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Chen-Burr XII Model as a Competing Risks Model with Applications to Real-Life Data Sets

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Abstract: In this paper Chen-Burr XII distribution is constructed and graphical description of the probability density function, hazard rate and reversed hazard rate functions of the proposed model is obtained. Also, some statistical characteristics of the Chen-Burr XII distribution are discussed and some new models as sub-models from the Chen-Burr XII distribution are introduced. Moreover, maximum likelihood estimation of the parameters, reliability, hazard rate and reversed hazard rate functions of the Chen-Burr XII distribution are considered. Also, the asymptotic confidence intervals of the distribution parameters, reliability, hazard rate and reversed hazard rate functions are presented. Finally, three real life data sets are applied to prove how the Chen-Burr XII distribution can be applied in real life and to confirm its superiority over some existing distributions.

Keywords: competing risks; additive model; Chen-Burr XII distribution; maximum likelihood estimation; hazard shape

MSC: 60G30; 62E10; 62F10; 62F12; 62G30; 62N05

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1. Introduction

Many techniques are available in statistical literature for constructing, expanding, and generalizing lifetime distributions. These include compound distributions, finite and infinite mixed distributions, probability integral transforms, and transformations of variables and distribution functions, see [1]. Researchers have proposed these methods to provide more flexible modifications, extensions or generalizations of the existing lifetime distributions. In life testing, human mortality research, engineering modelling, electronic sciences, biological surveys, and reliability studies, there is a need for different shapes of lifetime distributions with hazard rate functions (hrfs) that accommodate different patterns of failure.

Comparing risks is a useful technique for creating new lifetime distributions. The output model has a flexible hrf with bathtub shape and more intricate shapes, which highlights the significance of the competing risks approach. The concept of competing risks, which shows up in many life-testing studies where the failure of the tested item may be linked to multiple causes or modes of failure, is the basis of the competing risks approach. In a way, these failure modes compete to make the tested item fail. For this reason, this is commonly referred to as competing risks in statistical literature. Also, competing risks arise in series systems, in which the components are arranged in series. Each component possesses a specified distribution with specified parameters and these components are statistically independent of each other, therefore the lifetime of the series system is the minimum of its components lifetimes. In reliability studies, demographic, medical, and biological sciences, as well as in engineering applications, competing risks frequently happened. Additionally, the series model, additive model, and multi-risk model are other names for the competing risks concept.

Several lifetime distributions, such as the additive Weibull (AW) distribution presented by [2], have been introduced in literature based on the concept of competing risks, it is constructed by combining two Weibull (W) distributions; one has an increasing hrf and another has a decreasing hrf. The additive Burr XII (ABXII) distribution was presented by [3] by adding two hrfs of Burr XII (BXII) distribution; one has a decreasing hrf and another has an increasing hrf. Ref. [4] proposed a competing risks distribution, called the B distribution. A new modified W (NMW) distribution was derived by [5] by combining the W distribution with the modified W distribution presented by [6] in a series system. The exponential-W distribution was constructed by [7]. The additive modified W distribution was obtained by [8]. The log-logistic W distribution was introduced by [9], by considering a series system with two components; one has the W distribution and another has the log-logistic distribution. The additive Perks-W distribution was obtained by [10]. BXII modified W (BXII-MW) distribution was derived by [11]. W-Chen (W-C) distribution was introduced by [12]. The log-normal modified W distribution was proposed by [13]. Lomax-W distribution was constructed by [14]. The flexible W extension-BXII distribution was presented by [15] by combining the flexible W extension distribution obtained by [16] and BXII distribution in a series system. The additive Chen-W distribution was proposed by [17] and the flexible additive W distribution was developed by [18] via the combination of three W distributions. The Lindley-BXII distribution was suggested by [19]. The flexible additive Chen-Gompertz distribution was introduced by [20] by combining Chen and a special case of Gompertz distributions in a series system. The additive power-transformed half-logistic model was proposed by [21] by combining two power-transformed half-logistic distributions in a series system. The three-component additive W distribution was considered by [22]. Recently, the additive flexible W extension-Lomax distribution was presented by [23]. The additive Perks distribution was presented by [24] based on the sum of the positive and negative hrfs of Perks distributions. More recently, the additive Chen (AC) distribution was introduced by [25]. Also, ref. [26] suggested the additive Chen-Perks (AC-P) distribution. Most recently, the Additive Xgamma-BXII Distribution was introduced as a competing risks model between the xgamma and BXII distributions.

