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Article

Some Results on Certain Supercobalancing Numbers

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Abstract: In this work, supercobalancing numbers are considered and some properties of these numbers are investigated. In the first part of this work, it is shown that every supercobalancing number is also a subbalancer. More specifically, B_3-supercobalancing numbers which have not been considered before within the scope of this subject are examined. All the solution classes of the Diophantine equation of B_3-supercobalancing numbers are determined exactly.

Keywords: supercobalancing numbers; subbalancing numbers; Diophantine equations

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1. Introduction

One of the most attractive topics in number theory is the concept of integer sequences. Since ancient times, integer sequences have attracted the attention of the mathematicians. Some of the most studied integer sequences are Fibonacci, Lucas, Pell and Pell–Lucas sequences. In addition to these integer sequences, another integer sequence that has attracted attention recently is the sequence of balancing numbers. The terms of the sequence of balancing numbers n are the solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1}$$

for some positive integer r, which is called the balancer of n [1].

It is obvious from (1) that

$$n^2 = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \tag{2}$$

It follows from (2) that n is a balancing number if and only if n^2 is a triangular number, that is, balancing numbers are the square roots of square-triangular numbers. Additionally, from (2), n is a balancing number if and only if $8n^2 + 1$ is a perfect square [1].

Later, in [2], Panda and Ray introduced cobalancing numbers n which are the solutions of the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) \tag{3}$$

for some positive integer r, which is called the cobalancer of n.

It is obvious from (3) that

$$n(n+1) = \frac{(n+r)(n+r+1)}{2} \text{ and } r = \frac{-(2n+1) + \sqrt{8n^2+8n+1}}{2} \tag{4}$$

It follows from (4) that n is a cobalancing number if and only if $n(n+1)$ is a triangular number, that is, cobalancing numbers are related to pronic-triangular numbers. Additionally, from (4), n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square.

Let B_n denote the nth balancing number and let b_n denote the nth cobalancing number. Then, the recurrence relations of balancing and cobalancing numbers are

$$\begin{aligned} B_{n+1} &= 6B_n - B_{n-1} \qquad (n \geq 2) \\ b_{n+1} &= 6b_n - b_{n-1} + 2 \qquad (n \geq 2) \end{aligned}$$

where $B_1 = 1, B_2 = 6$ and $b_1 = 0, b_2 = 2$, respectively.

Since $8B_n^2 + 1$ and $8b_n^2 + 8b_n + 1$ are perfect squares, $\sqrt{8B_n^2+1}$ and $\sqrt{8b_n^2+8b_n+1}$ are integers. Thus, nth Lucas-balancing and nth Lucas-cobalancing numbers were defined as

$$C_n = \sqrt{8B_n^2+1} \text{ and } c_n = \sqrt{8b_n^2+8b_n+1}$$

respectively, in [3,4].

The recurrence relations of Lucas-balancing and Lucas-cobalancing numbers are

$$\begin{aligned} C_{n+1} &= 6C_n - C_{n-1} \qquad (n \geq 2) \\ c_{n+1} &= 6c_n - c_{n-1} \qquad (n \geq 2) \end{aligned}$$

where $C_1 = 3,\ C_2 = 17$ and $c_1 = 1,\ c_2 = 7$, respectively.

The Binet formulas for balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are

$$\begin{aligned} B_n &= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \qquad & b_n &= \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \\ C_n &= \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} \qquad & c_n &= \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \end{aligned}$$

where $\alpha_1 = 1+\sqrt{2}$ and $\alpha_2 = 1-\sqrt{2}$.

Since the first published article on balancing numbers, a lot of research has been conducted on this topic by many authors [5–16]. Additionally, in [17], a relationship between the Diophantine equation of balancing numbers and Fibonacci numbers was studied. In [18], Rihane investigated the existence of balancing and Lucas-balancing numbers in the terms of a k-generalized Fibonacci sequence. Moreover, in [19], balancing numbers were generalized to t-balancing numbers and some algebraic identities regarding these numbers were obtained. Furthermore, in [20], reciprocal sums involving balancing and Lucas-balancing numbers were studied. Later, in [21], general identities were obtained related to reciprocal sums of products of balancing and Lucas-balancing numbers. In addition to these, several combinatorial expressions for balancing and Lucas-balancing numbers were obtained in [22].

In [23], Panda G. and Panda A. defined almost balancing numbers and showed that there are two types of almost balancing numbers: A_1-balancing and A_2-balancing numbers. They called n an A_1-balancing number if

$$1+2+\cdots+(n-1)+1 = (n+1)+(n+2)+\cdots+(n+r) \tag{5}$$

for some positive integer r, which is called the A_1-balancer of n.

