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# Optimality Conditions for Mathematical Programs with Vanishing Constraints Using Directional Convexificators

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**Abstract:** This article deals with mathematical programs with vanishing constraints (MPVCs) involving lower semi-continuous functions. We introduce generalized Abadie constraint qualification (ACQ) and MPVC-ACQ in terms of directional convexificators and derive necessary KKT-type optimality conditions. We also derive sufficient conditions for global optimality for the MPVC under convexity utilizing directional convexificators. Further, we introduce a Wolfe-type dual model in terms of directional convexificators and derive duality results. The results are well illustrated by examples.

**Keywords:** mathematical programs with vanishing constraints; directional convexificators; optimality conditions; constraint qualifications; generalized convexity

**MSC:** 49J52; 90C25; 90C26; 90C30; 90C46

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## 1. Introduction

Achtziger and Kanzow [1] initially proposed MPVC in 2008. Its roots are in optimization topology design problems related to mechanical structures. MPVC is widely used in several fields, like the robot motion planning problem [2], the economic dispatch problem [3], and nonlinear integer optimal control [4,5]. In solving MPVC, it is very difficult to satisfy standard constraint qualifications such as Mangasarian–Fromovitz constraint qualification (MFCQ) and linearly independent constraint qualification (LICQ), while Abadie constraint qualification (ACQ) is a very strong assumption for MPVC. MPVC is closely related to the class of mathematical programs with equilibrium constraints (MPECs) (see [6–8]). MPVC can always be formulated as an MPEC, but when solving the MPEC, this formulation presents certain difficulties because it violates the MPEC-type constraint qualifications. Thus, it is important to consider MPVC as an independent optimization problem. MPVC has garnered a lot of attention recently. Many authors have worked on MPVC with continuously differentiable functions (see [1,9–15]) and with nonsmooth functions (see [16–18]).

In 1994, Demyanov [19] introduced the notion of convexificators. Jeyakumar and Luc [20] proposed a revised version of convexificators comprising a closed set that is not necessarily bounded or convex. Several applications of convexificators are reported in [21–28]. By extending the notion of convexificators to a discontinuous case, Dempe and Pilecka [29] introduced the notion of directional convexificators based on the notion of continuity directions. Using directional convexificators, Gadhi et al. [30] established optimality conditions for a set-valued optimization problem. Using directional convexificators, Gadhi [31] developed Stampacchia and Minty variational inequalities of scalar optimization problems and used these inequalities to determine necessary and sufficient optimality conditions. Lafhim and Kalmoun [32] obtained optimality conditions for mathematical programs with equilibrium constraints using directional convexificators. For more applications of directional convexificators, we refer the reader to [33,34].

The concept of duality is very important in determining the lower bound of an objective function. Wolfe duality was introduced by Wolfe [35] for differentiable cases. Wolfe duality has also been studied in several fields, such as multiobjective programming problems [25], mathematical programs with vanishing constraints [12,36,37], mathematical programs with equilibrium constraints [24,38,39], semi-infinite programming [40,41], interval-valued programming [42–47], bilevel programming problems [48], etc.

Recently, Lafhim and Kalmoun [32] derived optimality conditions for mathematical programs with equilibrium constraints in terms of directional convexificators to also deal with nonsmooth discontinuous functions. Motivated by the work of Lafhim and Kalmoun [32], we wanted to develop analogous results for mathematical programs with vanishing constraints. Therefore, the main objective of this paper is to expand on the findings of Hu et al. [16] and the duality results of Mishra et al. [12] to include discontinuous functions with a nonempty set of continuity directions. Hence, we aim to establish optimality and duality theorems for mathematical programs with vanishing constraints in terms of directional convexificators that can tackle discontinuous functions with a nonempty set of continuity directions that cannot be handled by the results reported in [12,16]. As far as we are aware, no research has been conducted on mathematical programs with vanishing constraints in terms of directional convexificators for the purpose of resolving MPVCs with discontinuous functions. As such, this study represents an effort in this area.

The structure of our paper is summarized as follows. In Section 2, some basic definitions, preliminary information, and notations are provided. In Section 3, we introduce several stationary points under directional convexificators and nonsmooth Abadie constraint qualifications in terms of directional convexificators. Further, we derive the necessary optimality condition. We also obtain a sufficient optimality condition under the assumption of convexity in terms of directional convexificators. We illustrate the results with suitable examples. In Section 4, we present a Wolfe-type dual model in terms of directional convexificators and further report weak and strong duality results under the assumption of convexity using directional convexificators. In Section 5, we report the findings of this paper and discuss some future research possibilities.

## 2. Preliminaries

Let $\mathcal{R}^n$ be a usual $n$-dimensional Euclidean space with a norm of $\|.\|$. The convex hull and the closure of a nonempty subset ($A$) of $\mathcal{R}^n$ are denoted by $coA$ and $clA(\text{or}\bar{A})$, respectively, while $coneA$ represent the convex cone (containing the origin) generated by $A$. Let $\mathcal{R}^n_+$, $[a,b]$ and $\langle a,b\rangle$ be the non-negative orthant of $\mathcal{R}^n$, the closed line segment between $a,b \in \mathcal{R}^n$ and the inner product, respectively. The set

$$A^o := \{\vartheta \in \mathcal{R}^n : \langle \zeta, \vartheta \rangle \leq 0, \forall \zeta \in A\}$$

denotes the negative polar cone, which is a nonempty, closed, and convex cone. We recall the following property from [49].

Let $A_1$ and $A_2$ be two closed and convex cones in $\mathcal{R}^n$; then,

$$(A_1 \cap A_2)^o = cl(A_1^o + A_2^o). \tag{1}$$

The *cone of all feasible directions* and the *contingent cone* or *cone tangent* to $A$ at $\bar{\zeta} \in clA$ are expressed as follows [50]:

$$D(A,\bar{\zeta}) := \{d \in \mathcal{R}^n : \exists\delta > 0,\ \ \forall\lambda \in (0,\delta),\ \ \bar{\zeta} + \lambda d \in A\}$$

and

$$T(A,\bar{\zeta}) := \{d \in \mathcal{R}^n : \exists t_k \downarrow 0,\ \ \exists d_k \to d,\ \ \bar{\zeta} + t_k d_k \in A\},$$

The *cone normal* to $A$ at $\bar{\zeta} \in clA$ is defined by

$$N(A,\bar{\zeta}) := (T(A,\bar{\zeta}))^o.$$

The following proposition expresses the relation between a tangent cone and the cone of all feasible directions at a point of a locally star-shaped set. Recall that a set ($A \subseteq \mathcal{R}^n$) is *locally star-shaped* at $\bar{\zeta} \in A$ iff

$$\bar{\zeta} + \lambda(\zeta - \bar{\zeta}) \in A, \forall \zeta \in A.$$

**Proposition 1** ([51]). *If $A \subseteq \mathcal{R}^n$ is locally star-shaped at $\bar{\zeta} \in A$, then $T(A, \bar{\zeta}) = cl(D(A, \bar{\zeta}))$.*

The lower and upper Dini directional derivatives are defined as follows:

**Definition 1** ([20]). *Let $h\colon \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ be an extended real-valued function, and let $\zeta \in \mathcal{R}^n$ be such that $f(\zeta)$ is finite. The lower and upper Dini derivatives of $h$ at $\zeta$ in a direction $\vartheta \in \mathcal{R}^n$ are defined by*

$$h^-(\zeta, \vartheta) := \liminf_{t \downarrow 0} \frac{h(\zeta + t\vartheta) - h(\zeta)}{t}$$

*and*

$$h^+(\zeta, \vartheta) := \limsup_{t \downarrow 0} \frac{h(\zeta + t\vartheta) - h(\zeta)}{t},$$

*respectively.*

The concept of continuity directions is very important for the subsequent analysis.

**Definition 2** ([29]). *A vector $d \in \mathcal{R}^n$ is a continuity direction of $h\colon \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ at $\zeta \in \mathcal{R}^n$ iff, for all sequences, $\{t_k\} \subset ]0, +\infty[$ with $\{t_k\} \searrow 0$, and we have*

$$\lim_{k \to \infty} f(\zeta + t_k d) = f(\zeta).$$

*The set of all continuity directions of $h$ at $\zeta$ is denoted by $D_h(\zeta)$.*

The notion of directional convexificators is based on the notion of continuity directions.

**Definition 3** ([29]). *Let $D$ be a nonempty cone of $\mathcal{R}^n$. Let $D_h(\zeta)$ be the set of all continuity directions of $h$ at $\zeta$. The function $h\colon \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ admits*

*(a)* *A directional upper convexificator $\partial^*_D h(\zeta) \subset \mathcal{R}^n$ at $\zeta$ iff $D \subseteq D_h(\zeta)$ such that the set $(\partial^*_D h(\zeta))$ is closed, and for each $d \in D$, one has*

$$h^-(\zeta, d) \leq \sup_{\zeta^* \in \partial^*_D h(\zeta)} \langle \zeta^*, d \rangle;$$

*(b)* *A directional lower convexificator $\partial^*_D h(\zeta) \subset \mathcal{R}^n$ at $\zeta$ iff $D \subseteq D_h(\zeta)$ such that the set $(\partial^*_D h(\zeta))$ is closed, and for each $d \in D$, one has*

$$h^+(\zeta, d) \geq \inf_{\zeta^* \in \partial^*_D h(\zeta)} \langle \zeta^*, d \rangle;$$

*(c)* *A directional convexificator $\partial^*_D h(\zeta) \subset \mathcal{R}^n$ at $\zeta$ iff it is both an upper and lower directional convexificator of $h$ at $\zeta$;*

*(d)* *An upper regular directional convexificator of $h$ at $\zeta$ iff $D \subseteq D_h(\zeta)$ such that the set $(\partial^*_D h(\zeta))$ is closed, and for each $d \in D$, one has*

$$h^+(\zeta, d) = \sup_{\zeta^* \in \partial^*_D h(\zeta)} \langle \zeta^*, d \rangle;$$

*(e)* *A lower regular directional convexificator of $h$ at $\zeta$ iff $D \subseteq D_h(\zeta)$ such the set $\partial^*_D h(\zeta)$ is closed, and for each $d \in D$, one has*

$$h^-(\zeta, d) = \inf_{\zeta^* \in \partial^*_{\mathcal{D}} h(\zeta)} \langle \zeta^*, d \rangle.$$

**Remark 1.** *The notion of directional convexificators merges with the notion of convexificators introduced in [20] when $D = \mathcal{R}^n$.*

The following notion of convexity in terms of directional convexificators was introduced by Lafhim and Kalmoun [32].