An important concept for lifetime distributions and lifetime data modelling is the hrf. There are different shapes of the hrf, the most common are an increasing hrf, decreasing hrf, constant hrf, unimodal (upside bathtub) and bathtub hrf. The most important hrf shapes, that are very useful in practice and play an important role in reliability, are the bathtub, unimodal and modified bathtub shapes. In statistical literature there are many lifetime distributions that are introduced with bathtub and unimodal (upside bathtub) hrf, see [27]. A more complex pattern of the hrf is the generalized/modified bathtub hrf, which is a modification of bathtub hrf. The behavior of bathtub and modified bathtub hrfs throughout their infant mortality phase is the main distinction between them. The infant mortality phase in the modified bathtub hrf was split into two stages by [28]. The first stage exhibits an increasing failure rate, signifying failures resulting from rough faults, such as those arising from inappropriate handling, inadequate manufacture, or defective control systems. In this initial stage, the hrf pattern swiftly peaks, is followed by a period of decreasing hrf, and subsequently increases hrf. (for more details see [27,29]).

This paper introduces the Chen-BXII (C-BXII) distribution, a new competing risks model, by considering a series system consisting of two independent components operating in series. The first component, X_1's lifetime, has a Chen distribution while the second component, X_2's lifetime has a BXII distribution. The system's lifespan $X = \min\{X_1, X_2\}$ has the C-BXII distribution as a result. This distribution's great degree of flexibility and diversity in shape appear in its pdf and hrf, which indicates its significance. Three essential shapes of the hrf are displayed: decreasing-unimodal, modified bathtub, and bathtub. These shapes make the C-BXII distribution more useful for modelling lifetime data. Additionally, the constructed distribution includes some novel additive models that are special cases that haven't been presented in statistical literature. As special cases, it also includes some well-known models.

A two-parameter lifetime distribution with bathtub or increasing hrf was presented by [30]. The reliability function (rf) and the hrf of Chen distribution are given, respectively, by:

$$R_1(x;\alpha,\beta) = e^{\alpha(1-e^{x^\beta})}, \qquad x > 0; \alpha, \beta > 0, \tag{1}$$

and

$$h_1(x;\alpha,\beta) = \alpha\beta x^{\beta-1} e^{x^\beta}, \qquad x > 0; \alpha, \beta > 0, \tag{2}$$

where α and β are shape parameters.

The BXII distribution is a component of the Burr continuous distribution system which was proposed by [31]. The BXII distribution's rf and hrf are provided, respectively, by:

$$R_2(x;c,k) = (1+x^c)^{-k}, \; x > 0; c, k > 0, \tag{3}$$

and

$$h_2(x;c,k) = \frac{ckx^{c-1}}{(1+x^c)}, \; x > 0; c, k > 0, \tag{4}$$

where c and k are shape parameters.

The structure of this paper is as follows: the C-BXII distribution is constructed in Section 2. In Section 3, the pdf, hrf, and rhrf of the constructed model are graphically described. Also, statistical characteristics of the C-BXII distribution are studied in Section 4 and some sub models as particular cases from the C-BXII distribution are introduced. Maximum likelihood (ML) estimation and the asymptotic confidence intervals (ACI) of the parameters, rf, hrf and rhrf of the C-BXII distribution are considered in Section 5. A simulation study is conducted in Section 6 for evaluating the performance of the ML estimates. Finally, In Section 7 three applications are considered to demonstrate the applicability of the proposed model and its superiority over some existing distributions.

2. Model Construction

This section presents the construction of the suggested model using the idea of competing risks. A Graphical description of the pdf, hrf and reversed hrf (rhrf) of the proposed model is introduced. Furthermore, an explanation of the behavior of the hrf is provided.

The rf of the proposed model can be obtained by multiplying the rfs of Chen and BXII distributions as follows:

$$R(x;\underline{\theta}) = \prod_{i=1}^{2} R_i(x) = \frac{e^{\alpha(1-e^{x^\beta})}}{(1+x^c)^k}, x > 0; \underline{\theta} > \underline{0}, \tag{5}$$

where $\underline{\theta} = (\alpha, \beta, c, k)$ is a parameter vector, $R_1(x)$ and $R_2(x)$ are the rfs of Chen and BXII distributions, respectively.

The corresponding cumulative distribution function (cdf) of the C-BXII distribution is given by:

$$F(x;\underline{\theta}) = 1 - R(x;\underline{\theta}) = 1 - \frac{e^{\alpha(1-e^{x^\beta})}}{(1+x^c)^k}, x > 0; \underline{\theta} > \underline{0}. \tag{6}$$

Also, the hrf of the C-BXII distribution can be expressed as the sum of the hrfs of Chen and BXII distributions as using Equations (2) and (4) one obtains

$$\begin{aligned} h(x;\underline{\theta}) &= h_1(x;\alpha,\beta) + h_2(x;c,k) \\ &= \alpha\beta x^{\beta-1} e^{x^\beta} + \tfrac{ckx^{c-1}}{(1+x^c)}, x > 0; \underline{\theta} > \underline{0}. \end{aligned} \tag{7}$$

The pdf of the C-BXII distribution can be derived using the following relationship between the pdf, rf and hrf (see [5])

$$f(x;\underline{\theta}) = h(x;\underline{\theta})R(x;\underline{\theta}).$$

Then

$$f(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1}e^{x^{\beta}} + \frac{ckx^{c-1}}{(1+x^{c})}\right]\frac{e^{\alpha(1-e^{x^{\beta}})}}{(1+x^{c})^{k}}, x > 0; \underline{\theta} > \underline{0}. \quad (8)$$