They called n an A_2-balancing number if

$$1+2+\cdots+(n-1)-1 = (n+1)+(n+2)+\cdots+(n+r) \tag{6}$$

for some positive integer r, which is called the A_2-balancer of n.

Later, in [24], D-subbalancing numbers were introduced by substituting an arbitrary positive integer D instead of 1 and -1, which are the last terms of the left side of Equations (5) and (6).

Moreover, positive integers r in the Diophantine equations of D-subbalancing numbers were called D-subbalancers of D-subbalancing numbers.

Davala and Panda [25] introduced D-supercobalancing numbers, which are the solutions of Diophantine equations obtained by adding the positive integer D to the right side of the Diophantine equation of cobalancing numbers.

Further, it was pointed out that the choice of the positive integer D in the definition of subbalancing and supercobalancing numbers has crucial importance in [24,25]. This is due to the fact that D-subbalancing and D-supercobalancing numbers do not exist for every positive integer D.

In [24], Davala and Panda showed that D-subbalancing numbers exist when the positive integers D in the definition of subbalancing numbers are chosen as cobalancing numbers and obtained at least two solution classes of the Diophantine equation of b_m-subbalancing numbers for $m \geq 2$.

Similarly, in [25], Davala and Panda showed that D-supercobalancing numbers exist when the positive integers D in the definition of supercobalancing numbers are chosen as balancing numbers and obtained at least two solution classes of the Diophantine equation of B_m-supercobalancing numbers for $m \geq 2$.

Later, Rayaguru and Panda [26] showed that T_k-subbalancing and T_k-supercobalancing numbers exist when the positive integers D in the definition of subbalancing and supercobalancing numbers are chosen as triangular numbers and obtained several algebraic relations related to these numbers.

Sarı and Karadeniz-Gözeri [27] examined b_3-subbalancing numbers obtained by taking $D = b_3$ in the definition of subbalancing numbers and obtained various new identities related to these numbers. Further, they introduced b_3-Lucas subbalancing numbers and obtained some algebraic identities between b_3-Lucas subbalancing numbers and b_3-subbalancing numbers.

Recently, in addition to the fact that the positive integer D in the definition of subbalancing numbers can be chosen as cobalancing and triangular numbers, Sarı and Karadeniz-Gözeri [28] proved that D-subbalancing numbers are obtained when the values of D are chosen as the terms of the sequence of balancing numbers. Thus, they showed that for every positive integer m, B_m-subbalancing numbers exist. Furthermore, they obtained at least two solution classes of the Diophantine equation of B_m-subbalancing numbers for every positive integer m and dealt with B_3-subbalancing numbers obtained by taking $D = B_3$ in the definition of subbalancing numbers. They obtained several algebraic identities related to these numbers and derived several algebraic relations between B_3-subbalancing and b_3-subbalancing numbers.

In the present work, first we give an important result about the relationship between D-supercobalancing numbers and D-subbalancers that correspond to D-subbalancing numbers. Then, we examine B_3-supercobalancing numbers obtained by taking $D = B_3$ in the definition of supercobalancing numbers and obtain several new algebraic identities related to these numbers. Moreover, we show that the Diophantine equation of B_3-supercobalancing numbers has exactly two solution classes. We also give the recurrence relation and the Binet formula for B_3-supercobalancing numbers. Further, we obtain some algebraic relations between B_3-supercobalancing numbers and balancing, cobalancing, Lucas-balancing, Lucas-cobalancing and B_3-subbalancing numbers.

2. Relationship between Supercobalancing Numbers and Subbalancers

In this section, we show that D-supercobalancing numbers coincide with D-subbalancers for the proper values of D.

Definition 1 ([24]). *A positive integer n is a D-subbalancing number if*

$$1 + 2 + \cdots + (n-1) + D = (n+1) + (n+2) + \cdots + (n+r) \tag{7}$$

for some positive integer r, where D is a fixed positive integer. The positive integer r in (7) *is called D-subbalancer of n.*

As a result of this definition, we can give the following two corollaries.

