a (p,q) Laplacian operator and with a reaction term that depends not only on the solution and its gradient but also incorporates an intrinsic operator, which is its main novelty. This paper can be seen as the study of a quasilinear Neumann problem involving an elaborated perturbation with a Nemytskij operator. The approach proceeds through a version of the sub/supersolution method, exploiting an invariance property regarding the sub/supersolution ordered interval with respect to the intrinsic operator. An example illustrates the applicability of our result.

Keywords: neumann problem; sub/supersolution; intrinsic operator; gradient dependence; positive solution

MSC: 35B09; 35D30; 35J66

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Citation: Motreanu, D.; Sciammetta, A. On a Neumann Problem with an Intrinsic Operator. *Axioms* **2024**, *13*, 497. https://doi.org/10.3390/axioms13080497

Academic Editors: Deguang Han and Simeon Reich

Received: 9 June 2024
Revised: 14 July 2024
Accepted: 24 July 2024
Published: 25 July 2024

1. Introduction

Our objective is to study the following quasilinear elliptic problem with a Neumann boundary condition: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta_p u - \mu \Delta_q u + |u|^{p-2}u = f(x, B(u), \nabla(B(u))) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$ and outer unit normal ν to $\partial\Omega$. In the statement of problem (1), there is a real parameter μ for which we suppose that $\mu \geq 0$. The equation is driven by the (p,q)-type Laplacian operator $-\Delta_p u - \mu\Delta_q u$, which is perturbed with the power term $|u|^{p-2}u$. We assume that $1 < q < p < +\infty$. The reaction term in the equation is described by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, meaning that $f(\cdot, s, \xi)$ is measurable for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and that $f(x,\cdot,\cdot)$ is continuous for almost all $x \in \Omega$), which is composed with an intrinsic operator $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ in Sobolev space $W^{1,p}(\Omega)$, subject to certain hypotheses (see conditions (H_1) and (H_2) in Section 2).

For the sake of clarity, we recall that the negative $p-$Laplacian $-\Delta_p : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and the negative $q-$Laplacian $-\Delta_q : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ are defined, respectively, by

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) dx \quad \text{for all } u, v \in W^{1,p}(\Omega),$$

and

$$\langle -\Delta_q u, v \rangle = \int_\Omega |\nabla u(x)|^{q-2} \nabla u(x) \cdot \nabla v(x) dx \quad \text{for all } u, v \in W^{1,q}(\Omega).$$

We refer to [1,2] for the background related to the p-Laplacian operator. We note that the operator $-\Delta_p - \mu\Delta_q$ is well defined in the space $W^{1,p}(\Omega)$. This is true because $\Omega \subset \mathbb{R}^N$

is bounded and $p > q$, so $W^{1,p}(\Omega)$ is continuously embedded in $W^{1,q}(\Omega)$, as can be seen through Hölder's inequality. Consequently, due to the continuity of $-\Delta_p$ and $-\Delta_q$, we have the continuous operator $-\Delta_p - \mu\Delta_q : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$. Two cases are extremely important. If $\mu = 0$, we obtain the negative $p-$Laplacian, while if $\mu = 1$, we obtain the negative $(p,q)-$Laplacian. These two cases are essentially different. Notice that the negative $(p,q)-$Laplacian $-\Delta_p - \Delta_q$ is a non-homogeneous operator, whereas the negative $p-$Laplacian is a homogeneous operator of order $p-1$.

The main feature of the present work is that the reaction term of the equation in (1)—that is, $f(x, B(u), \nabla(B(u)))$—is subject to the combined effects of both convection and the intrinsic operator. Taking into account that the right-hand side of the equation depends on the solution u, on its gradient ∇u, and on the intrinsic operator B, the problem does not possess a variational structure, so the variational methods are not applicable. Comprehensive information about variational methods can be found in [3–5].

The key contribution of this work consists of handling the intrinsic operator $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ within the sub/supersolution method for the Neumann problem (1). This fact represents a novel development in the field of non-linear elliptic boundary value problems. Problems involving such an intrinsic operator B have until now been considered in [6] in the context of Dirichlet boundary condition and for Dirichlet problems driven by a competing operator in [7]. If the intrinsic operator B is the identity map, problem (1) reduces to a quasilinear elliptic equation with a convection term. It exhibits full dependence on the solution and its gradient. For various problems involving convection, we refer to [8–10]).