Also, the rhrf and the cumulative hazard rate function (chrf) of the C-BXII distribution are given, respectively, as follows:

$$r(x;\underline{\theta}) = \frac{f(x;\underline{\theta})}{F(x;\underline{\theta})} = \frac{\left[\alpha\beta x^{\beta-1}e^{x^{\beta}} + \frac{ckx^{c-1}}{(1+x^{c})}\right]e^{\alpha(1-e^{x^{\beta}})}}{(1+x^{c})^{k} - e^{\alpha(1-e^{x^{\beta}})}}, x > 0; \underline{\theta} > \underline{0}, \quad (9)$$

and

$$H(x;\underline{\theta}) = -\ln R(x;\underline{\theta}) = k\ln(1+x^{c}) - \alpha\left(1 - e^{x^{\beta}}\right), x > 0; \underline{\theta} > \underline{0}. \quad (10)$$

The suggested model can be used to simulate the lifespan of an item or individual which is subject to two independent failure modes, acting simultaneously on it and one of these failure modes will cause the failure of this item or individual. The lifetime of one of these failure modes has Chen distribution and the other has BXII distribution. Also, this additive model can be interpreted as a lifetime of a series system with two parts functioning independently, the lifetime of the first part has the Chen distribution and the latter has BXII distribution. Hence, the lifetime of the series system is the minimum of the lifetimes of the two items.

3. Graphical Description

In this subsection, graphical description of the pdf, hrf and rhrf of the C-BXII distribution are presented to demonstrate the adaptability of the proposed model.

Figure 1 displays the pdf of the C-BXII distribution for different parameter values, from which one can observe that the pdf of the C-BXII distribution can be bimodal, decreasing-unimodal, unimodal or decreasing. Figure 1 displays the pdf of C-BXII distribution for different parameter values, from which one can observe that the pdf of C-BXII distribution can be bimodal, decreasing-unimodal, unimodal or decreasing. Figure 2 exhibits the hrf of C-BXII distribution for certain selected parameter values. The hrf of C-BXII distribution exhibits different and important shapes, which are: increasing, decreasing, bathtub, modified bathtub and decreasing-unimodal shapes. Figure 3a,b show that the hrf has the modified bathtub shape, since the hrf increases then decreases and finally increases again. Also, Figure 3a exhibits the hrf of C-BXII distribution is a decreasing hrf followed by unimodal shape or a decreasing hrf and Figure 3b displays the bathtub hrf of C-BXII distribution.

While Figure 4 displays the rhrf of the C-BXII distribution for given parameter values. The rhrf can be decreasing or decreasing followed by a unimodal shape. From these plots, the flexibility and diversity in the shapes of the pdf and hrf of the C-BXII distribution can be demonstrated. Therefore, the C-BXII distribution can present a better fit for several types of lifespan data.

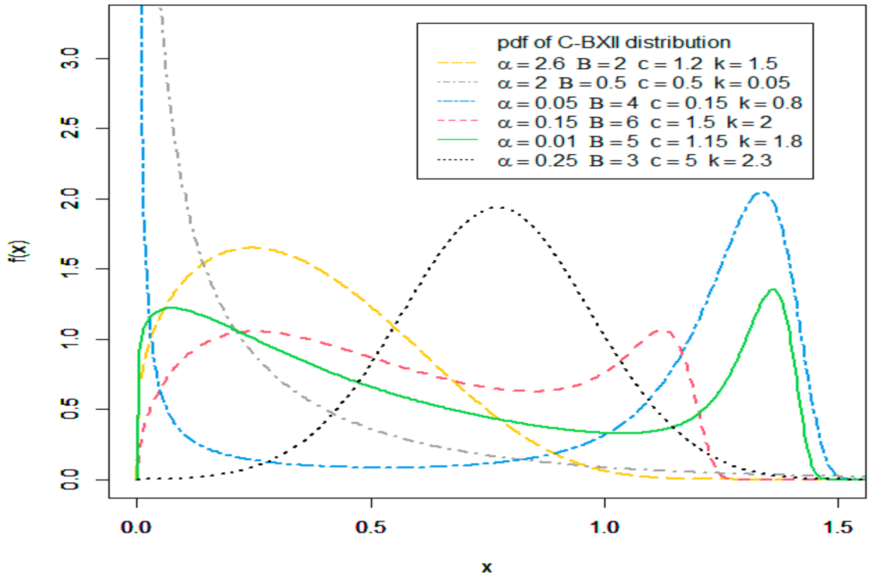


Figure 1. Plots of the C-BXII pdf.