Corollary 1. *Let n be a D-subbalancing number and r a D-subbalancer of n. Then,*

$$n^2 + D = \frac{(n+r)(n+r+1)}{2} \quad \textit{and} \quad n = \frac{(2r+1) \pm \sqrt{8r^2+8r-8D+1}}{2}.$$

Proof. By using (7), we obtain

$$\begin{aligned} \frac{(n-1)n}{2} + D &= \frac{(n+r)(n+r+1)}{2} - \frac{n(n+1)}{2} \\ &= \frac{(n+r)(n+r+1)}{2} - \frac{n^2+n}{2}. \end{aligned}$$

Thus, we obtain

$$n^2 + D = \frac{(n+r)(n+r+1)}{2} \tag{8}$$

On the other hand, from (8) we obtain

$$2n^2 + 2D = n^2 + 2nr + n + r^2 + r$$

Thus, we obtain the quadratic equation

$$n^2 - n(2r+1) - (r^2 + r - 2D) = 0 \tag{9}$$

We obtain the solutions of the quadratic Equation (9) as follows:

$$n = \frac{(2r+1) \pm \sqrt{8r^2+8r-8D+1}}{2}.$$

□

Corollary 2. *r is the D-subbalancer of a D-subbalancing number if and only if* $8r^2 + 8r - 8D + 1$ *is a perfect square.*

Proof. Suppose that r is a D-subbalancer and $8r^2 + 8r - 8D + 1$ is not a perfect square. Since r is the D-subbalancer of a D-subbalancing number, we obtain from Corollary 1

$$n = \frac{(2r+1) \pm \sqrt{8r^2+8r-8D+1}}{2}.$$

Thus, we obtain

$$2n - 2r - 1 = \pm\sqrt{8r^2+8r-8D+1} \tag{10}$$

The left side of Equation (10) is an integer, but since $\pm\sqrt{8r^2+8r-8D+1}$ is not a perfect square, we find a contradiction.

Conversely, suppose that $8r^2 + 8r - 8D + 1$ is a perfect square. Then,

$$n = \frac{(2r+1) \pm \sqrt{8r^2+8r-8D+1}}{2}$$

are integers. It can be seen from Corollary 1 that r is a D-subbalancer of the D-subbalancing number n. □

Definition 2 ([25]). *A positive integer n is a D-supercobalancing number if*

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + D \tag{11}$$

for some positive integer r, where D is a fixed positive integer. The positive integer r in (11) *is called D-supercobalancer of n.*

As a result of this definition, we can give the following two corollaries.

Corollary 3. *Let n be a D-supercobalancing number and r a D-supercobalancer of n. Then*

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 8n - 8D + 1}}{2}.$$

Proof. By using (11), we obtain

$$\frac{n(n+1)}{2} = \frac{(n+r)(n+r+1)}{2} - \frac{n(n+1)}{2} + D.$$

Thus, we obtain

$$n^2 + n = \frac{(n+r)(n+r+1)}{2} + D \tag{12}$$

On the other hand, from (12) we obtain

$$2n^2 + 2n - 2D = n^2 + 2nr + n + r^2 + r$$

Thus, we obtain the quadratic equation

$$r^2 + r(2n+1) - (n^2 + n - 2D) = 0 \tag{13}$$

We obtain the solutions of the quadratic equation (13) as follows:

$$r = \frac{-(2n+1) \pm \sqrt{8n^2 + 8n - 8D + 1}}{2}.$$

Since r is a positive integer, we obtain

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 8n - 8D + 1}}{2}.$$

□

Corollary 4. *n is a D-supercobalancing number if and only if* $8n^2 + 8n - 8D + 1$ *is a perfect square.*

Proof. Suppose that n is a D-supercobalancing number and $8n^2 + 8n - 8D + 1$ is not a perfect square. Since n is a D-supercobalancing number, we obtain from Corollary 3

$$2r + 2n + 1 = \sqrt{8n^2 + 8n - 8D + 1} \tag{14}$$

The left side of Equation (14) is an integer, but since $\sqrt{8n^2 + 8n - 8D + 1}$ is not a perfect square, we find a contradiction.

Conversely, suppose that $8n^2 + 8n - 8D + 1$ is a perfect square. Then,

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 8n - 8D + 1}}{2}$$

is an integer. It can be seen from Definition 2 that n is a D-supercobalancing number. □

It can be seen from Corollary 2 and Corollary 4 that D-supercobalancing numbers and D-subbalancers coincide.

We now give the following examples that will be extremely useful in analyzing the relationship between D-supercobalancing numbers and D-subbalancers.

If $D = B_1$, then B_1-supercobalancing numbers satisfy the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + 1.$$

It is obvious from Corollary 4 that since n is a B_1-supercobalancing number, $8n^2 + 8n - 7$ is a perfect square. On the other hand, it is obvious from Corollary 2 that since $8n^2 + 8n - 7$ is a perfect square, n is a B_1-subbalancer. Consequently, n is both a B_1-supercobalancing number and a B_1-subbalancer. For example, $1, 7, 43$ and 253 are all B_1-supercobalancing numbers and B_1-subbalancers.