In order to study problem (1), we build a non-variational approach based on a sub,supersolution method adapted to the presence of the intrinsic operator. The method of sub/supersolution for quasilinear elliptic problems with a Neumann boundary condition with a convection term has been implemented in [9]. In the case of the Robin boundary condition, this has been accomplished in [10]. A detailed treatment in the broad sense of the sub/supersolution method combined with set-valued analysis is developed in [1]. In this respect, we aim to show that given a subsolution $\underline{u}$ and a supersolution $\overline{u}$, appropriately defined for problem (1), that satisfy the pointwise order $\underline{u} \le \overline{u}$ almost everywhere in Ω, there exists a weak solution $u \in W^{1,p}(\Omega)$ to problem (1) with the location property $\underline{u} \le u \le \overline{u}$ almost everywhere in Ω. Notice that the location $\underline{u} \le u \le \overline{u}$ provides a priori estimates for the weak solution u. The main difficulty to be overcome pertains to how to handle the intrinsic operator B in the framework of sub/supersolution. We resolve this difficulty by requiring the invariance of the ordered interval

$$[\underline{u}, \overline{u}] := \{u \in W^{1,p}(\Omega) \ : \ \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ a.e. in } \Omega\}$$

with respect to the intrinsic operator B (see hypothesis (H_1)). A relevant tool in the proof of the main result is an auxiliary perturbed problem formulated by means of a truncation operator and a cut-off function corresponding to the ordered interval $[\underline{u}, \overline{u}]$. The existence of a weak solution to the auxiliary problem is shown by applying the following abstract result on the surjectivity of pseudomonotone operators. A systematic presentation of the theory of monotone and pseudomonotone operators is given in [1,11,12].

Theorem 1 ([1] Theorem 2.99)**.** *Let X be a real, reflexive Banach space, and let $A : X \to X^*$ be a bounded, coercive, and pseudomonotone operator. Then, for every $b \in X^*$, the equation $Ax = b$ has at least one solution $x \in X$.*

Finally, making use of comparison arguments, we show that the solution $u \in W^{1,p}(\Omega)$ of the auxiliary problem is, in fact, a solution to the original problem (1), satisfying the location property $u \in [\underline{u}, \overline{u}]$.

An example of problem (1) containing the explicit description of an intrinsic operator that satisfies all the required conditions demonstrates the applicability of our result.

2. Preliminaries and Hypotheses

The functional space associated to problem (1) with a Neumann boundary condition is the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \tag{2}$$

where $\|\cdot\|_{L^p(\Omega)}$ denotes the usual norm of the Banach space $L^p(\Omega)$. The duality pairing between $W^{1,p}(\Omega)$ and its dual $(W^{1,p}(\Omega))^*$ will be denoted by $\langle\cdot,\cdot\rangle$. We refer to [13] for the background regarding Sobolev spaces.

In the sequel, in order to simplify the presentation, we suppose that $N > p$; thus, the Sobolev critical exponent is $p^* = \frac{Np}{N-p}$. The case $N \leq p$ is easier and we omit it.

We first record some important properties of the operator $-\Delta_p - \mu\Delta_q$.

Proposition 1 (see [1] Section 2.3.2)**.** *The operator $-\Delta_p - \mu\Delta_q : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ with $1 < q < p < +\infty$ and $\mu \geq 0$ is continuous, strictly monotone (so pseudomonotone), and satisfies the (S_+)-property; that is, any sequence $\{u_n\} \subset W^{1,p}(\Omega)$ for which $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle -\Delta_p u_n - \mu\Delta_q u_n, u_n - u\rangle \leq 0 \tag{3}$$

fulfills $u_n \to u$ in $W^{1,p}(\Omega)$.