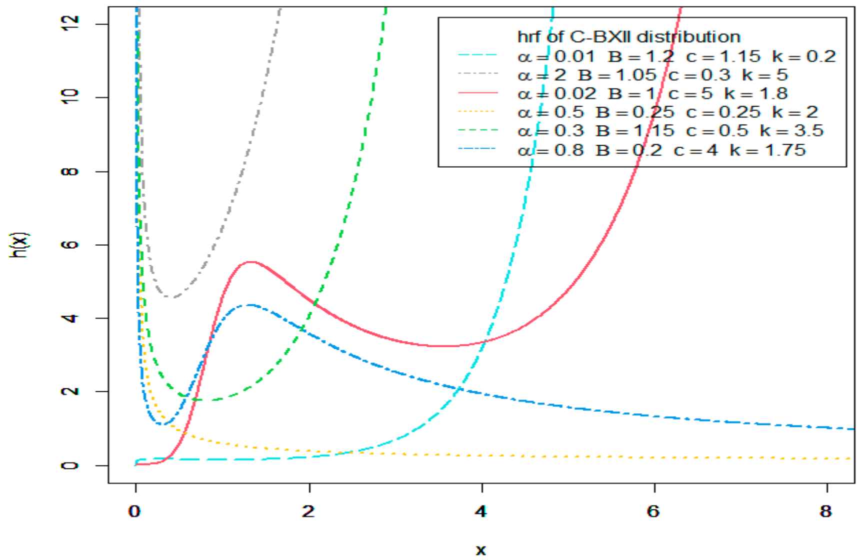
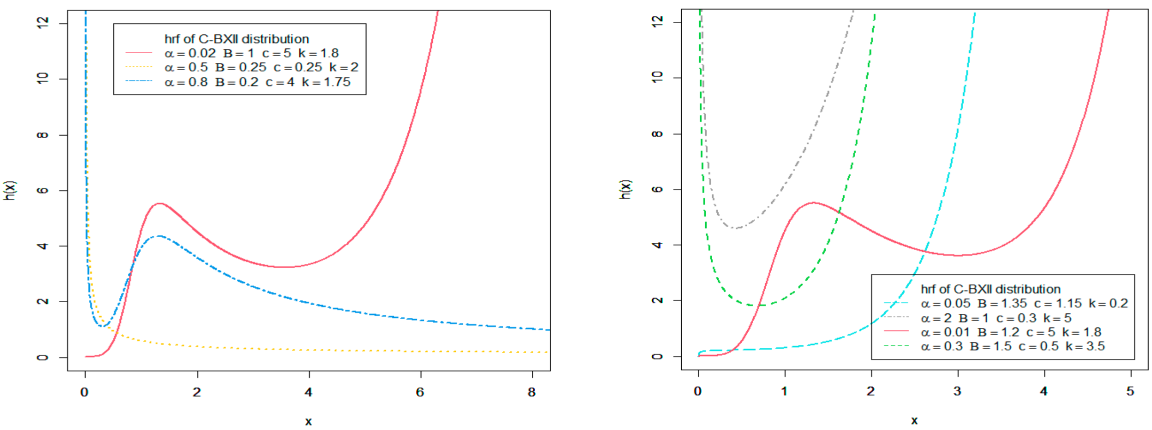


Figure 2. Plots of the C-BXII hrf.



(**a**) (**b**)

Figure 3. Plots of the C-BXII hrf.

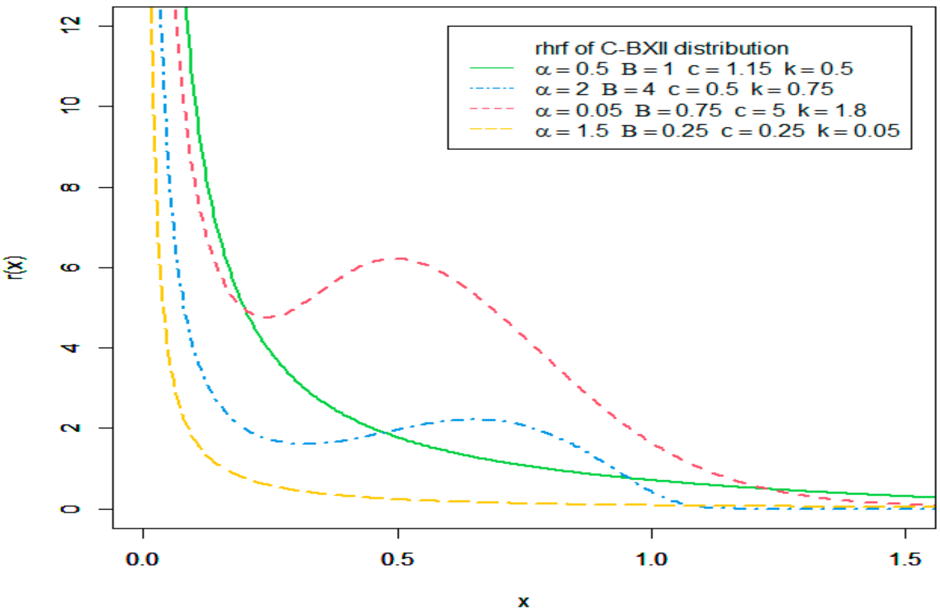


Figure 4. Plots of the C-BXII rhrf.