If $D = B_2$, then B_2-supercobalancing numbers satisfy the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + 6.$$

It is obvious that n is both a B_2-supercobalancing number and a B_2-subbalancer. For example, $2, 3, 6$ and 8 are all B_2-supercobalancing numbers and B_2-subbalancers.

If $D = B_3$, then B_3-supercobalancing numbers satisfy the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r) + 35.$$

It is obvious that n is both a B_3-supercobalancing number and a B_3-subbalancer. For example, $7, 9, 35$ and 49 are all B_3-supercobalancing numbers and B_3-subbalancers.

3. Some Properties of B_3-Supercobalancing Numbers

In [25], at least two solution classes of the Diophantine equation of B_m-supercobalancing numbers were given for $m \geq 2$. In this section, first we show that there are exactly two solution classes of the Diophantine equation of B_3-supercobalancing numbers. Then, we obtain some algebraic identities regarding these numbers.

Throughout this paper, the nth B_3-supercobalancing number is denoted by $(sB_3)_n$. For the notation of B_3-supercobalancing numbers, we use the property of cobalancing numbers to be the balancers of corresponding to balancing numbers. Since the nth B_3-subbalancing number is denoted by $(SB_3)_n$ in [28], and B_3-subbalancers corresponding to B_3-subbalancing numbers are B_3-supercobalancing numbers, we prefer this notation, with a relationship similar to the relationship between the notation of balancing and cobalancing numbers. Moreover, the nth balancing number is denoted by B_n, the nth cobalancing number is denoted by b_n, the nth Lucas-balancing number is denoted by C_n and the nth Lucas-cobalancing number is denoted by c_n.

We give the following theorem to use in order to prove the theorem that gives the solution classes of the Diophantine equation of B_3-supercobalancing numbers. We also use similar techniques included in [23,29,30] in order to prove the theorem that gives these solution classes.

Theorem 1. *For every positive integer m, the relationship between balancing, Lucas-balancing numbers and cobalancing, Lucas-cobalancing numbers are*

$$15C_m - 26B_m = 34b_m + 2c_m + 17$$

and

$$15C_m + 26B_m = 34b_{m+1} - 2c_{m+1} + 17.$$

Proof. From the Binet formulas for balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers, we deduce that

$$\begin{aligned}
15C_m - 26B_m &= 15\left(\frac{\alpha_1^{2m}+\alpha_2^{2m}}{2}\right) - 26\left(\frac{\alpha_1^{2m}-\alpha_2^{2m}}{4\sqrt{2}}\right) \\
&= \alpha_1^{2m}\left(\frac{30\sqrt{2}-26}{4\sqrt{2}}\right) + \alpha_2^{2m}\left(\frac{30\sqrt{2}+26}{4\sqrt{2}}\right) \\
&= \alpha_1^{2m-1}\left(\frac{34}{4\sqrt{2}}+1\right) + \alpha_2^{2m-1}\left(\frac{-34}{4\sqrt{2}}+1\right) \\
&= 34\left(\frac{\alpha_1^{2m-1}-\alpha_2^{2m-1}}{4\sqrt{2}} - \frac{1}{2}\right) + 2\left(\frac{\alpha_1^{2m-1}+\alpha_2^{2m-1}}{2}\right) + 17 \\
&= 34b_m + 2c_m + 17.
\end{aligned}$$

The other case can be proved similarly. □

Theorem 2. *The Diophantine equation of B_3-supercobalancing numbers has exactly two solution classes, that is, B_3-supercobalancing numbers are in the form*

$$(17b_k + c_k + 8) \quad \textit{and} \quad (17b_{k+1} - c_{k+1} + 8)$$

for $k \geq 1$.

Proof. Let n be a B_3-supercobalancing number. Then, $8n^2 + 8n - 279$ is a perfect square. Thus, it is necessary to solve the generalized Pell equation $y^2 - 2x^2 = -281$ $(y \in \mathbb{Z})$ in order to obtain all the B_3-supercobalancing numbers, where $x = 2n + 1$.