**Definition 4** ([32])**.** *Assume $\mathfrak{f}_1, \ldots, \mathfrak{f}_q \colon \mathcal{R}^n \to \mathcal{R} \cup \{\infty\}$ are such that $\mathfrak{f}_1$ has an upper regular directional convexificator and $\mathfrak{f}_2, \ldots, \mathfrak{f}_q$ have a directional convexificator at $\bar{\zeta}$. The $q-$tuple$(\mathfrak{f}_1, \ldots, \mathfrak{f}_q)$ is called $\partial^*_D-$ convex at $\bar{\zeta}$ with respect to $D_{\mathfrak{f}_1}$ if for each $\zeta \in \mathcal{R}^n$, $\xi_i \in \partial^*_{D_{\mathfrak{f}_i}} \mathfrak{f}_i(\bar{\zeta})$, $i \in Q := \{1, 2, \ldots, q\}$ there exists $\vartheta \in [N_{D_{\mathfrak{f}_1}}(0_n)]^o$ such that*

$$\begin{aligned} &\mathfrak{f}_1(\zeta) - \mathfrak{f}_1(\bar{\zeta}) \geq \langle \xi_1, \vartheta \rangle, \\ &\mathfrak{f}_i(\zeta) - \mathfrak{f}_i(\bar{\zeta}) \geq \langle \xi_i, \vartheta \rangle, \ \ i \in Q \setminus \{1\}. \end{aligned}$$

**Lemma 1** ([32])**.** *Assume that $A_1 \subset \mathcal{R}^n$ is a proper closed, convex cone that contains $0_{\mathcal{R}^n}$ and $\emptyset \neq A_2 \subseteq \mathcal{R}^n$ is a bounded set such that*

$$\sup_{s \in A_2} \langle s, \vartheta \rangle \geq 0, \ \ \forall \vartheta \in A_1.$$

*Then, $0 \in \overline{co} A_2 + A_1^o$.*

## 3. Problem Formulation and Optimality Conditions

In this section, we derive optimality conditions for nonsmooth mathematical programs with vanishing constraints involving discontinuous functions using directional convexificators.

### 3.1. Problem Formulation

We consider the following mathematical program with vanishing constraints (MPVC):

$$\begin{aligned} &\min \ \ f(\zeta) \\ &\text{s.t.} \ \ g_i(\zeta) \leq 0, \ \ \forall i \in M := \{1, 2, \ldots, m\}, \\ &\qquad h_j(\zeta) = 0, \ \ \forall j \in I_h := \{1, 2, \ldots, p\}, \\ &\qquad H_k(\zeta) \geq 0, \ \ \forall k \in L := \{1, 2, \ldots, l\}, \\ &\qquad G_k(\zeta) H_k(\zeta) \leq 0, \ \ \forall k \in L, \end{aligned} \tag{2}$$

where $f : \mathcal{R}^n \to \mathcal{R}$, $g_i : \mathcal{R}^n \to \mathcal{R}$, $i \in M$, $h_j : \mathcal{R}^n \to \mathcal{R}$, $j \in I_h$ and $G_k, H_k : \mathcal{R}^n \to \mathcal{R}$, $k \in L$ are the given functions may or may not be continuously differentiable.

The set of all feasible points of the MPVC is expressed as

$$\begin{aligned} S := \{\zeta \in \mathcal{R}^n : g_i(\zeta) \leq 0, \ \forall i \in M, \ h_j(\zeta) = 0, \ \forall j \in I_h, \\ H_k(\zeta) \geq 0, \ \forall k \in L, \ G_k(\zeta) H_k(\zeta) \leq 0, \ \forall k \in L\}. \end{aligned}$$

For $\bar{\zeta} \in \mathcal{R}^n$, we define the following indices:

$$
\begin{aligned}
&I_g := \{i \in M : g_i(\bar{\zeta}) = 0\},\\
&I_+ := \{k \in L : H_k(\bar{\zeta}) > 0\},\\
&I_0 := \{k \in L : H_k(\bar{\zeta}) = 0\},\\
&I_{+0} := \{k \in L : H_k(\bar{\zeta}) > 0, G_k(\bar{\zeta}) = 0\},\\
&I_{+-} := \{k \in L : H_k(\bar{\zeta}) > 0, G_k(\bar{\zeta}) < 0\},\\
&I_{0+} := \{k \in L : H_k(\bar{\zeta}) = 0, G_k(\bar{\zeta}) > 0\},\\
&I_{0-} := \{k \in L : H_k(\bar{\zeta}) = 0, G_k(\bar{\zeta}) < 0\},\\
&I_{00} := \{k \in L : H_k(\bar{\zeta}) = 0, G_k(\bar{\zeta}) = 0\}.
\end{aligned}
$$

We denote the set of all continuity directions of $f, g_i, i \in I_g, hj(\text{and} - h_j), j \in I_h, G_k(\text{and} -G_k), H_k(\text{and} - H_k), k \in L$ at $\bar{\zeta}$ as $\mathcal{D}_f, D_{g_i}, i \in I_g, D_{h_j}, j \in I_h, D_{G_k}, D_{H_k}, k \in L$, respectively, and we let

$$
\mathcal{D}_g := \bigcap_{i \in I_g} D_{g_i}, \;\; \mathcal{D}_h := \bigcap_{j=1}^{p} D_{h_j},
$$

$$
\mathcal{D}_G := \bigcap_{k=1}^{l} D_{G_k}, \;\; \mathcal{D}_H := \bigcap_{k=1}^{l} D_{H_k}.
$$

Consider that all the functions have a directional upper convexificator at $\bar{\zeta} \in S$. Therefore, we have the following notations:

$$
\mathfrak{g} = \bigcup_{i \in I_g} co\partial^*_{D_{g_i}} g_i(\bar{\zeta}), \;\; \mathfrak{h} = \bigcup_{j=1}^{p} co\partial^*_{D_{h_j}} h_j(\bar{\zeta}) \cup co\partial^*_{D_{h_j}} (-h_j)(\bar{\zeta}),
$$

$$
\mathcal{G}_{I_{+0}} = \bigcup_{k \in I_{+0}} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}), \;\; \mathcal{H}_{I_{0+}} = \bigcup_{k \in I_{0+}} co\partial^*_{D_{H_k}} H_k(\bar{\zeta}) \cup co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}),
$$

$$
\mathcal{H}_{I_{0-}} = \bigcup_{k \in I_{0-}} co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}), \;\; \mathcal{H}_{I_{00}} = \bigcup_{k \in I_{00}} co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}),
$$

$$
(\mathcal{GH})_{I_{00}} = \bigcup_{k \in I_{00}} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}) \cup co\partial^*_{D_{H_k}} H_k(\bar{\zeta}),
$$

$$
\Gamma(\bar{\zeta}) = \mathfrak{g} \cup \mathfrak{h} \cup \mathcal{G}_{I_{+0}} \cup \mathcal{H}_{I_{0+}} \cup \mathcal{H}_{I_{0-}} \cup \mathcal{H}_{I_{00}},
$$

$$
\Pi(\bar{\zeta}) = \mathfrak{g} \cup \mathfrak{h} \cup \mathcal{G}_{I_{+0}} \cup \mathcal{H}_{I_{0+}} \cup \mathcal{H}_{I_{0-}} \cup \mathcal{H}_{I_{00}} \cup (\mathcal{GH})_{I_{00}}.
$$

Using the above notations, motivated by [32], we introduce Abadie-type constraint qualification (ACQ) for the MPVC using directional convexificators, which are very useful to derive further optimality conditions.

**Definition 5.** *Let $\bar{\zeta} \in S$ and let all the functions associated with the MPVC (2) have directional upper convexificators at $\bar{\zeta}$. Then,*

*(a) The generalized standard ACQ, denoted by $\partial^*_D$—GS ACQ, is satisfied at $\bar{\zeta}$ iff $\Gamma^o(\bar{\zeta}) \subseteq T(S, \bar{\zeta})$;*

*(b) The generalized MPVC ACQ, denoted by $\partial^*_D$—MPVC ACQ, is satisfied at $\bar{\zeta}$ iff $\Pi^o(\bar{\zeta}) \subseteq T(S, \bar{\zeta})$.*

**Remark 2.** *Since $\Gamma(\bar{\zeta}) \subseteq \Pi(\bar{\zeta})$, $\partial^*_D$—GS ACQ implies $\partial^*_D$—MPVC ACQ. If all the involved functions are continuous, then $\partial^*_D$—GS ACQ and $\partial^*_D$—MPVC ACQ coincide with ([16], Definition 3.1) and ([16], Definition 3.2), respectively. If all the functions are continuously differentiable, then $\partial^*_D-$GS ACQ and $\partial^*_D-$MPVC ACQ coincide with ([1], p. 77) and ([1], Definition 3), respectively.*

Now, we formulate an extended version of stationary points for MPVC in the context of directional convexificators by generalizing ([14], Definitions 6.1.12, 6.1.9 and 6.1.1) and ([16], Definitions 3.3, 3.5 and 3.6).

**Definition 6.** *A feasible point ($\bar{\zeta} \in S$) of the MPVC is called a*

*(a)* $\partial^*_D$*—generalized weak stationary point (GW-stationary point) iff there are vectors ($\lambda = (\lambda_g, \lambda_h, \lambda_H, \lambda_G) \in \mathcal{R}^{m+p+2l}$ and $\lambda_- = (\lambda_{-h}, \lambda_{-H}) \in \mathcal{R}^{p+l}$) such that the following conditions hold:*

$$0 \in co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \sum_{i \in I_g} \lambda_{g_i} co\partial^*_{D_{g_i}} g_i(\bar{\zeta}) +$$
$$\sum_{j=1}^{p} [\lambda_{h_j} co\partial^*_{D_{h_j}} h_j(\bar{\zeta}) + \lambda_{(-h_j)} co\partial^*_{D_{h_j}} (-h_j)(\bar{\zeta})] + \sum_{k=1}^{l} \lambda_{(-H_k)} co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}) \tag{3}$$
$$+ \sum_{k=1}^{l} [\lambda_{G_k} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}) + \lambda_{H_k} co\partial^*_{D_{H_k}} H_k(\bar{\zeta})] + N_{D_f}(0_{\mathcal{R}^n}),$$

$$\lambda_{g_i} \geq 0, i \in I_g, \quad \lambda_{h_j}, \lambda_{(-h_j)} \geq 0, j \in I_h, \quad \lambda_{(-H_k)}, \lambda_{G_k}, \lambda_{H_k} \geq 0, k \in L, \tag{4}$$

$$\lambda_{(-H)_{I_{+0} \cup I_{+-}}} = 0, \ \lambda_{H_{I_{+0} \cup I_{+-}}} = 0, \ \lambda_{G_{I_{0+} \cup I_{+-} \cup I_{0-}}} = 0, \ \lambda_{(-H_k)} - \lambda_{H_k} \geq 0, \ k \in I_{0-}; \tag{5}$$

*(b)* $\partial^*_D$*—generalized Mordukhovich stationary point (GM-stationary point) iff conditions (3)–(5) and the following conditions hold true:*

$$\forall k \in I_{00}, \quad \lambda_{G_k}(\lambda_{(-H_k)} - \lambda_{H_k}) = 0; \tag{6}$$

*(c)* $\partial^*_D$*—generalized strong stationary point (GS-stationary point) iff conditions (3)–(5) and the following conditions hold true:*

$$\forall k \in I_{00}, \quad \lambda_{G_k} = 0, \quad (\lambda_{(-H_k)} - \lambda_{H_k}) \geq 0. \tag{7}$$

**Remark 3.** *Since different stationary concepts for MPVC involving differentiable functions given in [14] are generalized to a nonsmooth case involving discontinuous functions in Definition 6, stationary points are called generalized stationary points. If all the involved functions are continuous, then these stationary concepts coincide with [16]. If all the functions are continuously differentiable, then these stationary concepts reduce to the stationary concepts defined in [14]. Directly from the definitions, we obtain following relationship between stationary points:*

$$\partial^*_D\text{—GS-stationary point} \implies \partial^*_D\text{—GM-stationary point} \implies \partial^*_D\text{—GW-stationary point}.$$

*3.2. Necessary Optimality Conditions*

Now, prove the following necessary optimality condition for a strong stationary point.