A solution to problem (1) is understood in the weak sense. Namely, a weak solution to problem (1) is any $u \in W^{1,p}(\Omega)$ such that $f(x, B(u), \nabla(B(u)))v \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx &+ \mu\int_\Omega |\nabla u(x)|^{q-2}\nabla u(x)\cdot\nabla v(x)dx + \int_\Omega |u|^{p-2}uvdx\\ &= \int_\Omega f(x, B(u), \nabla(B(u)))vdx\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$. We introduce the fundamental notions of subsolution and supersolution for problem (1).

A supersolution to problem (1) is any $\overline{u} \in W^{1,p}(\Omega)$ such that $f(x, B(\overline{u}), \nabla(B(\overline{u})))v \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x)\cdot\nabla v(x)dx &+ \mu\int_\Omega |\nabla \overline{u}(x)|^{q-2}\nabla \overline{u}(x)\cdot\nabla v(x)dx + \int_\Omega |\overline{u}|^{p-2}\overline{u}vdx\\ &\geq \int_\Omega f(x, B(\overline{u}), \nabla(B(\overline{u})))vdx\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$, with $v \geq 0$ a.e. in Ω.

A subsolution to problem (1) is any $\underline{u} \in W^{1,p}(\Omega)$ such that $f(x, B(\underline{u}), \nabla(B(\underline{u})))v \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x)\cdot\nabla v(x)dx &+ \mu\int_\Omega |\nabla \underline{u}(x)|^{q-2}\nabla \underline{u}(x)\cdot\nabla v(x) + \int_\Omega |\underline{u}|^{p-2}\underline{u}vdx\\ &\leq \int_\Omega f(x, B(\underline{u}), \nabla(B(\underline{u})))vdx\end{aligned}$$

for all $v \in W^{1,p}(\Omega)$, with $v \geq 0$ a.e. in Ω. It is clear that a weak solution $u \in W^{1,p}(\Omega)$ to problem (1) is simultaneously a subsolution and supersolution.

For a given real number $r \in]1, +\infty[$, set $r' := \dfrac{r}{r-1}$ (the Hölder conjugate of r).

We proceed to formulate the assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and the intrinsic operator $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. We assume that there exist a subsolution $\underline{u} \in W^{1,p}(\Omega)$ and a supersolution $\overline{u} \in W^{1,p}(\Omega)$ of problem (1) with $\underline{u} \leq \overline{u}$ a.e. in Ω for which the following conditions hold:

(H_0) There exist a function $\sigma \in L^{r'}(\Omega)$ with $r \in]1, p^*[$ and constants $a > 0$ and $\beta \in [0, \frac{p}{(p^*)'}[$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a|\xi|^{\beta} \ \text{ for a.e. } x \in \Omega, \ \forall s \in [\underline{u}(x), \overline{u}(x)], \ \forall \xi \in \mathbb{R}^N.$$

(H_1) The map $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is continuous and fulfils

$$\underline{u} \leq B(v) \leq \overline{u}$$

a.e. in Ω and for all $\underline{u} \leq v \leq \overline{u}$ almost everywhere.

(H_2) There exist positive constants K_1 and K_2 such that

$$\|B(u)\| \leq K_1 \|u\| + K_2 \quad \text{a.e.}, \ \forall u \in W^{1,p}(\Omega).$$

We note that under assumption (H_0), the integrals in the definitions of subsolution $\underline{u}$ and supersolution $\overline{u}$ exist. This can be easily checked by means of Hölder's inequality. We also point out that under the assumption (H_2), the map B is bounded in the sense that it maps bounded sets into bounded sets.

Consider the Nemytskij type operator $N_{f,B} : [\underline{u}, \overline{u}] \to (W^{1,p}(\Omega))^*$ defined with f and B, as above, by

$$\left\langle N_{f,B} u, v \right\rangle = \int_{\Omega} f(x, B(u), \nabla(B(u))) v dx, \ \forall u \in [\underline{u}, \overline{u}], \ \forall v \in W^{1,p}(\Omega).$$

This is well defined on the ordered interval$[\underline{u}, \overline{u}]$ by virtue of hypotheses (H_0) and (H_1) and in conjunction with Hölder's inequality. The properties of various Nemytskij operators are discussed in [4].