Limiting behavior of the hrf, $h(x;\underline{\theta})$, and its first derivative with respect to x can be taken into consideration to investigate the behavior of the hrf of the C-BXII distribution, which is shown in Figure 3.

The first derivative of $h(x;\underline{\theta})$ with respect to x can be derived using (7) as given below:

$$\overset{'}{h}(x;\underline{\theta}) = \frac{dh(x;\underline{\theta})}{dx} = \overset{'}{h}_1(x;\alpha,\beta) + \overset{'}{h}_2(x;c,k),$$

where

$$\overset{'}{h}_1(x;\alpha,\beta) = \alpha\beta x^{\beta-2}\left(\beta x^{\beta} + \beta - 1\right)e^{x^{\beta}},$$

and

$$\overset{'}{h}_2(x;c,k) = \frac{ck(c-1)x^{c-2}}{(1+x^c)} - \frac{c^2kx^{2c-2}}{(1+x^c)^2}.$$

The shape of the hrf of the C-BXII distribution can be characterized as follows:

a. For $\beta > 1$ and $c > 1$:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = 0, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = \infty,$$

and for $\beta = 1$ and $c > 1$:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = \alpha, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = \infty.$$

In this case $h_1(x;\alpha,\beta)$ is an increasing hrf while $h_2(x;c,k)$ is a unimodal hrf. So, the hrf of C-BXII distribution can be either increasing hrf so that the positive term for $\overset{'}{h}_1(x;\alpha,\beta)$ dominates the negative one for $\overset{'}{h}_2(x;c,k)$ and $\overset{'}{h}(x;\underline{\theta}) > 0$. Also, the hrf of C-BXII distribution can show the modified bathtub shape. In this case let x_0^* and x_0^{**} be two critical values of $h(x;\underline{\theta})$, so therefore for $x < x_0^*$, both $\overset{'}{h}_1(x;\alpha,\beta) > 0$ and $\overset{'}{h}_2(x;c,k) > 0$. So, $h(x;\underline{\theta})$ is increasing for $x < x_0^*$, that is $\overset{'}{h}(x;\underline{\theta}) > 0$. For $x_0^* < x \leq x_0^{**}$, there are two possibilities for $h_1(x;\alpha,\beta)$ and $h_2(x;c,k)$. The first is that the two functions are negative, $\overset{'}{h}_1(x;\alpha,\beta) < 0$ and $\overset{'}{h}_2(x;c,k) < 0$, or one function of them is negative and the other is positive, but the negative one dominates the positive one, therefore $h'(x;\underline{\theta}) < 0$, that is $h(x;\underline{\theta})$ is a decreasing hrf. For $x_0^{**} < x$, $h(x;\underline{\theta})$ is increasing hrf again because $h_1'(x;\alpha,\beta) > 0$ is positive and dominates the negative one $h_2'(x;c,k) < 0$. In brief, the hrf of C-BXII distribution presents the modified bathtub shape, that is, the hrf is increasing on $[0, x_0^*)$, decreasing on (x_0^*, x_0^{**}) and finally takes an increasing pattern on $(x_0^{**}, \infty]$. These two shapes of the hrf are displayed in Figure 3a,b.

b. For $\beta = 1$ and $c = 1$:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = \alpha + k, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = \infty,$$

and for $\beta > 1$ and $c = 1$:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = k, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = \infty,$$

In the present case, $h_1(x;\alpha,\beta)$ is an increasing hrf whereas $h_2(x;c,k)$ is a decreasing hrf. Therefore, the hrf of C-BXII distribution can be an increasing hrf so that the positive term for $h_1'(x;\alpha,\beta)$ dominates the negative one for $h_2'(x;c,k)$ and $h'(x;\underline{\theta}) > 0$.

c. For $\beta \geq 1$ and $c < 1$:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = \infty, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = \infty.$$

In the present case, $h_1'(x;\alpha,\beta) > 0$ and $h_2'(x;c,k) < 0$. For $x < x_0$, where x_0 is a critical point at which $h'(x;\underline{\theta}) = 0$, $h'(x;\underline{\theta}) < 0$. For $x > x_0$, $h'(x;\underline{\theta}) > 0$. Thus, the hrf of C-BXII distribution is a bathtub hrf as shown in Figure 3b.

d. For $\beta < 1$ and $c > 1$ and for $\beta < 1$ and $c \leq 1$:
Here in these two cases, the limiting behavior of the hrf, $h(x;\underline{\theta})$, is the same, that is:

$$\lim_{x\to 0^+} h(x;\underline{\theta}) = \infty, \quad \text{and} \quad \lim_{x\to\infty} h(x;\underline{\theta}) = 0,$$

and the hrf of C-BXII distribution is decreasing followed by unimodal shape or a decreasing hrf as shown in Figure 3a. For more explanation considers the case when:

- $\beta < 1$ and $c > 1$, In this case, $h(x;\underline{\theta})$ is a decreasing-unimodal hrf. Let x_0^* and x_0^{**} are two critical values for $h(x;\underline{\theta})$. When $x < x_0^*$, $h(x;\underline{\theta})$ is a decreasing hrf, $h'(x;\underline{\theta}) < 0$, and when $x_0^* < x \leq x_0^{**}$, the hrf of C-BXII distribution is increasing hrf, $h'(x;\underline{\theta}) > 0$. Finally, when $x_0^{**} < x$, $h(x;\underline{\theta})$ is decreasing again.
- $\beta < 1$ and $c \leq 1$, here $h(x;\underline{\theta})$ is a decreasing hrf, since $h_1(x;\alpha,\beta)$ is bathtub hrf and $h_2(x;c,k)$ is a decreasing hrf and $h_2(x;c,k)$ dominates the increasing part in $h_1(x;\alpha,\beta)$.

4. Statistical Properties

The quantile function and the mode, central and non-central moments, moment generating function, rth incomplete moment and inequality curves, mean residual life (MRL) and mean past life (MPL) are some important statistical characteristics of the C-BXII distribution that are investigated in this section. Also, mean time to failure (MTTF), mean time between failures (MTBF) and the availability (Av), Rényi entropy and Tsallis entropy (q-entropy), the order statistics and some sub-models of the proposed distribution are studied.

i. The quantile function and the mode
One can obtain the quantile function of the C-BXII distribution by inverting

$$R(x;\underline{\theta}) = 1 - q, 0 < q < 1.$$

So, the quantile function can be acquired through the solution of the following nonlinear equation

$$\alpha\left(1-e^{x_q^{\beta}}\right)-k\ln\left(1+x_q^{c}\right)-\ln(1-q)=0, 0<q<1. \quad (11)$$

The median of the C-BXII distribution, expressed by x_m, the first quartile, symbolized by $x_{0.25}$, and the third quartile, expressed by $x_{0.75}$, can be derived as particular cases of the quantile function by putting $q=0.5$, $q=0.25$ and $q=0.75$ into (11), respectively.

The mode of the C-BXII distribution is the value of x_0 which maximize $f(x;\underline{\theta})$. So, the following nonlinear equation can be solved numerically to get the mode of the C-BXII distribution,

$$\begin{aligned}&\left\{\alpha\beta x_0^{\beta-2}e^{x_0^{\beta}}\left[\beta x_0^{\beta}+\beta-1\right]+\frac{ckx_0^{c-2}}{\left(1+x_0^{c}\right)^2}\left[(c-1)\left(1+x_0^{c}\right)-cx_0^{c}\right]\right\}\\&\qquad-\left[\alpha\beta x_0^{\beta-1}e^{x_0^{\beta}}+\frac{ckx_0^{c-1}}{\left(1+x_0^{c}\right)}\right]^2\frac{e^{\alpha\left(1-e^{x_0^{\beta}}\right)}}{\left(1+x_0^{c}\right)^{k}}=0.\end{aligned} \quad (12)$$

The mathematical derivation of the mode of the C-BXII distribution is provided in Appendix A.

Table 1 exhibits some numerical outcomes of the first quartile, $x_{0.25}$, the median, x_m, and the third quartile, $x_{0.75}$, as particular cases of the quantile and the mode of the C-BXII distribution for various values for the parameter $\underline{\theta}$ using R programming language and software version 4.4.1. It is obvious from Table 1 that the C-BXII distribution can be unimodal or bimodal, this is shown clearly in Figure 1.

Table 1. Some quartiles and modes of the C-BXII distribution for various values for the parameter.

α	β	c	k	$x_{0.25}$	x_m	$x_{0.75}$	Mode	
2.6	2	1.2	1.8	0.1738	0.3213	0.4995	0.2057	-
0.05	4	0.15	0.8	0.0038	1.0691	1.2995	1.3373	-
0.15	6	1.5	2	0.2881	0.5529	0.9185	0.2511	1.1238
0.01	5	1.15	1.8	0.2178	0.5181	1.0980	0.0724	1.3608
0.25	3	5	2.3	0.6295	0.7694	0.9084	0.7712	-

ii. Central and non-central moments

The rth non-central moment of the C-BXII distribution is given by:

$$\mu_r'=\frac{r}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\left[\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+r}{c},k-\frac{\beta k+r}{c}\right)\right], r=1,2,\ldots, \quad (13)$$

where $\mathbf{B}(.,.)$ is the beta function and $0<\frac{\beta k+r}{c}<k$.

The derivation of the rth non-central moment of the C-BXII distribution is given in Appendix A.

By substituting $r=1$ into (13), the mean of the C-BXII distribution can be obtained as follows:

$$\mu=\frac{1}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\left[\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+1}{c},k-\frac{\beta k+1}{c}\right)\right], \quad (14)$$

where $0<\frac{\beta k+1}{c}<k$.