**Theorem 1.** *Let $\bar{\zeta} \in S$ be a local minimizer of (2). Assume that $f$ admits a bounded upper regular directional convexificator ($\partial^*_{\mathcal{D}_f} f(\bar{\zeta})$) at $\bar{\zeta}$ and that $g_i, i \in M$, $h_j, -h_j, j \in I_h$, $G_k, H_k, -H_k, k \in I_{00}$, $G_k, k \in I_{+0}$, $H_k, -H_k, k \in I_{0+}$ admit upper directional convexificators ($\partial_{D_{g_i}} g_i(\bar{\zeta})$, $i \in M$, $\partial_{D_{h_j}} h_j(\bar{\zeta}), \partial_{D_{h_j}}(-h_j)(\bar{\zeta}), j \in P, \partial_{D_{G_k}} G_k(\bar{\zeta}), \partial_{D_{H_k}} H_k(\bar{\zeta}),\ \partial_{D_{H_k}}(-H_k)(\bar{\zeta})$, $k \in I_{00}, \partial_{D_{G_k}} G_k(\bar{\zeta})$, $k \in I_{+0}, \partial_{D_{H_k}} H_k(\bar{\zeta}),\ \partial_{D_{H_k}}(-H_k)(\bar{\zeta}),\ k \in I_{0+}$) at $\bar{\zeta}$. Suppose that $S$ is locally star-shaped at $\bar{\zeta}$ and that the following assertions hold true:*

*(A1)* $\mathcal{D}_f$ *is closed and convex;*

*(A2) cone* $\Gamma(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n})$ *is closed;*

*(A3) cl* $(D(S, \bar{\zeta}) \cap \mathcal{D}_f) =$ *cl* $D(S, \bar{\zeta}) \cap \mathcal{D}_f$.

*If $\partial^*_D -$ GS ACQ holds at $\bar{\zeta}$, then $\bar{\zeta}$ is a $\partial^*_D$—GS stationary point.*

**Proof.** Suppose that $\bar{\zeta}$ is a local minimizer of the MPVC, that is,

$$f(\bar{\zeta}) \leq f(\zeta), \quad \forall \zeta \in U \cap S$$

for some neighborhood $(U)$ of $\bar{\zeta}$.

First, we show that

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0, \ \ \forall d \in D(S, \bar{\zeta}) \cap \mathcal{D}_f. \tag{8}$$

On the contrary, suppose that

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, \hat{d} \rangle < 0 \ \text{ for some } \ \hat{d} \in D(S, \bar{\zeta}) \cap \mathcal{D}_f.$$

Since $f$ has an upper regular directional convexificator $(\partial^*_{\mathcal{D}_f} f(\bar{\zeta}))$ at $\bar{\zeta}$,

$$f^+(\bar{\zeta}, \hat{d}) = \sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, \hat{d} \rangle < 0.$$

Consequently, there exists a sufficiently small $\hat{t}$ $(> 0)$ such that $\bar{\zeta} + \hat{t}d \in S$ and

$$f(\bar{\zeta} + \hat{t}\hat{d}) < f(\bar{\zeta}).$$

Since this contradicts the local optimality of $\bar{\zeta}$, (8) holds true. Hence, for any $d \in \text{cl}(D(S, \bar{\zeta}) \cap \mathcal{D}_f)$, one has

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0.$$

Now, from Proposition 1 and assumption $(A3)$, we obtain

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0, \ \ \forall d \in T(S, \bar{\zeta}) \cap \mathcal{D}_f.$$

Since $\partial^*_D-$GS ACQ holds at $\bar{\zeta}$,

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0, \ \ \forall d \in \Gamma^o(\bar{\zeta}) \cap \mathcal{D}_f.$$

Using Lemma 1, we have

$$0 \in \overline{co}\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + (\Gamma^o(\bar{\zeta}) \cap \mathcal{D}_f)^o.$$

According to property (1) and considering $\mathcal{D}^o_f = N_{\mathcal{D}_f}(0_{\mathcal{R}^n})$ ([29], Lemma 1) and $A^{oo} = \text{cl cone}A$ ([49], Proposition 2.3.3), we obtain

$$0 \in \overline{co}\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \text{cl}\Big[\text{cl cone } \Gamma(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n})\Big].$$

Using the closure property of subsets $A$ and $B$ in $\mathbb{R}^n$, $cl(clA + clB) = cl(A + clB) = cl(A + B)$ ([32], Theorem 3.4), we conclude

$$0 \in \text{cl}\Big(co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \Big[\text{cone } \Gamma(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n})\Big]\Big).$$

Since $f$ has a bounded upper regular directional convexificator at $\bar{\zeta}$, $\text{co}\partial^*_{\mathcal{D}_f} f(\bar{\zeta})$ is compact and assumption $(A2)$ holds,

$$0 \in co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \text{cone } \Gamma(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n}).$$

Thus, there exist non-negative multipliers ($\lambda_{g_i}, i \in I_g$, $\lambda_{h_j}, \lambda_{(-h_j)}, j \in I_h$, $\lambda_{H_k}, k \in I_{0+}$, $\lambda_{G_k}, k \in I_{+0}$, $\lambda_{(-H_k)}, k \in I_{0+} \cup I_{0-} \cup I_{00}$) such that

$$
\begin{aligned}
0 \in co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \sum_{i \in I_g} \lambda_{g_i} co\partial^*_{D_{g_i}} g_i(\bar{\zeta}) + \sum_{j=1}^{p} [\lambda_{h_j} co\partial^*_{D_{h_j}} h_j(\bar{\zeta}) + \lambda_{(-h_j)} co\partial^*_{D_{h_j}} (-h_j)(\bar{\zeta})] + \\
\sum_{k \in I_{0+} \cup I_{0-} \cup I_{00}} \lambda_{(-H_k)} co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}) + \sum_{k \in I_{+0}} \lambda_{G_k} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}) \\
+ \sum_{k \in I_{0+}} \lambda_{H_k} co\partial^*_{D_{H_k}} H_k(\bar{\zeta}) + N_{D_f}(0_{\mathcal{R}^n}).
\end{aligned}
\tag{9}
$$

We set $\lambda_{(-H)_{I_{+0} \cup I_{+-}}} = 0$, $\lambda_{H_{I_{00} \cup I_{0-} \cup I_{+-} \cup I_{+0}}} = 0$, and $\lambda_{G_{I_{+-} \cup I_{0+} \cup I_{0-} \cup I_{00}}} = 0$. Then, we find that $\bar{\zeta}$ is a $\partial^*_D-$ GS-stationary point. □

**Remark 4.** *We can see that if $f$ is continuous at $\bar{\zeta}$, that is, $\mathcal{D}_f = \mathcal{R}^n$, then assumptions (A1) and (A2) are trivially satisfied, and Theorem 1 reduces to ([16], Theorem 3.1). If all the functions are continuously differentiable, then Theorem 1 reduces to ([1], Theorem 1).*

We illustrate Theorem 1 using the following example.

**Example 1.** *Consider the following two-dimension MPVC problem:*

$$
\begin{aligned}
& min \;\; f(\zeta_1, \zeta_2) \\
& s.t. \;\; g(\zeta_1, \zeta_2) \leq 0,\; H(\zeta_1, \zeta_2) \geq 0, \\
& \qquad G(\zeta_1, \zeta_2) H(\zeta_1, \zeta_2) \leq 0,
\end{aligned}
$$

*where*

$$
f(\zeta_1, \zeta_2) = \begin{cases} \dfrac{-\zeta_2^2}{\zeta_1}, & \zeta_1 < 0,\;\; \zeta_2 \in \mathcal{R} \\ \zeta_1, & \zeta_1 \geq 0,\;\; \zeta_2 \geq 0 \\ +\infty, & \zeta_1 \geq 0,\;\; \zeta_2 < 0 \end{cases},
$$

$$
g(\zeta_1, \zeta_2) = \begin{cases} -|\zeta_2|, & \zeta_1 \geq 0,\;\; \zeta_2 \in \mathcal{R} \\ \sqrt{-\zeta_1} + \sqrt{-\zeta_2} + 1, & \zeta_1 < 0,\;\; \zeta_2 < 0 \\ \dfrac{1}{2}, & \zeta_1 < 0,\;\; \zeta_2 \geq 0 \end{cases},
$$

$$
H(\zeta_1, \zeta_2) = \begin{cases} -1, & \zeta_1 \geq 0,\;\; \zeta_2 < 0 \\ -\dfrac{1}{4}, & \zeta_1 < 0,\;\; \zeta_2 < 0 \\ \zeta_1, & \zeta_1 \in \mathcal{R},\;\; \zeta_2 \geq 0 \end{cases}
$$

*and*

$$
G(\zeta_1, \zeta_2) = \begin{cases} -\dfrac{1}{2}, & \zeta_1 < 0,\;\; \zeta_2 \geq 0 \\ -1, & \zeta_1 < 0,\;\; \zeta_2 < 0 \\ \zeta_2, & \zeta_1 \geq 0,\;\; \zeta_2 \in \mathcal{R}. \end{cases}
$$

*It is obvious that $(\bar{\zeta}_1, \bar{\zeta}_2) = (0,0)$ is a global optimum solution of the above problem, and we have*

$$
\begin{aligned}
\mathcal{D}_f(0,0) &= \{d \in \mathcal{R}^2 : d_1 \geq 0,\; d_2 \geq 0\}, \\
D_g(0,0) &= \{d \in \mathcal{R}^2 : d_1 \geq 0\}, \\
D_H(0,0) &= \{d \in \mathcal{R}^2 : d_2 \geq 0\}, \\
D_G(0,0) &= \{d \in \mathcal{R}^2 : d_1 \geq 0\}.
\end{aligned}
$$

*Now, we can derive the following directional convexificators at* $(0,0)$ :

$$\begin{cases} \partial^*_{\mathcal{D}_f} f(0,0) = \{(1,0)\}, \ \partial^*_{D_g} g(0,0) = \{(0,1),(0,-1)\}, \ \partial^*_{D_H} H(0,0) = \{(1,0)\}, \\ \partial^*_{D_H}(-H)(0,0) = \{(-1,0)\}, \ \partial^*_{D_G} G(0,0) = \{(0,1)\}, \end{cases} \tag{10}$$

*where* $\partial^*_{\mathcal{D}_f} f(0,0)$ *is upper regular. The feasible set of the problem is*

$$S = \{(\zeta_1,\zeta_2) : \zeta_1 \geq 0, \ \zeta_2 \geq 0, \ \zeta_1\zeta_2 = 0\},$$