An important tool in our approach is the truncation operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ associated with the ordered interval $[\underline{u}, \overline{u}]$, which is defined by

$$Tu(x) = \begin{cases} \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x), \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \end{cases} \tag{4}$$

almost everywhere in Ω, for all $u \in W^{1,p}(\Omega)$. On the basis of (4), it is straightforward to verify that the operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is continuous and bounded in the sense that it maps bounded sets into bounded sets. Using the maps f and B, we introduce the composed operator $\mathcal{N}_{f,B} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$\mathcal{N}_{f,B} = N_{f,B} \circ T. \tag{5}$$

Explicitly, this is expressed as

$$\left\langle \mathcal{N}_{f,B} u, v \right\rangle = \int_{\Omega} f(x, B(Tu), \nabla(B(Tu))) v dx, \ \forall u, v \in W^{1,p}(\Omega).$$

In addition, we also need the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ associated with the ordered interval $[\underline{u}, \overline{u}]$, defined as

$$\pi(x,s) = \begin{cases} (s - \overline{u}(x))^{\frac{\beta}{p-\beta}} & \text{if} \quad s > \overline{u}(x), \\ 0 & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x), \\ -(\underline{u}(x) - s)^{\frac{\beta}{p-\beta}} & \text{if} \quad s < \underline{u}(x), \end{cases} \tag{6}$$

with the constant β in hypothesis (H_0). The function π in (6) has the growth

$$|\pi(x,s)| \leq c|s|^{\frac{\beta}{p-\beta}} + \varrho(x) \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \tag{7}$$

with a constant $c > 0$ and a function $\varrho \in L^{\frac{p^*(p-\beta)}{\beta}}(\Omega)$. Furthermore, from (6), we can infer the estimates

$$\int_\Omega \pi(x, u(x))u(x)\, dx \geq r_1 \|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - r_2 \text{ for all } u \in W^{1,p}(\Omega) \tag{8}$$

and

$$\int_\Omega |\pi(x, u(x))||v(x)|\, dx \leq r_3 \|u\|^{\frac{\beta}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} \|v\|_{L^{\frac{p}{p-\beta}}(\Omega)} + r_4 \|v\|_{L^{\frac{p}{p-\beta}}(\Omega)} \text{ for all } u, v \in W^{1,p}(\Omega), \tag{9}$$

with positive constants r_1, r_2, r_3, and r_4 (for more details, see [9]).

The Nemytskij operator $\Pi : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ corresponding to the function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ in (6) acts as

$$\langle \Pi(u), v \rangle = \int_\Omega \pi(x, u)v dx,\ \forall u, v \in W^{1,p}(\Omega). \tag{10}$$

Since $\beta < \frac{p}{(p^*)'}$, by (7) and the Rellich–Kondrachov compact embedding theorem, it follows that the map $\Pi : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is completely continuous. This is the consequence of the fact that $\frac{p^*(p-\beta)}{\beta} > (p^*)'$ if, and only if, $\beta < \frac{p}{(p^*)'}$.

3. Perturbed Problem

In order to find a solution to the Neumann problem (1) within the ordered interval $[\underline{u}, \overline{u}]$ determined by a subsolution $\underline{u}$ and a supersolution $\overline{u}$, we perturb problem (1) using the Nemytskij operator $\Pi : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ corresponding to the cut-off function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ defined in (6), as well as the truncation operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ and a parameter $\lambda > 0$. More precisely, we focus on the auxiliary Neumann problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u + |u|^{p-2}u + \lambda\Pi(u) = \mathcal{N}_{f,B}(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Here, Formulae (5) and (10) are utilized. In line with what has been said above, a weak solution to problem (11) is a function $u \in W^{1,p}(\Omega)$ such that $f(x, B(u), \nabla(B(u)))v \in L^1(\Omega)$ and

$$\begin{aligned} \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) dx \quad &+ \quad \mu \int_\Omega |\nabla u(x)|^{q-2}\nabla u(x) \cdot \nabla v(x) dx + \int_\Omega |u|^{p-2}uv dx \\ &+ \quad \lambda \int_\Omega \pi(x, u)v dx = \int_\Omega f(x, B(Tu), \nabla(B(Tu)))v dx \end{aligned}$$

for all $v \in W^{1,p}(\Omega)$.