Substituting $r=2$ in (13), the second non-central moment of the C-BXII distribution is given as:

$$\mu_2'=\frac{2}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\left[\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+2}{c},k-\frac{\beta k+2}{c}\right)\right], \quad (15)$$

where $0 < \frac{\beta k+2}{c} < k$.

The variance of the C-BXII distribution can be obtained by substituting (14) and (15) into the following equation:

$$V(X) = \mu_2 = \mu_2' - \mu^2. \tag{16}$$

The coefficients kurtosis (CK), skewness (CS) and variation (CV), are given, respectively, by

$$CV = \frac{\sqrt{\mu_2}}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \tag{17}$$

$$CS = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{\left(\mu_2' - \mu^2\right)^{3/2}}, \tag{18}$$

and

$$Ck = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{\left(\mu_2' - \mu^2\right)^2}, \tag{19}$$

where μ, μ_2' and μ_2 are obtained, respectively, in (14), (15) and (16) and μ_3' and μ_4' can be derived, respectively, by setting $r = 3$ and $r = 4$ into (13).

Table 2 provides numerical findings for the variance, CV, CS, CK, and first four non-central moments of the C-BXII distribution for a certain parameter values. This indicates that both left (negative) and right (positive) skewness are covered by the C-BXII distribution.

Table 2. Moments of the C-BXII distribution for different parameters values.

α	β	c	k	μ	μ_2'	μ_3'	μ_4'	μ_2	CV	CS	Ck
2.6	2	1.2	1.8	0.3513	0.1722	0.1010	0.0666	0.0487	0.6284	0.5808	2.7403
0.05	4	0.15	0.8	0.7455	0.9149	1.1585	1.4891	0.3591	0.8039	−0.2744	1.2299
0.15	6	1.5	2	0.5948	0.4774	0.4383	0.4317	0.1236	0.5912	0.1682	1.7465
0.01	5	1.15	1.8	0.6342	0.6180	0.7023	0.8536	0.2158	0.7325	0.3647	1.6800
0.25	3	5	2.3	0.7684	0.6331	0.5517	0.5041	0.0427	0.2689	−0.0323	2.8953

iii. The moment generating function

The moment generating function of a random variable X with the C-BXII distribution, symbolized by $M_X(t)$, can be obtained as follows:

$$\begin{aligned} M_X(t) &= E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x;\underline{\theta})dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{r}{c}\frac{t^r}{r!}\binom{i}{j}(-1)^j \left[\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k + r}{c}, k - \frac{\beta k + r}{c}\right)\right], \end{aligned} \tag{20}$$

where $0 < \frac{\beta k+r}{c} < k$.

iv. Incomplete moments and inequality curves

For a random variable X with the C-BXII distribution, its rth incomplete moment may be obtained using

$$\begin{aligned} &\mu_r(t) = \int_0^t x^r f(x;\underline{\theta})dx \\ &= -t^r R(t;\underline{\theta}) + \int_0^t r x^{r-1} R(x;\underline{\theta})dx \\ &= \frac{-t^r e^{\alpha(1-e^{t^\beta})}}{(1+t^c)^k} + \frac{r}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j \frac{\alpha^i j^k}{i!k!}\mathbf{IB}_{(t^c)}\left(\frac{\beta k + r}{c}, k - \frac{\beta k + r}{c}\right), \end{aligned} \tag{21}$$

where $\mathbf{IB}_{(t^c)}\left(\frac{\beta k+r}{c}, k - \frac{\beta k+r}{c}\right)$ is a lower incomplete beta function and $0 < \frac{\beta k+r}{c} < k$.

The well-known Lorenz and Bonferroni curves are applied widely in several areas, including insurance, demography, economics, reliability research, and life testing. These curves are significant applications of the first incomplete moment. $L(p)$ and $B(p)$; the corresponding symbols for the Lorenz and Bonferroni curves, are defined below:

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{\mu(q)}{\mu}, \tag{22}$$

and

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{L(p)}{p}. \tag{23}$$

by substituting $r=1$ and $t=q$ into (21) and $q=F^{-1}(p)$ for $0<p<1$, then $\mu(q)$ the first incomplete moment can be obtained and μ is gotten from (14). (For more details see [32]).

For the C-BXII distribution, Lorenz and Bonferroni curves can be obtained, respectively, by

$$L(p)=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{IB}_{(q^c)}\left(\frac{\beta k+r}{c},\,k-\frac{\beta k+r}{c}\right)}{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+1}{c},\,k-\frac{\beta k+1}{c}\right)}, \tag{24}$$

and

$$B(p)=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\mathrm{i}}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{IB}_{(q^c)}\left(\frac{\beta k+r}{c},k-\frac{\beta k+r}{c}\right)}{p\sum_{i=0}^{\infty}\sum_{j=0}^{\mathrm{i}}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+1}{c},k-\frac{\beta k+1}{c}\right)}, \tag{25}$$

where $0<\frac{\beta k+1}{c}<k$.