*which is shown in Figure 1.*

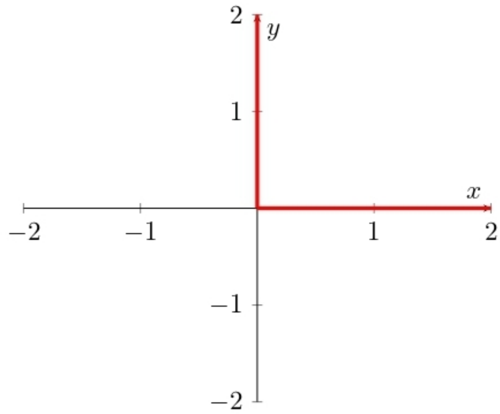


**Figure 1.** The red lines represent both the feasible region and the tangent cone of Example 1.

*The set*

$$\Gamma(0,0) = \{(0,1),(0,-1),(-1,0)\}, \ cone\ \Gamma(0,0) = \mathcal{R}_- \times \mathcal{R}.$$

*The negative polar cone of* $\Gamma(0,0)$ *is*

$$\Gamma^o(0,0) = \{\vartheta \in \mathcal{R}^2 : \vartheta_1 \geq 0, \ \ \vartheta_2 = 0\} = \mathcal{R}_+ \times \{0\}.$$

*We can see that*

$$T(S,(0,0)) = D(S,(0,0)) = S.$$

*Also,* $N_{\mathcal{D}_f}(0,0) = \mathcal{R}_- \times \mathcal{R}_-$. *Now, it is clear that* $\Gamma^o(0,0) \subseteq T(S,(0,0))$, $cl(D(S,(0,0)) \cap \mathcal{D}_f) = cl(D(S,(0,0))) \cap \mathcal{D}_f$ *and cone* $\Gamma(0,0) + N_{\mathcal{D}_f}$ *is closed. Then, there exist* $\lambda_g = \dfrac{1}{2}$, $\lambda_{(-H)} = 0$ *and* $\lambda_G = \lambda_H = 0$ *such that*

$$\begin{aligned} 0 \in co\partial^*_{\mathcal{D}_f} f(0,0) + &\lambda_g co\partial^*_{D_g} g(0,0) + \lambda_{(-H)} co\partial^*_{D_H}(-H)(0,0) + \lambda_G co\partial^*_{D_G} G(0,0) \\ &+ \lambda_H co\partial^*_{D_H} H(0,0) + N_{\mathcal{D}_f}(0,0), \end{aligned} \tag{11}$$

*as shown in Figure 2.*

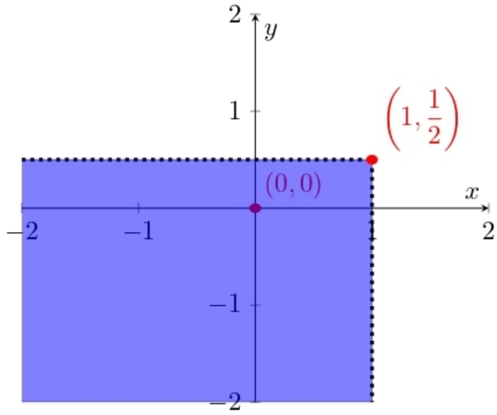


**Figure 2.** The set obtained by inclusion (11) containing the origin with $\lambda_g = \frac{1}{2}$ and $\lambda_{(-H)} = \lambda_G = \lambda_H = 0$.

Since $\partial^*_D$—ACQ may not always be satisfied at an optimal point, we derive the following result under a weaker condition of $\partial^*_D$—MPVC-ACQ to identify $\partial^*_D$—GM-stationary points.

**Theorem 2.** *Assume that all the conditions of Theorem 1 hold, except assertion (A2) is replaced by* $\Pi(\bar{\zeta}) + N_{\mathcal{D}_f}$ *and is closed. If* $\partial^*_D$*—MPVC-ACQ holds at* $\bar{\zeta}$*, then* $\bar{\zeta}$ *is a* $\partial^*_D$*—GM stationary point.*

**Proof.** The proof of the above theorem is the same as that of Theorem 1. Since $\bar{\zeta}$ is a local minimizer of MPVC,

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0, \;\; \forall d \in T(S, \bar{\zeta}) \cap \mathcal{D}_f.$$

On the other hand, $\partial^*_D$−MPVC ACQ at $\bar{\zeta}$ implies

$$\sup_{\zeta^* \in \partial^*_{\mathcal{D}_f} f(\bar{\zeta})} \langle \zeta^*, d \rangle \geq 0, \;\; \forall d \in \Pi^o(\bar{\zeta}) \cap \mathcal{D}_f.$$

Now, according to Lemma 1 and property (1),

$$0 \in \overline{co}\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \text{cl}\Big(\text{cl cone } \Pi(\bar{\zeta}) + \mathcal{D}^o_f\Big).$$

We know that $\mathcal{D}^o_f = N_{\mathcal{D}_f}(0_{\mathcal{R}^n})$, cone $\Pi(\bar{\zeta}) + N_{\mathcal{D}_f}$ is closed, and $\partial^*_{\mathcal{D}_f} f(\bar{\zeta})$ is compact; therefore, one has

$$0 \in co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + cone\ \Pi(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n}).$$

This implies that there exist non-negative multipliers ($\lambda_{g_i}$, $i \in I_g$, $\lambda_{h_j}, \lambda_{(-h_j)}$, $j \in P$, $\lambda_{H_k}$, $k \in I_{0+}$, $\lambda_{(-H_k)}$, $k \in I_{0+} \cup I_{0-} \cup I_{00}$, $\lambda_{G_k}$, $k \in I_{+0}$, $\lambda_{H_k}$, $k \in I_{00}$) such that

$$\begin{aligned} 0 \in\ & co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \sum_{i \in I_g} \lambda_{g_i} co\partial^*_{D_{g_i}} g_i(\bar{\zeta}) + \sum_{j=1}^{p} [\lambda_{h_j} co\partial^*_{D_{h_j}} h_j(\bar{\zeta}) + \lambda_{(-h_j)} co\partial^*_{D_{h_j}} (-h_j)(\bar{\zeta})] + \\ & \sum_{k \in I_{0+} \cup I_{0-} \cup I_{00}} \lambda_{(-H_k)} co\partial^*_{D_{H_k}} (-H_k)(\bar{\zeta}) + \sum_{k \in I_{0+} \cup I_{00}} \lambda_{H_k} co\partial^*_{D_{H_k}} H_k(\bar{\zeta}) \\ & + \sum_{k \in I_{+0}} \lambda_{G_k} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}) + N_{\mathcal{D}_f}(0_{\mathcal{R}^n}). \end{aligned} \tag{12}$$

Let $\lambda_{(-H)_{I_{+0}\cup I_{+-}}} = 0,\ \lambda_{H_{I_{0-}\cup I_{+-}\cup I_{+0}}} = 0$, and $\lambda_{G_{I_{+-}\cup I_{0+}\cup I_{0-}\cup I_{00}}} = 0$, according to which we obtain

$$
\begin{aligned}
&0 \in co\partial^*_{\mathcal{D}_f} f(\bar{\zeta}) + \sum_{i\in I_g} \lambda_{g_i} co\partial^*_{D_{g_i}} g_i(\bar{\zeta}) + \\
&\sum_{j=1}^{p}\left[\lambda_{h_j} co\partial^*_{D_{h_j}} h_j(\bar{\zeta}) + \lambda_{(-h_j)} co\partial^*_{D_{h_j}}(-h_j)(\bar{\zeta})\right] + \sum_{k=1}^{l}\lambda_{(-H_k)} co\partial^*_{D_{H_k}}(-H_k)(\bar{\zeta}) + \\
&\sum_{k=1}^{l}\left[\lambda_{G_k} co\partial^*_{D_{G_k}} G_k(\bar{\zeta}) + \lambda_{H_k} co\partial^*_{D_{H_k}} H_k(\bar{\zeta})\right] + N_{\mathcal{D}_f}(0_{\mathcal{R}^n}), \\
&\lambda_{g_i} \geq 0, i \in I_g,\ \lambda_{h_j}, \lambda_{(-h_j)} \geq 0,\ j \in I_h,\ \lambda_{(-H_k)}, \lambda_{G_k}, \lambda_{H_k} \geq 0, k \in L. \\
&\lambda_{(-H)_{I_{+0}\cup I_{+-}}} = 0,\ \lambda_{H_{I_{+0}\cup I_{+-}}} = 0,\ \lambda_{G_{I_{0+}\cup I_{+-}\cup I_{0-}}} = 0,\ \lambda_{-H_k} - \lambda_{H_k} \geq 0, k \in I_{0-}, \\
&\forall k \in I_{00}, \lambda_{G_k} = 0, \lambda_{G_k}(\lambda_{(-H_k)} - \lambda_{H_k}) = 0.
\end{aligned}
$$

Thus, $\bar{\zeta}$ is a $\partial^*_D$—GM stationary point. □

Next, we provide an example that illustrates the above theorem. We can see that $\partial^*_D$—MPVC-ACQ holds but not $\partial^*_D$—ACQ.

**Example 2.** *Consider the following two-dimensional MPVC problem:*

$$
\begin{aligned}
&min\ \ f(\zeta_1, \zeta_2) \\
&s.t.\ \ H(\zeta_1, \zeta_2) \geq 0,\ \ \ G(\zeta_1, \zeta_2)H(\zeta_1, \zeta_2) \leq 0,
\end{aligned}
$$

*where*

$$
f(\zeta_1, \zeta_2) = \begin{cases} \zeta_1 + \zeta_2, & \zeta_1 \geq 0,\ \zeta_2 \geq 0 \\ 2 + |\zeta_2|, & \zeta_1 \geq 0,\ \zeta_2 < 0 \\ \zeta_1^2 + 1, & \zeta_1 < 0,\ \zeta_2 \in \mathbb{R} \end{cases},
$$

$$
H(\zeta_1, \zeta_2) = \zeta_2,\ G(\zeta_1, \zeta_2) = \zeta_2 - |\zeta_1|.
$$

*It is obvious that $(\bar{\zeta}_1, \bar{\zeta}_2) = (0, 0)$ is the global optimum solution of the above problem; therefore, we have*

$$
\mathcal{D}_f(0, 0) = \{d \in \mathcal{R}^2 : d_1 \geq 0,\ d_2 \geq 0\}.
$$

*Now, we can propose the following directional convexificators at $(0, 0)$ :*

$$
\begin{cases} \partial^*_{\mathcal{D}_f} f(0, 0) = \{(1, 1)\},\ \partial^*_{D_H} H(0, 0) = \{(0, 1)\}, \\ \partial^*_{D_H}(-H)(0, 0) = \{(0, -1)\},\ \partial^*_{D_G} G(0, 0) = \{(-1, 1), (1, 1)\}, \end{cases} \tag{13}
$$

*where $\partial^*_{\mathcal{D}_f} f(0, 0)$ is upper regular. The feasible set of the problem is*

$$
S = \{(\zeta_1, \zeta_2) : \zeta_2 \geq 0,\ \zeta_2 - |\zeta_1| \leq 0\},
$$

*as shown in Figure 3.*

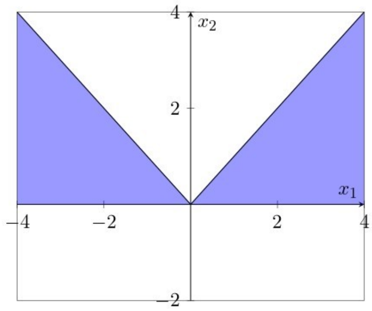


**Figure 3.** Blue represents both the feasible region and the tangent cone of Example 2.

$$\Pi(0,0) = \{(0,-1), ([-1,1] \times 1), (0,1)\} \text{ and } \Gamma(0,0) = \{(0,-1)\}.$$

*The negative polar cones of* $\Pi(0,0)$ *and* $\Gamma(0,0)$ *are*

$$\Pi^o(0,0) = \{(0,0)\}, \quad \text{and} \quad \Gamma^0(0,0) = \mathbb{R} \times \mathbb{R}_+,$$

*respectively. We can see that*

$$T(S,(0,0)) = D(S,(0,0)) = S.$$

*Now, it is clear that* $\Pi^o(0,0) \subseteq T(S,(0,0))$*, but* $\Gamma^o(0,0) \nsubseteq T(S,(0,0))$.