The fundamental solution of the Pell equation $y^2 - 2x^2 = 1$ is $(y, x) = (3, 2)$ and there are two solution classes of the generalized Pell equation $y^2 - 2x^2 = -281$ as $(y, x) = (13, 15)$ and $(y, x) = (-13, 15)$. Thus, the solutions corresponding to these two classes are given by

$$y_k + x_k\sqrt{2} = (13 + 15\sqrt{2})(3 + 2\sqrt{2})^k \qquad (k = 1, 2, \cdots) \tag{15}$$

$$y_k + x_k\sqrt{2} = (-13 + 15\sqrt{2})(3 + 2\sqrt{2})^k \qquad (k = 1, 2, \cdots) \tag{16}$$

respectively.

By solving Equations (15) and (16), we obtain

$$\begin{aligned}
x_k &= 26\left(\frac{\alpha_1^{2k}-\alpha_2^{2k}}{4\sqrt{2}}\right) + 15\left(\frac{\alpha_1^{2k}+\alpha_2^{2k}}{2}\right) \\
y_k &= 13\left(\frac{\alpha_1^{2k}+\alpha_2^{2k}}{2}\right) + 60\left(\frac{\alpha_1^{2k}-\alpha_2^{2k}}{4\sqrt{2}}\right) \\
x_k^{'} &= -26\left(\frac{\alpha_1^{2k}-\alpha_2^{2k}}{4\sqrt{2}}\right) + 15\left(\frac{\alpha_1^{2k}+\alpha_2^{2k}}{2}\right) \\
y_k^{'} &= -13\left(\frac{\alpha_1^{2k}+\alpha_2^{2k}}{2}\right) + 60\left(\frac{\alpha_1^{2k}-\alpha_2^{2k}}{4\sqrt{2}}\right)
\end{aligned}$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

Thus, using the Binet formulas for balancing and Lucas-balancing numbers, we finally obtain

$$x_k = 26B_k + 15C_k \qquad \text{and} \qquad x_k^{'} = -26B_k + 15C_k \qquad (k \geq 1).$$

Since $x = 2n+1$ and the values of n satisfying the generalized Pell equation $y^2 - 2x^2 = -281$ are B_3-supercobalancing numbers, we find that B_3-supercobalancing numbers are in the form

$$n_k = \frac{26B_k + 15C_k - 1}{2} \quad \text{and} \quad n_k^{'} = \frac{-26B_k + 15C_k - 1}{2} \qquad (k \geq 1).$$

By using Theorem 1, we obtain

$$n_k = 17b_{k+1} - c_{k+1} + 8 \quad \text{and} \quad n_k^{'} = 17b_k + c_k + 8.$$

Thus, B_3-supercobalancing numbers are in the form

$$(17b_k + c_k + 8) \quad \text{and} \quad (17b_{k+1} - c_{k+1} + 8) \qquad (k \geq 1).$$

□

In [25], at least two solution classes of B_{2t-1}-supercobalancing numbers were given as

$$\frac{(2B_t + C_{t-1})C_l + 2(C_t - 4B_{t-1})B_l - 1}{2} \quad \text{and} \quad \frac{(2B_t + C_{t-1})C_l - 2(C_t - 4B_{t-1})B_l - 1}{2}$$

for $l \geq 1$. In the case of $t = 2$, it can be seen that the solutions obtained from these formulas and the solutions given in Theorem 2 coincide.

The smallest positive integer satisfying the formulas of the solutions is 9. On the other hand, for $D = B_3$, we find that B_3-subbalancers that correspond to the first two terms of the sequence of B_3-subbalancing numbers are 7 from Corollary 1. Because of this, we call the first two terms of this sequence as 7.

As a result of this theorem, we can give the following corollary.

Corollary 5. *For any positive integer m, the relationship between B_3-supercobalancing numbers and cobalancing, Lucas-cobalancing numbers is*

$$\begin{aligned} (sB_3)_{2m} &= 17b_m + c_m + 8 \\ (sB_3)_{2m+1} &= 17b_{m+1} - c_{m+1} + 8. \end{aligned}$$

Theorem 3. *The recurrence relation of B_3-supercobalancing numbers is*

$$(sB_3)_{m+2} = 6(sB_3)_m - (sB_3)_{m-2} + 2 \qquad (m \geq 2)$$

where $(sB_3)_0 = 7, (sB_3)_1 = 7, (sB_3)_2 = 9$ and $(sB_3)_3 = 35$.

Proof. It can be obtained by using Corollary 5 and recurrence relations of cobalancing and Lucas-cobalancing numbers. □

Corollary 6. *For every $m \geq 2$, B_3-supercobalancing numbers satisfy*

$$(sB_3)_{2m} = 6(sB_3)_{2m-2} - (sB_3)_{2m-4} + 2$$

and

$$(sB_3)_{2m+1} = 6(sB_3)_{2m-1} - (sB_3)_{2m-3} + 2.$$

Proof. It can be proved by using Theorem 3. □

Corollary 7. *Every B_3-supercobalancing number is odd.*

Proof. Since the values of b_m are even and the values of c_m are odd for $m \geq 1$, it can be proved by using Corollary 5. □

In the following theorem, the Binet formula for B_3-supercobalancing numbers is given.