Next, we prove that the solvability of problem (11) can be guaranteed, provided that $\lambda > 0$ is sufficiently large.

Theorem 2. *Assume that $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of problem* (1)*, respectively, with $\underline{u} \leq \overline{u}$ a.e. in Ω such that hypotheses $(H_0) - (H_2)$ are fulfilled. Then, there exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$, there exists a solution to the auxiliary problem* (11)*.*

Proof. For each $\lambda > 0$, we introduce the non-linear operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, defined by

$$A_\lambda = -\Delta_p - \mu\Delta_q + \mathcal{E} + \lambda\Pi - \mathcal{N}_{f,B}, \tag{12}$$

where $\mathcal{E} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is defined as

$$\langle \mathcal{E}(u), v \rangle = \int_\Omega |u|^{p-2}uv dx, \quad \forall u, v \in W^{1,p}(\Omega).$$

It is known from [1] (Lemma 2.111) that the maps $-\Delta_p - \mu\Delta_q$ and $\mathcal{E}$ are bounded. Using (9), it readily follows the boundedness of the map Π (see [9]). By hypotheses $(H_0) - (H_2)$, we derive the boundedness of the operator $\mathcal{N}_{f,B}$. Therefore, the operator A_λ is bounded.

Now, we prove that the non-linear operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ in (12) is pseudomonotone. To this end, let a sequence $\{u_n\} \subset W^{1,p}(\Omega)$ satisfy

$$u_n \rightharpoonup u \ \text{ in } W^{1,p}(\Omega) \tag{13}$$

and

$$\limsup_{n\to\infty} \langle A_\lambda u_n, u_n - u\rangle \leq 0. \tag{14}$$

We aim to show that

$$\liminf_{n\to\infty} \langle A_\lambda u_n, u_n - v\rangle \geq \langle A_\lambda u, u - v\rangle,\ \forall v \in W^{1,p}(\Omega). \tag{15}$$

By Hölder's inequality (13) and Rellich–Kondrachov compact embedding theorem we obtain

$$\int_\Omega \sigma|u_n - u|\,dx \leq \|\sigma\|_{L^{r'}(\Omega)}\|u_n - u\|_{L^r(\Omega)} \to 0 \text{ as } n \to +\infty \tag{16}$$

and

$$\int_\Omega |u_n|^{p-1}|u_n - u|\,dx \leq \|u_n\|_{L^{p'}(\Omega)}\|u_n - u\|_{L^p(\Omega)} \to 0 \text{ as } n \to +\infty. \tag{17}$$

Hölder's inequality implies

$$\int_\Omega |\nabla(B(Tu_n))|^\beta |u_n - u| dx \leq \|\nabla(B(Tu_n))\|^\beta_{L^p(\Omega)} \|u_n - u\|_{L^{\frac{p}{p-\beta}}(\Omega)}.$$

Taking into account (13), Rellich–Kondrachov compact embedding theorem, and the inequality $\frac{p}{p-\beta} < p^*$ (thanks to the assumption $\beta < \frac{p}{(p^*)'}$ in (H_0)), the preceding estimate yields

$$\lim_{n\to\infty} \int_\Omega |\nabla(B(Tu_n))|^\beta |u_n - u|\,dx = 0. \tag{18}$$

Through the definition of the truncation operator $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ in (4), combined with (H_0), (H_1), (16), and (18), we find that

$$\lim_{n\to\infty} \int_\Omega f(x, B(Tu_n), \nabla(B(Tu_n)))(u_n - u)\,dx = 0. \tag{19}$$

In addition, from (9), the inequality $\frac{p}{p-\beta} < .p^*$ and the Rellich=-Kondrachov compact embedding theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega \pi(x, u_n)(u_n - u)\,dx = 0. \tag{20}$$

Gathering (12), (14), (17), (19), and (20), we arrive at (3).

We carry on the proof by referring to the $(S_+)-$property of the operator $-\Delta_p - \mu\Delta_q$ given in Proposition 1, which ensures the strong convergence $u_n \to u$ in $W^{1,p}(\Omega)$. In view of the expression of A_λ in (12), it follows that (15) holds true, whence the operator A_λ is pseudomonotone.