*Consequently,* $\partial_D^*-$*MPVC ACQ holds at* $(0,0)$*, but* $\partial_D^*-$*GS-ACQ does not hold at* $(0,0)$.

$$\textit{cone}\ \Pi(0,0) = \{(\zeta_1,\zeta_2) \in \mathbb{R}^2 : \zeta_2 \geq |\zeta_1|\} \cup \{(\zeta_1,\zeta_2) \in \mathbb{R}^2 : \zeta_1 = 0\}.$$

*Furthermore,* $N_{\mathcal{D}_f}(0,0) = \mathcal{R}_- \times \mathcal{R}_-$*,* $cl(D(S,(0,0)) \cap \mathcal{D}_f) = cl(D(S,(0,0))) \cap \mathcal{D}_f$*, and cone* $\Gamma(0,0) + N_{\mathcal{D}_f}$ *is closed. Then, there exist* $\lambda_{(-H)} = \lambda_G = \lambda_H = 1$ *such that*

$$\begin{aligned} 0 \in co\partial^*_{\mathcal{D}_f} f(0,0) + &\lambda_{(-H)} co\partial^*_{D_H}(-H)(0,0) + \lambda_G co\partial^*_{D_G} G(0,0) \\ &+ \lambda_H co\partial^*_{D_H} H(0,0) + N_{\mathcal{D}_f}(0,0), \end{aligned} \tag{14}$$

*as shown in the Figure 4.*

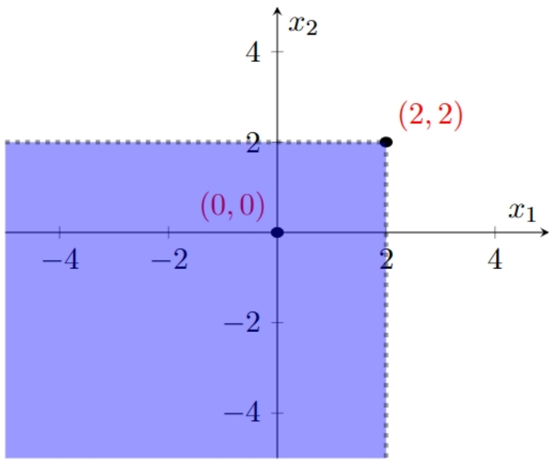


**Figure 4.** The set obtained by inclusion (14) containing the origin with $\lambda_{(-H)} = \lambda_G = \lambda_H = 1$.

### 3.3. Sufficient Optimality Condition

Now, we establish the following sufficient optimality conditions using the concept of generalized convexity. We define the following index sets:

$$\begin{cases} I_g^+(\zeta) := \{i \in M : \lambda_{g_i}^+ > 0\}, & \\ I_h^+ := \{j \in I_h : \lambda_{h_j}^+ > 0\}, & I_h^- := \{j \in I_h : \lambda_{(-h_j)}^- < 0\}, \\ \tilde{I}_+^+(\zeta) := \{k \in I_+(\zeta) : \lambda_{(-H_k)}^+ > 0\}, & \tilde{I}_0^+(\zeta) := \{k \in I_0(\zeta) : \lambda_{(-H_k)}^+ > 0\}, \\ \tilde{I}_0^-(\zeta) := \{k \in I_0(\zeta) : \lambda_{H_k}^- < 0\}, & \tilde{I}_{00}^+(\zeta) := \{k \in I_{00}(\zeta) : \lambda_{H_k}^+ > 0\}, \\ \tilde{I}_{0-}^+(\zeta) := \{k \in I_{0-}(\zeta) : \lambda_{H_k}^+ > 0\}, & \tilde{I}_{0+}^+(\zeta) := \{k \in I_{0+}(\zeta) : \lambda_{H_k}^+ > 0\}, \\ I_{+0}^+(\zeta) := \{k \in I_{+0}(\zeta) : \lambda_{G_k}^+ > 0\}, & I_{+0}^-(\zeta) := \{k \in I_{+0}(\zeta) : \lambda_{(-G_k)}^- < 0\}, \\ I_{+0}^-(\zeta) := \{k \in I_{+0}(\zeta) : \lambda_{(-G_k)}^- < 0\}, & I_{+-}^+(\zeta) := \{k \in I_{+-}(\zeta) : \lambda_{G_k}^+ > 0\}, \\ I_{0+}^-(\zeta) := \{k \in I_{0+}(\zeta) : \lambda_{(-G_k)}^- < 0\}, & I_{0-}^+(\zeta) := \{k \in I_{0-}(\zeta) : \lambda_{G_k}^+ > 0\}, \\ I_{00}^-(\zeta) := \{k \in I_{00}(\zeta) : \lambda_{(-G_k)}^- < 0\}, & I_{00}^+(\zeta) := \{k \in I_{00}(\zeta) : \lambda_{G_k}^+ > 0\}. \end{cases} \tag{15}$$

**Theorem 3.** *Let $\bar{\zeta}$ be a $\partial_D^*$—GW-stationary point of (2) and suppose that $\left(f, (g_i)_{i\in I_g}, (\pm h_j)_{(j\in I_h)},\right.$ $\left.(-H_k)_{k\in I_{0+}\cup I_{0-}\cup I_{00}}\right)$ is $\partial_D^*$—convex at $\bar{\zeta}$ with respect to $\mathcal{D}_f$. If $I_{00}^+ \cup \tilde{I}_{00}^+ \cup \tilde{I}_{0+}^+ \cup \tilde{I}_{0-}^+ \cup I_{+0}^+ = \emptyset$, then $\bar{\zeta}$ is a global optimal solution of the MPVC.*

**Proof.** On the contrary, assume that $\bar{\zeta}$ is not a global optimal solution of the MPVC. Thus, there exists a point ($\tilde{\zeta} \in S$) such that

$$f(\bar{\zeta}) > f(\tilde{\zeta}).$$

Since $\left(f, (g_i)_{i\in I_g}, (\pm h_j)_{(j\in I_h)}, (-H_k)_{k\in I_{0+}\cup I_{0-}\cup I_{00}}\right)$ is $\partial_D^*-$ convex at $\bar{\zeta}$ with respect to $\mathcal{D}_f$, for all $\zeta_f^* \in \mathrm{co}\partial_{\mathcal{D}_f}^* f(\bar{\zeta})$, $\zeta_{g_i}^* \in \mathrm{co}\partial_{D_{g_i}}^* g_i(\bar{\zeta})$, $\zeta_{h_j}^* \in \mathrm{co}\partial_{D_{h_j}}^* h_j(\bar{\zeta})$, $\zeta_{(-h_j)}^* \in \mathrm{co}\partial_{D_{h_j}}^*(-h_j)(\bar{\zeta})$, $\zeta_{(-H_k)}^* \in \mathrm{co}\partial_{D_{H_k}}^*(-H_k)(\bar{\zeta})$, there exists $\vartheta \in [N_{\mathcal{D}_f}(0_{\mathcal{R}^n})]^o$ such that

$$\langle \zeta_f^*, \vartheta\rangle \le f(\tilde{\zeta}) - f(\bar{\zeta}), \tag{16}$$

$$\langle \zeta_{g_i}^*, \vartheta\rangle \le g_i(\tilde{\zeta}) - g_i(\bar{\zeta}), \ \forall i \in I_g, \tag{17}$$

$$\langle \zeta_{h_j}^*, \vartheta\rangle \le h_j(\tilde{\zeta}) - h_j(\bar{\zeta}), \ \forall j \in P, \tag{18}$$

$$\langle \zeta_{(-h_j)}^*, \vartheta\rangle \le -h_j(\tilde{\zeta}) + h_j(\bar{\zeta}), \ \forall j \in P, \tag{19}$$

$$\langle \zeta_{(-H_k)}^*, \vartheta\rangle \le -H_k(\tilde{\zeta}) + H_k(\bar{\zeta}), \ \forall k \in I_{0+} \cup I_{0-} \cup I_{00}. \tag{20}$$

By multiplying both sides of the inequalities (17)–(20) by positive scalars $\lambda_{g_i}, i \in I_g$, $\lambda_{h_j}, \lambda_{(-h_j)}, j \in I_h$ and $\lambda_{(-H_k)}, k \in I_{0+} \cup I_{0-} \cup I_{00}$ and adding (16) to (20), we obtain

$$\begin{aligned} \left\langle \zeta_f^* + \sum_{i\in I_g} \lambda_{g_i}\zeta_{g_i}^* + \sum_{j=1}^{p}\left[\lambda_{h_j}\zeta_{h_j}^* + \lambda_{(-h_j)}\zeta_{(-h_j)}^*\right] + \sum_{I_{0+}\cup I_{0-}\cup I_{00}} \lambda_{(-H_k)}\zeta_{(-H_k)}^*, \vartheta \right\rangle \le f(\tilde{\zeta}) - f(\bar{\zeta}) \\ + \sum_{i\in I_g} \lambda_{g_i}(g_i(\tilde{\zeta}) - g_i(\bar{\zeta})) + \sum_{j=1}^{p}\Big(\lambda_{h_j}(h_j(\tilde{\zeta}) - h_j(\bar{\zeta})) + \lambda_{(-h_j)}(-h_j(\tilde{\zeta}) + h_j(\bar{\zeta}))\Big) \\ + \sum_{I_{0+}\cup I_{0-}\cup I_{00}} \lambda_{(-H_k)}(-H_k(\tilde{\zeta}) + H_k(\bar{\zeta})). \end{aligned} \tag{21}$$

Since $I_{00}^{+} \cup \tilde{I}_{00}^{+} \cup \tilde{I}_{0+}^{+} \cup \tilde{I}_{0-}^{+} \cup I_{+0}^{+} = \emptyset$, taking into account the $\partial_D^*$—GW-stationarity of $\bar{\zeta}$, one obtains

$$-\left(\zeta_f^* + \sum_{i \in I_g} \lambda_{g_i}\zeta_{g_i}^* + \sum_{j=1}^{p}\left[\lambda_{h_j}\zeta_{h_j}^* + \lambda_{(-h_j)}\zeta_{(-h_j)}^*\right] + \sum_{k=1}^{l}\lambda_{(-H_k)}\zeta_{(-H_k)}^* + \sum_{k=1}^{l}\left[\lambda_{G_k}\zeta_{G_k}^* + \lambda_{H_k}\zeta_{H_k}^*\right]\right) \in N_{D_f}(0_{\mathcal{R}^n}), \ \ \forall \zeta_{G_k}^* \in \partial_{D_{G_k}}^* G_k(\bar{\zeta}), \ \zeta_{H_k}^* \in \partial_{D_{H_k}}^* H_k(\bar{\zeta}).$$