Theorem 4. *For every positive integer m,*

$$(sB_3)_{2m} = \frac{\left(17+2\sqrt{2}\right)\alpha_1^{2m-1} - (17-2\sqrt{2})\alpha_2^{2m-1}}{4\sqrt{2}} - \frac{1}{2}$$

and

$$(sB_3)_{2m+1} = \frac{\left(17-2\sqrt{2}\right)\alpha_1^{2m+1} - (17+2\sqrt{2})\alpha_2^{2m+1}}{4\sqrt{2}} - \frac{1}{2}$$

where $\alpha_1 = 1+\sqrt{2}$ *and* $\alpha_2 = 1-\sqrt{2}$.

Proof. From Corollary 5 and the Binet formulas for cobalancing and Lucas-cobalancing numbers, we obtain

$$\begin{aligned} (sB_3)_{2m} &= 17\left(\frac{\alpha_1^{2m-1}-\alpha_2^{2m-1}}{4\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{\alpha_1^{2m-1}+\alpha_2^{2m-1}}{2}\right) + 8 \\ &= \frac{\left(17+2\sqrt{2}\right)\alpha_1^{2m-1} - (17-2\sqrt{2})\alpha_2^{2m-1}}{4\sqrt{2}} - \frac{1}{2}. \end{aligned}$$

The other case can be proved similarly. □

Theorem 5. *For every $m \geq 2$, B_3-supercobalancing numbers satisfy*

$$(sB_3)_{2m} = \frac{281(sB_3)_{2m-1} - 34(sB_3)_{2m-2} + 26}{195}$$

and

$$(sB_3)_{2m+1} = \frac{281(sB_3)_{2m} - 195(sB_3)_{2m-1} + 26}{34}.$$

Proof. It follows from Corollary 5 that

$$(sB_3)_{2m} = b_{m+1} + 14b_m + 7 \tag{17}$$

and

$$(sB_3)_{2m-1} = 14b_m + b_{m-1} + 7 \tag{18}$$

By using (17) and (18), we obtain

$$\begin{aligned} (sB_3)_{2m} &= \frac{3900b_m - 195b_{m-1} + 1755}{195} \\ &= \frac{281(14b_m + b_{m-1} + 7) - 34(b_m + 14b_{m-1} + 7) + 26}{195} \\ &= \frac{281(sB_3)_{2m-1} - 34(sB_3)_{2m-2} + 26}{195}. \end{aligned}$$

The other case can be proved similarly. □

Theorem 6. *For every $m \geq 2$, B_3-supercobalancing numbers satisfy*

$$[(sB_3)_m - 1]^2 = (sB_3)_{m-2}(sB_3)_{m+2} - 279.$$

Proof. This theorem is proved by induction. It is easily seen that the assertion is true for $m = 2$.

Assuming the assertion is true for $m \leq k$, we have

$$\begin{aligned}
[(sB_3)_{k+1} - 1]^2 &= (sB_3)^2_{k+1} - 2(sB_3)_{k+1} + 1 \\
&= (sB_3)^2_{k+1} + (sB_3)_{k-3}(sB_3)_{k+1} - 2(sB_3)_{k+1} - [(sB_3)_{k-1} - 1]^2 - 278 \\
&= [(sB_3)_{k+1} + (sB_3)_{k-3} - 2](sB_3)_{k+1} - [(sB_3)_{k-1} - 1]^2 - 278 \\
&= 6(sB_3)_{k-1}(sB_3)_{k+1} - (sB_3)^2_{k-1} + 2(sB_3)_{k-1} - 279 \\
&= (sB_3)_{k-1}(sB_3)_{k+3} - 279.
\end{aligned}$$

Thus, it is shown that the assertion is true for $m = k + 1$. □

4. Some Relations between the Sequence of B_3-Supercobalancing Numbers and the Other Integer Sequences

In this section, we give several algebraic identities between B_3-supercobalancing numbers and balancing, cobalancing, Lucas-balancing, Lucas-cobalancing and B_3-subbalancing numbers. We can give the following theorem on the relationship between B_3-supercobalancing and cobalancing numbers which can be proved by using (17) and (18).