Now , e claim that operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is coercive; that is,

$$\lim_{\|u\|\to+\infty} \frac{\langle A_\lambda u, u\rangle}{\|u\|} = +\infty. \tag{21}$$

By virtue of (4), it holds that $\underline{u} \le Tu \le \overline{u}$ a.e. in Ω for every $u \in W^{1,p}(\Omega)$; whereas by (H_1), we have $\underline{u} \le B(Tu) \le \overline{u}$ a.e. in Ω. Consequently, we are allowed to address hypothesis (H_0) with $s = B(Tu)(x)$ for a.e. $x \in \Omega$. Here, the invariance property of the ordered interval $[\underline{u}, \overline{u}]$, as postulated in (H_1) for the operator B, is essential in our argument. Then, (H_0), Hölder's and Young's inequalities, (H_2), the Sobolev embedding theorem, and (2) enable us to infer for each $\varepsilon > 0$ that

$$\begin{aligned}\left|\int_\Omega f(x, B(Tu), \nabla(B(Tu)))u dx\right| &\le \int_\Omega \left(\sigma|u| + a|\nabla(B(Tu))|^\beta |u|\right) dx \\ &\le \|\sigma\|_{L^{r'}(\Omega)}\|u\|_{L^r(\Omega)} + \varepsilon\|\nabla(B(Tu))\|^p_{L^p(\Omega)} + c(\varepsilon)\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} \\ &\le \varepsilon(K_1\|Tu\| + K_2)^p + c(\varepsilon)\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} + c_0\|u\| \\ &\le 2^{p-1}K_1^p\varepsilon\|u\|^p + c(\varepsilon)\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} + c_0\|u\| + d(\varepsilon),\end{aligned} \tag{22}$$

with positive constants $c(\varepsilon)$, $d(\varepsilon)$ (depending on ε), and c_0.

Then (8), (12), and (22) result in

$$\langle A_\lambda u, u\rangle \ge (1 - 2^{p-1}K_1^p\varepsilon)\|u\|^p + (\lambda r_1 - c(\varepsilon))\|u\|^{\frac{p}{p-\beta}}_{L^{\frac{p}{p-\beta}}(\Omega)} - d\|u\| - \lambda r_2 - d(\varepsilon). \tag{23}$$

Choose $\varepsilon < \frac{1}{2^{p-1}K_1^p}$, and after that, once that ε is fixed, $\lambda > \frac{c(\varepsilon)}{r_1}$. We see that (23) implies (21) because $p > 1$, which amounts to saying that the operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is coercive.

Since the operator $A_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded, pseudomonotone, and coercive, we are able to apply Theorem 1, ensuring the existence of $u \in W^{1,p}(\Omega)$ such that $A_\lambda u = 0$. Therefore, u is a (weak) solution to perturbed problem (11), and the proof is completed. □

4. Main Result

We are in a now position to state our main result with respect to the Neumann problem (P).

Theorem 3. *Let a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of the Neumann problem (1) be $\underline{u} \le \overline{u}$ a.e. in Ω, such that hypotheses $(H_0) - (H_2)$ are fulfilled. Then, problem (11) possesses a weak solution $u \in W^{1,p}(\Omega)$ with $u \in [\underline{u}, \overline{u}]$.*

Proof. By Theorem 2, there exists a solution $u \in W^{1,p}(\Omega)$ to the perturbed problem (11), provided that $\lambda > 0$ is sufficiently large. We fix such an admissible $\lambda > 0$.

We claim that u is a weak solution of the original Neumann problem (1). To prove this assertion, we use a comparison argument comparing u with the subsolution $\underline{u}$ and the supersolution $\overline{u}$.