Using the definition of polar cone, it follows from $\vartheta \in [N_{\mathcal{D}_f}(0_{\mathcal{R}^n})]^o$ that

$$\left\langle \zeta_f^* + \sum_{i \in I_g} \lambda_{g_i}\zeta_{g_i}^* + \sum_{j=1}^{p}\left[\lambda_{h_j}\zeta_{h_j}^* + \lambda_{(-h_j)}\zeta_{(-h_j)}^*\right] + \sum_{I_{0+}\cup I_{0-}\cup I_{00}} \lambda_{(-H_k)}\zeta_{(-H_k)}^*, \vartheta \right\rangle \geq 0.$$

Thus, according to (21),

$$0 \leq f(\tilde{\zeta}) - f(\bar{\zeta}) + \sum_{i \in I_g} \lambda_{g_i}(g_i(\tilde{\zeta}) - g_i(\bar{\zeta})) + \sum_{j=1}^{p}\Big(\lambda_{h_j}(h_j(\tilde{\zeta}) - h_j(\bar{\zeta})) + \lambda_{(-h_j)}(-h_j(\tilde{\zeta}) + h_j(\bar{\zeta}))\Big) + \sum_{I_{0+}\cup I_{0-}\cup I_{00}} \lambda_{-H_k}(-H_k(\tilde{\zeta}) + H_k(\bar{\zeta}))$$

Since $\bar{\zeta}, \tilde{\zeta} \in S$,

$$f(\bar{\zeta}) \leq f(\tilde{\zeta}).$$

This contradicts our assumption; hence, $\bar{\zeta}$ is a global optimal solution. □

## 4. Duality

In this section, we formulate a Wolfe-type dual model for MPVC (2) using directional convexificators. Now, we present the following Wolfe-type dual (WD) to MPVC (2) depending on a feasible point ($\zeta \in S$) denoted by VC-WD($\zeta$):

$$\partial_D^* - VC - WD(\zeta) \quad \max_{(u,\lambda_g,\lambda_h,\lambda_H,\lambda_G)} \Bigg\{ f(u) + \sum_{i \in I_g^+} \lambda_{g_i}^+ g_i(u) + \sum_{j \in I_h^+} \lambda_{h_j}^+ h_j(u) + \sum_{j \in I_h^-} \lambda_{(-h_j)}^- h_j(u) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \lambda_{(-H_k)}^+ H_k(u) - \sum_{k \in \tilde{I}_0^-} \lambda_{H_k}^- H_k(u) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \lambda_{G_k}^+ G_k(u) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \lambda_{(-G_k)}^- G_k(u) \Bigg\}$$

subject to :

$$0 \in co\partial_{D_f}^* f(u) + \sum_{i \in I_g^+} \lambda_{g_i}^+ co\partial_{D_{g_i}}^* g_i(u) + \sum_{j \in I_h^+} \lambda_{h_j}^+ co\partial_{D_{h_j}}^* h_j(u) - \sum_{j \in I_h^-} \lambda_{(-h_j)}^- co\partial_{D_{h_j}}^* (-h_j)(u) + \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \lambda_{(-H_k)}^+ co\partial_{D_{H_k}}^* (-H_k)(u) - \sum_{k \in \tilde{I}_0^-} \lambda_{H_k}^- co\partial_{D_{H_k}}^* H_k(u) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \lambda_{G_k}^+ co\partial_{D_{G_k}}^* G_k(u) - \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \lambda_{(-G_k)}^- co\partial_{D_{G_k}}^* (-G_k)(u) + N_{D_f}(0_{\mathcal{R}^n}), \quad (22)$$

where the indices are defined by the expression (15), and $u := (u_1, \ldots, u_n) \in \mathbb{R}^n, \lambda_g := (\lambda_{g_1}, \ldots, \lambda_{g_m}) \in \mathbb{R}^m, \lambda_h := (\lambda_{h_1}, \ldots, \lambda_{h_p}) \in \mathbb{R}^p, \lambda_H := (\lambda_{H_1}, \ldots, \lambda_{H_l}) \in \mathbb{R}^l, \lambda_G := (\lambda_{G_1}, \ldots, \lambda_{G_l}) \in \mathbb{R}^l$ are expressed as

$$\lambda_{g_i} := \begin{cases} \lambda_{g_i}^+, & i \in I_g^+, \\ 0, & \text{otherwise}; \end{cases}$$

$$\lambda_{h_j} := \begin{cases} \lambda^+_{h_j}, & j \in I^+_h, \\ \lambda^-_{(-h_j)}, & j \in I^-_h, \\ 0, & \text{otherwise}; \end{cases}$$

$$\lambda_{H_k} := \begin{cases} \lambda^+_{(-H_k)}, & k \in \tilde{I}^+_+ \cup \tilde{I}^+_0, \\ \lambda^-_{H_k}, & k \in \tilde{I}^-_0, \\ 0, & \text{otherwise}; \end{cases}$$

and

$$\lambda_{G_k} := \begin{cases} \lambda^-_{(-G_k)}, & k \in I^-_{0+} \cup I^-_{00} \cup I^-_{+0}, \\ \lambda^+_{G_k}, & k \in I^+_{00} \cup I^+_{+0} \cup I^+_{0-} \cup I^+_{+-}, \\ 0, & \text{otherwise}. \end{cases}$$

The set of all feasible points of $\partial^*_D-$VC-WD$(\zeta)$ is denoted by $S_W(\zeta)$. The projection of set $S_W(\zeta)$ on $\mathbb{R}^n$ is denoted by ${}_{pr\mathbb{R}^n}S_W(\zeta)$ and is defined as follows:

$${}_{pr\mathbb{R}^n}S_W(\zeta) := \{u \in \mathbb{R}^n : (u, \lambda_g, \lambda_h, \lambda_H, \lambda_G) \in S_W(\zeta)\}.$$

Now, we consider another duality problem that is independent of the primal (2), denoted by $\partial^*_D$—VC-WD, which is defined as follows:

$$\partial^*_D - VC - WD \quad \max_{(u,\lambda_g,\lambda_h,\lambda_H,\lambda_G)} \Bigg\{ f(u) + \sum_{i \in I^+_g} \lambda^+_{g_i} g_i(u) + \sum_{j \in I^+_h} \lambda^+_{h_j} h_j(u) + \sum_{j \in I^-_h} \lambda^-_{(-h_j)} h_j(u)$$
$$- \sum_{k \in \tilde{I}^+_+ \cup \tilde{I}^+_0} \lambda^+_{(-H_k)} H_k(u) - \sum_{k \in \tilde{I}^-_0} \lambda^-_{H_k} H_k(u) +$$
$$\sum_{k \in I^+_{00} \cup I^+_{+0} \cup I^+_{0-} \cup I^+_{+-}} \lambda^+_{G_k} G_k(u) + \sum_{k \in I^-_{0+} \cup I^-_{00} \cup I^-_{+0}} \lambda^-_{(-G_k)} G_k(u) \Bigg\}$$
$$\text{subject to}: \quad (u, \lambda_g, \lambda_h, \lambda_H, \lambda_G) \in \cap_{\zeta \in S} S_W(\zeta). \tag{23}$$

The set of all feasible points of $\partial^*_D$—VC-WD is denoted by $S_W := \cap_{\zeta \in S} S_W(\zeta)$.

**Remark 5.** *If all the involved functions are continuously differentiable, then the above dual models coincide with the models defined in [12] (Section 3). If all the continuity directions are $\mathbb{R}^n$,, then we have a dual model in terms of convexificators. In addition, if all the the functions are locally Lipschitz, then we have a dual model in terms of Clarke subdifferentials, as studied in [36].*

**Example 3.** *Consider the primal MPVC of Example 1. The directional convexificators of the involved functions at $\bar{\zeta} := (0, 0)$ are given in (17). For $\bar{\zeta} := (0, 0)$, the dual model is expressed as*

$$\partial^*_D - VC - WD(\bar{\zeta}) \quad \max_{(u,\lambda_g,\lambda_H,\lambda_G)} \Big\{ f(u) + \lambda^+_{g_1} g_1(u) - \lambda^+_{(-H_1)} H_1(u) - \lambda^-_{H_1} H_1(u) +$$
$$\lambda^+_{G_1} G_1(u) + \lambda^-_{(-G_1)} G_1(u) \Big\}$$

*subject to* :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda^+_{g_1} \begin{pmatrix} 0 \\ \bar{\zeta}^* \end{pmatrix} + \lambda^+_{(-H_1)} \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \lambda^-_{H_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda^+_{G_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \lambda^-_{(-G_1)} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + N_{D_f}(0_{\mathcal{R}^2}),$$

*and*

$$\lambda^+_{g_1} > 0, \ \text{if } 1 \in I^+_g(\bar{\zeta}), \ \lambda^+_{(-H_1)} > 0, \ \text{if } 1 \in \tilde{I}^+_0(\bar{\zeta}),$$
$$\lambda^-_{H_1} < 0, \ \text{if } 1 \in \tilde{I}^-_0(\bar{\zeta}), \ \lambda^+_{G_1} > 0, \ \text{if } 1 \in I^+_{00}(\bar{\zeta}), \ \lambda^-_{(-G_1)} < 0, \ \text{if } 1 \in I^-_{00}(\bar{\zeta}),$$

*where $\bar{\zeta}^* \in [-1, 1]$ and $N_{D_f}(0_{\mathcal{R}^2}) = \mathcal{R}_- \times \mathcal{R}_-$.*

**Example 4.** *Consider the primal MPVC of Example 2. The directional convexificators of the involved functions at* $\bar{\zeta} := (0,0)$ *are given in (13). For* $\bar{\zeta} := (0,0)$*, the dual model is expressed as*

$$\partial^*_D - VC - WD(\bar{\zeta}) \quad \max_{(u,\lambda_H,\lambda_G)} \Big\{ f(u) - \lambda^+_{(-H_1)} H_1(u) - \lambda^-_{H_1} H_1(u) + \lambda^+_{G_1} G_1(u) + \lambda^-_{(-G_1)} G_1(u) \Big\}$$

*subject to*

$$\begin{pmatrix}0\\0\end{pmatrix} \in \begin{pmatrix}1\\1\end{pmatrix} + \lambda^+_{(-H_1)}\begin{pmatrix}0\\-1\end{pmatrix} - \lambda^-_{H_1}\begin{pmatrix}0\\1\end{pmatrix} + \lambda^+_{G_1}\begin{pmatrix}\bar{\zeta}^*\\1\end{pmatrix} - \lambda^-_{(-G_1)}\begin{pmatrix}\bar{\zeta}^*\\-1\end{pmatrix} + N_{D_f}(0_{\mathcal{R}^2}),$$

*and*

$$\lambda^+_{(-H_1)} > 0, \text{ if } 1 \in \tilde{I}^+_0(\bar{\zeta}), \ \lambda^-_{H_1} < 0, \text{ if } 1 \in \tilde{I}^-_0(\bar{\zeta}),$$
$$\lambda^+_{G_1} > 0, \text{ if } 1 \in I^+_{00}(\bar{\zeta}), \ \lambda^-_{(-G_1)} < 0, \text{ if } 1 \in I^-_{00}(\bar{\zeta}),$$

*where* $\bar{\zeta}^* \in [-1,1]$ *and* $N_{D_f}(0_{\mathcal{R}^2}) = \mathcal{R}_- \times \mathcal{R}_-$.