Theorem 7. *For any positive integer m,*

$$(sB_3)_{2m} = 34b_m - (sB_3)_{2m-1} + 16$$

and

$$(sB_3)_{2m+1} = 14(sB_3)_{2m} - 195b_m - 91.$$

As a result of this theorem, we can give the following four corollaries.

Corollary 8. *For every $m \geq 2$, the relationship between the terms of the sequence of B_3-supercobalancing numbers and balancing numbers is*

$$[(sB_3)_{2m+2} + (sB_3)_{2m+1}] - [(sB_3)_{2m} + (sB_3)_{2m-1}] = 68B_m.$$

Corollary 9. *The sum of any even term of the sequence of B_3-supercobalancing numbers and the preceding one is a multiple of 4. Equivalently to this, the sum of any odd term of the sequence of B_3-supercobalancing numbers and the next one is a multiple of 4.*

Corollary 10. *For any positive integer m, the relations between B_3-supercobalancing numbers and balancing, cobalancing numbers are*

$$(sB_3)_{2m} = 2B_m + 15b_m + 7$$

and

$$(sB_3)_{2m+1} = 28B_m + 15b_m + 7.$$

Corollary 11. *For every $m \geq 2$, the relationship between the terms of the sequence of B_3-supercobalancing numbers and cobalancing numbers is*

$$7(sB_3)_{2m} = 10(sB_3)_{2m-1} - 17b_{m-1} - 7.$$

Theorem 8. *For every $m \geq 2$, the relationship between the terms of the sequence of B_3-supercobalancing numbers and Lucas-cobalancing numbers is*

$$(sB_3)_{2m} = (sB_3)_{2m-1} + 2c_m.$$

Proof. It can be proved by using Corollary 5. □

Theorem 9. *For any positive integer m, the relationship between the terms of the sequence of B_3-supercobalancing numbers and Lucas-balancing numbers is*

$$(sB_3)_{2m+1} = 15C_m - (sB_3)_{2m} - 1.$$

Proof. From Corollary 10, we obtain

$$(sB_3)_{2m+1} = 30B_m + 30b_m + 14 - (sB_3)_{2m}. \tag{19}$$

Then, by using the relations between balancing and cobalancing and Lucas-balancing numbers, we obtain

$$\begin{aligned}
(sB_3)_{2m+1} &= 15b_{m+1} + 15b_m + 14 - (sB_3)_{2m} \\
&= 15\left(\frac{B_{m+1} - B_m - 1}{2}\right) + 15\left(\frac{B_m - B_{m-1} - 1}{2}\right) + 14 - (sB_3)_{2m} \\
&= \frac{15}{2}(B_{m+1} - B_{m-1}) - (sB_3)_{2m} - 1 \\
&= 15(3B_m - B_{m-1}) - (sB_3)_{2m} - 1 \\
&= 15C_m - (sB_3)_{2m} - 1.
\end{aligned}$$

□

Corollary 12. *For every $m \geq 2$, the relationship between the terms of the sequence of B_3-supercobalancing numbers and balancing numbers is*

$$(sB_3)_{2m+1} + (sB_3)_{2m} = 60\sum_{i=1}^{m-1} B_i + 30B_m + 14.$$

Proof. From (19) and the relation between balancing and cobalancing numbers, we obtain

$$\begin{aligned}
(sB_3)_{2m+1} + (sB_3)_{2m} &= 30[2(B_1 + B_2 + \cdots + B_{m-1})] + 30B_m + 14 \\
&= 60\sum_{i=1}^{m-1} B_i + 30B_m + 14.
\end{aligned}$$

□

Theorem 10. *For any positive integer m, the relationship between the terms of the sequence of B_3-supercobalancing numbers and balancing numbers is*

$$(sB_3)_{2m+1} = (sB_3)_{2m} + 26B_m.$$

Proof. From (17), (18) and the relation between balancing and cobalancing numbers, we obtain

$$\begin{aligned}
(sB_3)_{2m+1} &= 13b_{m+1} - 13b_m + (sB_3)_{2m} \\
&= 13(b_{m+1} - b_m) + (sB_3)_{2m} \\
&= 26B_m + (sB_3)_{2m}.
\end{aligned}$$

□

From the above theorem, we can give the following corollary on the sums of B_3-supercobalancing numbers.