We are going to prove that $u \in [\underline{u}, \overline{u}]$. Let us show that $u \le \overline{u}$ a.e. in Ω. To this end, we act with $v = (u - \overline{u})^+ := \max\{u - \overline{u}, 0\} \in W^{1,p}(\Omega)$ as a test function in the supersolution $\overline{u}$ definition of the Neumann problem (1) and in the definition of the weak solution u for the perturbed problem (11). This gives

$$\begin{aligned}\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla(u-\overline{u})^+dx \quad &+ \quad \mu\int_\Omega |\nabla\overline{u}|^{q-2}\nabla\overline{u}\cdot\nabla(u-\overline{u})^+dx + \int_\Omega |\overline{u}|^{p-2}\overline{u}(u-\overline{u})^+dx \\ &\ge \quad \int_\Omega f(x, B(\overline{u}), \nabla(B(\overline{u})))(u-\overline{u})^+dx\end{aligned} \tag{24}$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u-\overline{u})^+dx + \mu\int_\Omega|\nabla u|^{q-2}\nabla u\cdot\nabla(u-\overline{u})^+dx+\int_\Omega|u|^{p-2}u(u-\overline{u})^+dx$$
$$+\lambda\int_\Omega\pi(x,u)(u-\overline{u})^+dx=\int_\Omega f(x,B(Tu),\nabla(B(Tu)))(u-\overline{u})^+dx. \quad (25)$$

From (24), (25), and (4), and since $\mu \geq 0$, we derive

$$\begin{aligned}
&\int_\Omega\Big(|\nabla u|^{p-2}\nabla u-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx\\
&+\mu\int_\Omega\Big(|\nabla u|^{q-2}\nabla u-|\nabla\overline{u}|^{q-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx\\
&+\int_\Omega\Big(|u|^{p-2}u-|\overline{u}|^{p-2}\overline{u}\Big)(u-\overline{u})^+\,dx+\lambda\int_\Omega\pi(x,u)(u-\overline{u})^+\,dx\\
&\leq\int_\Omega(f(x,B(Tu),\nabla(B(Tu)))-f(x,B(\overline{u}),\nabla(B(\overline{u}))))(u-\overline{u})^+\,dx\\
&=\int_{\{u>\overline{u}\}}(f(x,B(\overline{u}),\nabla(B(\overline{u})))-f(x,B(\overline{u}),\nabla(B(\overline{u}))))(u-\overline{u})\,dx=0.
\end{aligned} \quad (26)$$

Thanks to the monotonicity under the integrals, we also have

$$\int_\Omega\Big(|\nabla u|^{p-2}\nabla u-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx=\int_{\{u>\overline{u}\}}\Big(|\nabla u|^{p-2}\nabla u-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})\geq 0,$$

$$\int_\Omega\Big(|\nabla u|^{q-2}\nabla u-|\nabla\overline{u}|^{q-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx=\int_{\{u>\overline{u}\}}\Big(|\nabla u|^{q-2}\nabla u-|\nabla\overline{u}|^{q-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})\geq 0,$$

$$\int_\Omega(|u|^{p-2}u-|\overline{u}|^{p-2}\overline{u})(u-\overline{u})^+\,dx=\int_{\{u>\overline{u}\}}(|u|^{p-2}u-|\overline{u}|^{p-2}\overline{u})(u-\overline{u})\,dx\geq 0.$$

Then, according to (6), inequality (26) leads to

$$\int_{\{u>\overline{u}\}}(u-\overline{u})^{\frac{p}{p-\beta}}\,dx=\int_\Omega\pi(x,u)(u-\overline{u})^+\,dx\leq 0,$$

which confirms that $u \leq \overline{u}$ a.e in Ω.

The proof of the inequality $\underline{u} \leq u$ a.e. in Ω proceeds along a similar comparison argument, involving, in this case, the functions $\underline{u}$ and u. In this way, the enclosure property $u \in [\underline{u}, \overline{u}]$ is established.