The following theorem is a weak duality theorem that expresses the relationship between a feasible point of the primal problem and a feasible point of the corresponding Wolfe-type dual problem.

**Theorem 4** (Weak duality theorem). *Let* $\zeta \in S$ *and* $(u, \lambda_g, \lambda_h, \lambda_H, \lambda_G) \in S_W$*. Suppose that* $f$ *admits a bounded upper regular directional convexificator and the constraint functions admit upper directional convexificators at* $u$*. If* $(f, g_i(i \in I^+_g), h_j(j \in I^+_h(\zeta)), -h_j(j \in I^-_h(\zeta)), -H_k(k \in \tilde{I}^+_+(\zeta) \cup \tilde{I}^+_0(\zeta)), H_k(k \in \tilde{I}^-_0(\zeta)), -G_k(k \in I^-_{0+}(\zeta) \cup I^-_{00}(\zeta) \cup I^-_{+0}(\zeta)), G_k(k \in I^+_{00}(\zeta) \cup I^+_{0-}(\zeta) \cup I^+_{+0}(\zeta) \cup I^+_{+-}(\zeta))$ *is* $\partial^*_D$*— convex at* $u \in S \cup_{pre\mathbb{R}^n} S_W$ *with respect to* $D_f$*, then*

$$\begin{aligned} f(\zeta) \geq & f(u) + \sum_{i \in I^+_{g_i}} \lambda^g_i g_i(u) + \sum_{j \in I^+_h} \lambda^+_{h_j} h_j(u) + \sum_{j \in I^-_h} \lambda^-_{(-h_j)} h_j(u) - \sum_{k \in \tilde{I}^+_+ \cup \tilde{I}^+_0} \lambda^+_{(-H_k)} H_k(u) \\ & - \sum_{k \in \tilde{I}^-_0} \lambda^-_{H_k} H_k(u) + \sum_{k \in I^-_{0+} \cup I^-_{00} \cup I^-_{+0}} \lambda^-_{(-G_k)} G_k(u) + \sum_{k \in I^+_{00} \cup I^+_{+0} \cup I^+_{0-} \cup I^+_{+-}} \lambda^+_{G_k} G_k(u). \end{aligned} \tag{24}$$

**Proof.** According to the feasibility of MPVC (2) and VC-WD and the $\partial^*_D-$ convexity of $(f, g_i(i \in I^+_g(\zeta)), h_j(j \in I^+_h(\zeta)), -h_j(j \in I^-_h) - H_k(k \in \tilde{I}^+_+(\zeta) \cup \tilde{I}^+_0(\zeta)), H_k(k \in \tilde{I}^-_0(\zeta)), -G_k$ $(k \in I^-_{0+}(\zeta) \cup I^-_{00}(\zeta) \cup I^-_{+0}(\zeta)), G_k(k \in I^+_{00}(\zeta) \cup I^+_{0-}(\zeta) \cup I^+_{+0}(\zeta) \cup I^+_{+-}(\zeta))$ at $u \in S \cup_{pre\mathbb{R}^n} S_W$ with respect to $D_f$, then for all $u^*_f \in \text{co}\partial^*_{\mathcal{D}_f} f(u)$, $u^*_{g_i} \in \text{co}\partial^*_{D_{g_i}} g_i(u), i \in I^+_g(\zeta)$, $u^*_{h_j} \in \text{co}\partial^*_{D_{h_j}} h_j(u), j \in I^+_h$, $u^*_{(-h_j)} \in \text{co}\partial^*_{D_{h_j}}(-h_j)(u), j \in I^-_h$, $u^*_{(-H_k)} \in \text{co}\partial^*_{D_{H_k}}(-H_k)(u)$, $k \in I^+_+(\zeta) \cup I^+_0(\zeta)$, $u^*_{H_k} \in \text{co}\partial^*_{D_{H_k}}(H_k)(u)$, $k \in I^-_0(\zeta), u^*_{(-G_k)} \in \text{co}\partial^*_{D_{G_k}}(-G_k)(u)$, $k \in I^-_{0+}(\zeta) \cup I^-_{00}(\zeta), u^*_{G_k} \in \text{co}\partial^*_{D_{G_k}}(G_k)(u)$, $k \in I^+_{00}(\zeta) \cup I^+_{0-}(\zeta) \cup I^+_{+0}(\zeta) \cup I^+_{+-}(\zeta)$, there exists $\vartheta \in [N_{\mathcal{D}_f}(0_{\mathcal{R}^n})]^o$ such that

$$\begin{array}{lll}
f(u)+\langle u_f^*,\vartheta\rangle \le f(\zeta), & & \\
g_i(u)+\langle u_{g_i}^*,\vartheta\rangle \le g_i(\zeta)\le 0, & \lambda_{g_i}>0, & \forall i\in I_g^+(\zeta),\\
h_j(u)+\langle u_{h_j}^*,\vartheta\rangle \le h_j(\zeta)=0, & \lambda_{h_j}^+>0, & \forall j\in I_h^+,\\
h_j(u)-\langle u_{(-h_j)}^*,\vartheta\rangle \ge h_j(\zeta)=0, & \lambda_{(-h_j)}^-<0, & \forall j\in I_h^-,\\
-H_k(u)+\langle u_{(-H_k)}^*,\vartheta\rangle \le -H_k(\zeta)<0, & \lambda_{(-H_k)}^+>0, & \forall k\in \tilde{I}_+^+(\zeta),\\
-H_k(u)+\langle u_{(-H_k)}^*,\vartheta\rangle \le -H_k(\zeta)=0, & \lambda_{(-H_k)}^+>0, & \forall k\in \tilde{I}_0^+(\zeta),\\
-H_k(u)-\langle u_{H_k}^*,\vartheta\rangle \ge -H_k(\zeta)=0, & \lambda_{H_k}^-<0, & \forall k\in \tilde{I}_0^-(\zeta),\\
G_k(u)-\langle u_{(-G_k)}^*,\vartheta\rangle \ge G_k(\zeta)>0, & \lambda_{(-G_k)}^-<0, & \forall k\in I_{0+}^-(\zeta),\\
G_k(u)-\langle u_{(-G_k)}^*,\vartheta\rangle \ge G_k(\zeta)=0, & \lambda_{(-G_k)}^-<0, & \forall k\in I_{00}^-(\zeta)\cup I_{+0}^-,\\
G_k(u)+\langle u_{G_k}^*,\vartheta\rangle \le G_k(\zeta)=0, & \lambda_{G_k}^+>0, & \forall k\in I_{00}^+(\zeta)\cup I_{+0}^+(\zeta),\\
G_k(u)+\langle u_{G_k}^*,\vartheta\rangle \le G_k(\zeta)<0, & \lambda_{G_k}^+>0, & \forall k\in I_{0-}^+(\zeta)\cup I_{+-}^+(\zeta),
\end{array}$$

which implies that

$$\begin{aligned}
f(u)+&\sum_{i\in I_g^+}\lambda_{g_i}^+g_i(u)+\sum_{j\in I_h^+}\lambda_{h_j}^+h_j(u)+\sum_{j\in I_h^-}\lambda_{(-h_j)}^-h_j(u)-\sum_{k\in\tilde{I}_+^+\cup\tilde{I}_0^+}\lambda_{(-H_k)}^+H_k(u)\\
&-\sum_{k\in\tilde{I}_0^-}\lambda_{H_k}^-H_k(u)+\sum_{k\in I_{0+}^-\cup I_{00}^-\cup I_{+0}^-}\lambda_{(-G_k)}^-G_k(u)+\sum_{k\in I_{00}^+\cup I_{+0}^+\cup I_{0-}^+\cup I_{+-}^+}\lambda_{G_k}^+G_k(u)+\\
&\Big\langle u_f^*+\sum_{i\in I_g^+}\lambda_{g_i}u_{g_i}^*+\sum_{j\in I_h^+}\lambda_{h_j}^+u_{h_j}^*-\sum_{j\in I_h^-}\lambda_{(-h_j)}^-u_{(-h_j)}^*+\sum_{k\in\tilde{I}_+^+\cup\tilde{I}_0^+}\lambda_{(-H_k)}^+u_{(-H_k)}^*\\
&-\sum_{k\in\tilde{I}_0^-}\lambda_{H_k}^-u_{H_k}^*-\sum_{k\in I_{0+}^-\cup I_{00}^-\cup I_{+0}^-}\lambda_{(-G_k)}^-u_{(-G_k)}^*+\sum_{k\in I_{00}^+\cup I_{+0}^+\cup I_{0-}^+\cup I_{+-}^+}\lambda_{G_k}^+u_{G_k}^*,\vartheta\Big\rangle\le f(\zeta).
\end{aligned}\tag{25}$$

By using duality constraint (22) of the VC-WD, we have

$$\begin{aligned}
-\Bigg(&u_f^*+\sum_{i\in I_g^+}\lambda_{g_i}u_{g_i}^*+\sum_{j\in I_h^+}\lambda_{h_j}^+u_{h_j}^*-\sum_{j\in I_h^-}\lambda_{(-h_j)}^-u_{(-h_j)}^*+\sum_{k\in\tilde{I}_+^+\cup\tilde{I}_0^+}\lambda_{(-H_k)}^+u_{(-H_k)}^*\\
&-\sum_{k\in\tilde{I}_0^-}\lambda_{H_k}^-u_{H_k}^*-\sum_{k\in I_{0+}^-\cup I_{00}^-\cup I_{+0}^-}\lambda_{(-G_k)}^-u_{(-G_k)}^*+\sum_{k\in I_{00}^+\cup I_{+0}^+\cup I_{0-}^+\cup I_{+-}^+}\lambda_{G_k}^+u_{G_k}^*\Bigg)\in N_{D_f}(0_{\mathcal{R}^n}).
\end{aligned}$$

Using the definition of polar cone for $\vartheta\in[N_{D_f}(0_{\mathcal{R}^n})]^0$,

$$\begin{aligned}
\Big\langle &u_f^*+\sum_{i\in I_g^+}\lambda_{g_i}u_{g_i}^*+\sum_{j\in I_h^+}\lambda_{h_j}^+u_{h_j}^*-\sum_{j\in I_h^-}\lambda_{(-h_j)}^-u_{(-h_j)}^*+\sum_{k\in\tilde{I}_+^+\cup\tilde{I}_0^+}\lambda_{(-H_k)}^+u_{(-H_k)}^*-\sum_{k\in\tilde{I}_0^-}\lambda_{H_k}^-u_{H_k}^*\\
&-\sum_{k\in I_{0+}^-\cup I_{00}^-\cup I_{+0}^-}\lambda_{(-G_k)}^-u_{(-G_k)}^*+\sum_{k\in I_{00}^+\cup I_{+0}^+\cup I_{0-}^+\cup I_{+-}^+}\lambda_{G_k}^+u_{G_k}^*\Big\rangle\ge 0.
\end{aligned}$$

Now, it follows from (25) that

$$\begin{aligned}
f(\zeta)-\Bigg(&f(u)+\sum_{i\in I_g^+}\lambda_{g_i}g_i(u)+\sum_{j\in I_h^+}\lambda_{h_j}^+h_j(u)+\sum_{j\in I_h^-}\lambda_{(-h_j)}^-h_j(u)-\sum_{k\in\tilde{I}_+^+\cup\tilde{I}_0^+}\lambda_{(-H_k)}^+H_k(u)-\\
&\sum_{k\in\tilde{I}_0^-}\lambda_{H_k}^-H_k(u)+\sum_{k\in I_{0+}^-\cup I_{00}^-\cup I_{+0}^-}\lambda_{(-G_k)}^-G_k(u)+\sum_{k\in I_{00}^+\cup I_{+0}^+\cup I_{0-}^+\cup I_{+-}^+}\lambda_{G_k}^+G_k(u)\Bigg)\ge 0,
\end{aligned}$$

which implies that

$$f(\zeta) \geq f(u) + \sum_{i \in I_g^+} \lambda_{g_i} g_i(u) + \sum_{j \in I_h^+} \lambda_{h_j}^+ h_j(u) + \sum_{j \in I_h^-} \lambda_{(-h_j)}^- h_j(u) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \lambda_{(-H_k)}^+ H_k(u) - \sum_{k \in \tilde{I}_0^-} \lambda_{H_k}^- H_k(u) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \lambda_{(-G_k)}^- G_k(u) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \lambda_{G_k}^+ G_k(u).$$

□

Next, we propose a strong duality theorem.