Corollary 13. *For any positive integer m,*

$$\sum_{i=1}^{m}[(sB_3)_{2i+1} - (sB_3)_{2i}] = 13b_{m+1}.$$

Corollary 14. *For every $m \geq 2$, the relationship between the terms of the sequence of B_3-supercobalancing numbers and balancing numbers is*

$$(sB_3)_{2m+1} - (sB_3)_{2m-1} = 2(14B_m + B_{m-1}).$$

Proof. From Theorem 8, Theorem 10 and the relation between balancing and Lucas-cobalancing numbers, we obtain

$$\begin{aligned}(sB_3)_{2m+1} - (sB_3)_{2m-1} &= 2(13B_m + c_m) \\ &= 2(13B_m + B_m + B_{m-1}) \\ &= 2(14B_m + B_{m-1}).\end{aligned}$$

□

Theorem 11. *The relations between the terms of the sequence of B_3-supercobalancing numbers and balancing, Lucas-balancing numbers are*

$$(sB_3)_{2m} = \frac{C_{m+1} - 2B_{m-2} - 1}{2} \qquad (m \geq 2)$$

and

$$(sB_3)_{2m+1} = \frac{5C_{m+1} - 14B_m - 1}{2} \qquad (m \geq 1).$$

Proof. From (17) and the relation between balancing and cobalancing numbers, we obtain

$$\begin{aligned}(sB_3)_{2m} &= \left(\frac{B_{m+1} - B_m - 1}{2}\right) + 14\left(\frac{B_m - B_{m-1} - 1}{2}\right) + 7 \\ &= \frac{17B_m - 3B_{m-1} - 2B_{m-2} - 1}{2} \\ &= \frac{C_{m+1} - 2B_{m-2} - 1}{2}.\end{aligned}$$

The other case can be proved similarly. □

Theorem 12. *For every $m \geq 2$, the relationship between the even terms of the sequence of B_3-supercobalancing numbers and balancing numbers is*

$$15(sB_3)_{2m} - (sB_3)_{2m+2} = 13B_{m+1} - 15B_{m-2} - 7.$$

Proof. From Theorem 9, Theorem 10 and Theorem 11, we obtain

$$\begin{aligned}(sB_3)_{2m+2} &= 15C_{m+1} - (sB_3)_{2m+3} - 1 \\ &= 15[2(sB_3)_{2m} + 2B_{m-2} + 1] - [(sB_3)_{2m+2} + 26B_{m+1}] - 1 \\ &= 30(sB_3)_{2m} + 30B_{m-2} - (sB_3)_{2m+2} - 26B_{m+1} + 14.\end{aligned}$$

Thus, we deduce that

$$15(sB_3)_{2m} - (sB_3)_{2m+2} = 13B_{m+1} - 15B_{m-2} - 7.$$

□

In the following theorem and corollary, some identities between B_3-supercobalancing and B_3-subbalancing numbers are given.

Theorem 13. *For every $m \geq 0$, the relations between the terms of the sequence of B_3-supercobalancing numbers are*

$$(SB_3)_{2m} = \frac{(sB_3)_{2m+2} - (sB_3)_{2m}}{2}$$

and

$$(SB_3)_{2m+1} = \frac{(sB_3)_{2m+3} - (sB_3)_{2m+1}}{2}.$$

Proof. From (17) and the relation between B_3-subbalancing and balancing numbers, we obtain

$$\begin{aligned}
(SB_3)_{2m} &= 14B_m + B_{m+1} \\
&= \frac{b_{m+2} + 13b_{m+1} - 14b_m}{2} \\
&= \frac{(sB_3)_{2m+2} - (sB_3)_{2m}}{2}.
\end{aligned}$$

The other case can be proved by using Corollary 13 and the relation between B_3-subbalancing and balancing numbers. □

Corollary 15. *B_3-supercobalancing and B_3-subbalancing numbers satisfy*

$$\sum_{i=0}^{m} (SB_3)_{2i} = \frac{(sB_3)_{2m+2} - 7}{2}$$

and

$$\sum_{i=0}^{m} (SB_3)_{2i+1} = \frac{(sB_3)_{2m+3} - 7}{2}.$$

Proof. It can be proved by using Theorem 13. □

5. Conclusions

In this work, we deduced some new results on supercobalancing numbers. One of the important results obtained in this paper is about the relationship between supercobalancing numbers and subbalancers. Besides this, we determined all the solution classes of the Diophantine equation related with B_3-supercobalancing numbers. We also investigated the relationship between these numbers and the other integer sequences. In addition to these, some similarities between B_3-supercobalancing and cobalancing numbers can be noticed by examining Theorem 3 and Theorem 6. Furthermore, by examining Theorem 10 and Corollary 15, it can be observed that similar relations between balancing and cobalancing numbers also exist between B_3-subbalancing and B_3-supercobalancing numbers.

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