Now, exploiting the already-shown inclusion $u \in [\underline{u}, \overline{u}]$, we find from (4) and (6) that $Tu = u$ and $\Pi(u) = 0$. Consequently, the fact that u solves problem (11) indicates that u is a solution of the original problem (1), which concludes the proof. □

5. An Example

Given $1 < q < p < +\infty$ and $\mu \geq 0$, we formulate the following problem

$$\begin{cases}-\Delta_p u-\mu\Delta_q u+|u|^{p-2}u=\Big(|h(u)|^{p-2}h(u)+g(h(u))\Big)\Big(1+|h'(u)|^\beta|\nabla u|^\beta\Big) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases} \quad (27)$$

with a Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}$; a continuous function $g : \mathbb{R} \to \mathbb{R}$; and a constant $\beta \in \Big[0, \frac{p}{(p^*)'}\Big[$. Here, the notation h' stands for the derivative of h that exists almost everywhere. We assume that there exist constants $c_1, c_2 \in \mathbb{R}$ with $0 < c_1 < c_2$ such that $c_1 \leq h(t) \leq c_2$ if $c_1 \leq t \leq c_2$ and $g(h(c_1)) = g(h(c_2)) = 0$.

Set $\underline{u} = c_1$ and $\overline{u} = c_2$, which are elements of $W^{1,p}(\Omega)$. Let $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ be defined by $Bu = h \circ u$ for all $u \in W^{1,p}(\Omega)$. Since $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, it turns out that $h \circ u \in W^{1,p}(\Omega)$ whenever $u \in W^{1,p}(\Omega)$; so, the map B is well defined and continuous. If $u \in W^{1,p}(\Omega)$ satisfies $c_1 \leq u \leq c_2$ almost everywhere in Ω, we have

$c_1 \le Bu \le c_2$ a.e. in Ω. Furthermore, from (2) and the chain rule for Sobolev functions, and since h is Lipschitz-continuous, the derivative h' is bounded, and we obtain

$$\|Bu\| = \|h \circ u\| = \left(\|h'(u)\nabla u\|^p_{L^p(\Omega)} + \|h \circ u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \le M(\|u\| + 1)$$

for all $u \in W^{1,p}(\Omega)$, with a constant $M > 0$.

Define $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$f(s, \xi) = \left(|s|^{p-2}s + g(s)\right)\left(1 + |\xi|^\beta\right) \quad \text{for all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

It follows that $|f(s, \xi)| \le c(1 + |\xi|^\beta)$ for all $s \in [c_1, c_2]$ and $\xi \in \mathbb{R}^N$, with a constant $c > 0$. The conditions imposed on the functions g and h ensure

$$c_1^{p-1} \le h(c_1)^{p-1} = |h(c_1)|^{p-2}h(c_1) + g(h(c_1))$$

and

$$c_2^{p-1} \ge h(c_2)^{p-1} = |h(c_2)|^{p-2}h(c_2) + g(h(c_2)).$$

We deduce that $\underline{u} = c_1$ is a subsolution and $\overline{u} = c_2$ is a supersolution for problem (27). Therefore all the assumptions of Theorem 3 are fulfilled. Accordingly, we can infer that problem (27) admits at least one positive solution $u \in W^{1,p}(\Omega)$ that satisfies the a priori estimate $c_1 \le u(x) \le c_2$ for almost all $x \in \Omega$.

Remark 1. *Arguing as in [9] (Theorem 5.1), we can strengthen the assumptions, allowing us to establish the existence of multiple solutions to problem* (27)*.*

Remark 2. *The central point of our work is to provide a clear result guaranteeing that the method of sub/supersolution works in the case of a non-linear Neumann problem incorporating an intrinsic operator, as described in* (27)*. We have demonstrated with our application that Theorem 3 can be effectively used, and our hypotheses are verifiable. Many other examples of possible intrinsic operators can be considered. For instance, in the case of a Dirichlet problem, we indicated in [6] two other examples of relevant intrinsic operators: a truncation operator (for example, the positive part of a Sobolev function) and the composition of the inverse* $(-\Delta_p)^{-1}$ *with a superposition map. These examples can be adapted to a Neumann problem. Another major example of an intrinsic operator that we plan to deal with is a convolution product, observing that hypotheses* $(H_0) - (H_2)$ *are consistent with its action.*

Author Contributions: Conceptualization, D.M. and A.S.; Formal analysis, D.M. and A.S.; Investigation, D.M. and A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), grant number 2017AYM8XW; PRIN 2017—Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods", FFR-2023-Sciammetta.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The authors thank the referees for their helpful comments. The authors have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW), and by FFR-2023-Sciammetta.

Conflicts of Interest: The authors declare no conflicts of interest.

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