**Theorem 5** (Strong duality theorem)**.** *Let $\bar{\zeta}$ be a local minimizer of MPVC (2). Suppose that $f$ admits a bounded upper regular directional convexificator and constraint functions admit upper directional convexificators at $\bar{\zeta}$. Assume that $(f, g_i(i \in I_g^+), h_j(j \in I_h^+(\zeta)), -h_j(j \in I_h^-(\zeta)), -H_k(k \in \tilde{I}_+^+(\zeta) \cup \tilde{I}_0^+(\zeta)), H_k(k \in \tilde{I}_0^-(\zeta)), -G_k(k \in I_{0+}^-(\zeta) \cup I_{00}^-(\zeta) \cup I_{+0}^-(\zeta)), G_k(k \in I_{00}^+(\zeta) \cup I_{0-}^+(\zeta) \cup I_{+0}^+(\zeta) \cup I_{+-}^+(\zeta))$ is $\partial_D^*-$ convex at $u \in S \cup_{pre\mathbb{R}^n} S_W$ with respect to $D_f$. Furthermore, suppose that $S$ is star-shaped at $\bar{\zeta}$ and assertions $(A_1), (A_2), (A_3)$ hold. If $\partial_D^*$-ACQ holds at $\bar{\zeta}$, then there exist Lagrange multipliers $(\bar{\lambda}_g \in \mathbb{R}^m,\ \bar{\lambda}_h \in \mathbb{R}^p, \bar{\lambda}_H, \bar{\lambda}_G \in \mathbb{R}^l)$ such that $(\bar{\zeta}, \bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_H, \bar{\lambda}_G)$ is a feasible point of VC-WD$(\bar{\zeta})$. Then, $(\bar{\zeta}, \bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_H, \bar{\lambda}_G)$ is a global optimal solution of the dual VC-WD$(\bar{\zeta})$, and the respective objective values are equal.*

**Proof.** Since $\bar{\zeta}$ is a local minimizer of problem (2), assertions $(A_1), (A_2)$, and $(A_3)$ hold, and $\partial_D^*-$ACQ holds at $\bar{\zeta}$. Then, according to Theorem 1, it follows that there exist Lagrange multipliers $(\bar{\lambda}_g \in \mathbb{R}^m,\ \bar{\lambda}_h \in \mathbb{R}^p, \bar{\lambda}_H, \bar{\lambda}_G \in \mathbb{R}^l)$ such that conditions (3)–(5) and (7) hold. Then, $(\bar{\zeta}, \bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_H, \bar{\lambda}_G)$ is a feasible point of the VC-WD$(\bar{\zeta})$, and

$$\sum_{i \in I_g^+} \bar{\lambda}_{g_i}^+ g_i(\bar{\zeta}) + \sum_{j \in I_h^+} \bar{\lambda}_{h_j}^+ h_j(\bar{\zeta}) + \sum_{j \in I_h^-} \bar{\lambda}_{(-h_j)}^- h_j(\bar{\zeta}) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \bar{\lambda}_{(-H_k)}^+ H_k(\bar{\zeta}) - \sum_{k \in \tilde{I}_0^-} \bar{\lambda}_{H_k}^- H_k(\bar{\zeta}) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \bar{\lambda}_{(-G_k)}^- G_k(\bar{\zeta}) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \bar{\lambda}_{G_k}^+ G_k(\bar{\zeta}) = 0. \tag{26}$$

Furthermore, according to Theorem 4, we obtain

$$f(\zeta) \geq f(u) + \sum_{i \in I_g^+} \lambda_{g_i}^+ g_i(u) + \sum_{j \in I_h^+} \lambda_{h_j}^+ h_j(u) + \sum_{j \in I_h^-} \lambda_{(-h_j)}^- h_j(u) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \lambda_{(-H_k)}^+ H_k(u) - \sum_{k \in \tilde{I}_0^-} \lambda_{H_k}^- H_k(u) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \lambda_{(-G_k)}^- G_k(u) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \lambda_{G_k}^+ G_k(u). \tag{27}$$

Using (26) and (27), we obtain

$$f(\bar{\zeta}) + \sum_{i \in I_g^+} \bar{\lambda}_{g_i}^+ g_i(\bar{\zeta}) + \sum_{j \in I_h^+} \bar{\lambda}_{h_j}^+ h_j(\bar{\zeta}) + \sum_{j \in I_h^-} \bar{\lambda}_{(-h_j)}^- h_j(\bar{\zeta}) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \bar{\lambda}_{(-H_k)}^+ H_k(\bar{\zeta}) - \sum_{k \in \tilde{I}_0^-} \bar{\lambda}_{H_k}^- H_k(\bar{\zeta}) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \bar{\lambda}_{(-G_k)}^- G_k(\bar{\zeta}) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \bar{\lambda}_{G_k}^+ G_k(\bar{\zeta}) \geq f(u) + \sum_{i \in I_g^+} \lambda_{g_i}^+ g_i(u) + \sum_{j \in I_h^+} \lambda_{h_j}^+ h_j(u) + \sum_{j \in I_h^-} \lambda_{(-h_j)}^- h_j(u) - \sum_{k \in \tilde{I}_+^+ \cup \tilde{I}_0^+} \lambda_{(-H_k)}^+ H_k(u) - \sum_{k \in \tilde{I}_0^-} \lambda_{H_k}^- H_k(u) + \sum_{k \in I_{0+}^- \cup I_{00}^- \cup I_{+0}^-} \lambda_{(-G_k)}^- G_k(u) + \sum_{k \in I_{00}^+ \cup I_{+0}^+ \cup I_{0-}^+ \cup I_{+-}^+} \lambda_{G_k}^+ G_k(u).$$

Hence, $(\bar{\zeta}, \bar{\lambda}_g, \bar{\lambda}_h, \bar{\lambda}_H, \bar{\lambda}_G)$ is a global maximum of the VC-WD$(\bar{\zeta})$, and the respective objective values are equal. □

Now, we illustrate Theorems 4 and 5 in the following example.

**Example 5.** *Consider the primal MPVC of Example 1. The point $\bar{\zeta} = (0,0) \in S$ is a local minimizer of the MPVC. Its dual model (VC-WD) is given in Example 3. We can easily see that $(A_1), (A_2)$, and $(A_3)$, as well as $\partial_D^*$-ACQ, hold. According to Theorem 5, there exist Lagrange multipliers ($\bar{\lambda}_g \in \mathbb{R}^m$, $\bar{\lambda}_h \in \mathbb{R}^p, \bar{\lambda}_H, \bar{\lambda}_G \in \mathbb{R}^l$) such that $(0,0,\bar{\lambda}_g,\bar{\lambda}_h,\bar{\lambda}_H,\bar{\lambda}_G)$ is a feasible point of the VC-WD$(\bar{\zeta})$ and*

$$\bar{\lambda}_g^+ g(\bar{\zeta}) - \bar{\lambda}_{(-H)}^+ H(\bar{\zeta}) - \bar{\lambda}_H^- H(\bar{\zeta}) + \bar{\lambda}_{(-G)}^- G(\bar{\zeta}) + \bar{\lambda}_G^+ G(\bar{\zeta}) = 0.$$

*Furthermore, since the weak duality between MPVC (2) and the VC-WD $(\bar{\zeta})$ holds as in Theorem 4, according to Theorem 5, $(\bar{\zeta},\bar{\lambda}_g,\bar{\lambda}_h,\bar{\lambda}_H,\bar{\lambda}_G)$ is a global maximum of the VC-WD$(\bar{\zeta})$, and the respective objective values are equal.*

## 5. Conclusions

In this study, we focused on nonsmooth mathematical programs with vanishing constraints involving functions without the necessity of being continuous. The idea of directional convexificators is the main tool used in our proofs. We established several nonsmooth stationary conditions and proposed standard and MPVC Abadie constraint qualifications based on this novel concept. We derived necessary optimality conditions under generalized Abadie constraint qualifications. We proposed sufficient optimality conditions in terms of directional convexificators under generalized convexity. Lastly, we proposed a Wolfe-type dual model for MPVC and a weak duality theorem and strong duality theorem using directional convexificators. The results reported in this paper extend several results previously reported in literature [1,12–14,16,36].

It is generally known that under general nonlinear constraints, there is no feasible method that can be defined that always reaches global minimizers of the problem. It is not feasible to provide even local minimizers, at least not when convexity is not taken into account. Finding an acceptable stationary point—that is, a computable point exhibiting essential minimizer properties—is the goal of practical algorithms. In this regard, the Karush–Kuhn–Tucker (KKT) conditions are the most crucial instrument for characterizing minimizers of a problem. They have been specialized or adjusted to several specific situations, such as multi-objective optimization and nonsmooth optimization, among others, and are used to state the theoretical convergence of almost every approach in restricted optimization. In addition, KKT conditions provide useful stopping criteria for various algorithms. Since in MPVC, the standard constraint qualifications are not satisfied and stationary conditions differ due to nonlinear reformulations, the results reported in this paper are not only useful to locate local minimizers but also to provide stopping criteria for various algorithms. Moreover, they are useful in tackling nonsmooth discontinuous functions with nonempty sets of continuity directions at a stationary point using directional convexificators.

In future research, we can extend the results reported by Laha and Dwivedi [22] for interval-valued optimization problems to lower semi-continuous cases using directional convexificators and the saddle-point criteria proposed by Jaiswal and Laha [25] for multi-objective optimizations to lower semi-continuous cases. Some other dual models for primal MPVC (2), like the Mond–Weir-type dual model and mixed-type dual models, may be introduced by using directional convexificators. It will also be interesting to explore the impact of the results on the problem; each nonconvex domain (or function) can be presented as a limit of the difference between two sequences of convex domains (or functions). These are some possible extensions of our results.

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