v. The mean residual life and mean past life

The predicted extra life duration for a system or unit that is alive at age x_0 is represented by the MRL function, also known as the life expectation at age t, which is denoted by $m(x_0)$, it is provided by:

$$\begin{aligned} m(x_0) &= E(X-x_0|X>x_0)=\frac{1}{R(x_0;\underline{\theta})}\int_{x_0}^{\infty}R(x;\underline{\theta})dx \\ &= \frac{\left(1+x_0^c\right)^k}{c\exp\left[\alpha\left(1-e^{x_0^c}\right)\right]}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{IB}^{(x_0^c)}\left(\frac{\beta k+1}{c},\,k-\frac{\beta k+1}{c}\right), \end{aligned} \tag{26}$$

where $\mathbf{IB}^{(x_0^c)}\left(\frac{\beta k+1}{c},k-\frac{\beta k+1}{c}\right)$ is an upper incomplete beta and $0<\frac{\beta k+1}{c}<k$.

The mean waiting time or mean inactivity time, which is known as the mean reversed residual life function, or MPL, is denoted by $M(x_0)$. It is the amount of the time that has passed since a system or unit has failed, if the failure happened in $(0,x_0)$ and is given by:

$$\begin{aligned} M(x_0) &= E[(x_0-X)|X\le x_0]=\frac{1}{F(x_0;\underline{\theta})}\int_0^{x_0}F(x;\underline{\theta})dx \\ &= \frac{\left(1+x_0^c\right)^k}{\left(1+x_0^c\right)^k-exp\left[\alpha\left(1-e^{x_0^c}\right)\right]}\left[x_0-\frac{1}{c}\sum_{i=0}^{\infty}\sum_{j=0}^{\mathrm{i}}\sum_{k=0}^{\infty}\binom{i}{j}(-1)^j\frac{\alpha^i j^k}{i!k!}\mathbf{IB}_{(x_0^c)}\left(\frac{\beta k+1}{c},\,k-\frac{\beta k+1}{c}\right)\right], \end{aligned} \tag{27}$$

where $\mathbf{IB}_{(x_0^c)}\left(\frac{\beta k+1}{c},\,k-\frac{\beta k+1}{c}\right)$ is a lower incomplete beta function and $0<\frac{\beta k+1}{c}<k$.

The MRL and MPL of the C-BXII distribution are computed numerically using R software for some certain values of the parameters $\underline{\theta}=(\alpha,\beta,c,k)$ and for $x_0=0.1$ and $x_0=0.5$. Table 3 presents the numerical values. It is clear from Table 3 that the results of the MRL in the case of $x_0=0.5$ are smaller than their corresponding for $x_0=0.1$, whereas the results of the MPL are larger for $x_0=0.5$ than for $x_0=0.1$.

Table 3. Mean residual life and mean past life for selected values of the parameters and x_0.

α	β	c	k	x_0=0.1 *MRL*	x_0=0.1 *MPL*	x_0=0.5 *MRL*	x_0=0.5 *MPL*
2.6	2	1.2	1.5	0.3094	0.0441	0.1621	0.2423
			0.75	0.3523	0.0419	0.1738	0.2178
			0.5	0.3687	0.0405	0.1780	0.2077
		0.9	1.5	0.3035	0.0528	0.1660	0.2722
		0.7		0.3049	0.0606	0.1693	0.2984
	1.5	1.2		0.2643	0.0444	0.1582	0.2675
	1.0			0.2152	0.0514	0.1650	0.3184
1.8	2			0.3541	0.0447	0.2014	0.2421
0.9				0.4400	0.0456	0.2865	0.2468

vi. Mean time to failure, mean time between failures and availability

Reliability terminologies for predicting the lifespan of components include the MTTF, MTBF and the Av. These methods measure a failure rate and the consequent time of expected performance based on a set of data by presenting numerical results. Forecasting the MTTF, MTBF, and Av is also essential for developing and producing a maintainable system. Furthermore, customers can apply these reliability terms to help them choose which products to purchase.

The MTTF and MTBF for the C-BXII are determined, respectively, by:

$$MTTF = \int_0^\infty R(x;\underline{\theta})dx = \mu \tag{28}$$

and

$$MTBF = \frac{-x}{\ln R(x;\underline{\theta})} = \frac{x}{H(x;\underline{\theta})} = \frac{x}{k\ln(1+x^c) - \alpha\left(1 - e^{x^\beta}\right)}. \tag{29}$$

The Av is the probability that a component is effective at time x_0 and is defined as:

$$Av = \frac{MTTF}{MTBF}. \tag{30}$$

(See [33]).

Table 4 presents numerical results of MTTF, rf, MTBF, Av of the C-BXII for a certain specific parameter values $\underline{\theta} = (\alpha, \beta, c, k)$ and for $x_0 = 0.1$ and $x_0 = 0.5$. From Table 4 it is observed that:

Table 4. Mean time to failure, reliability, mean time between failure and availability for some values of the parameters and x_0.

α	β	c	k	*MTTF*	x_0=0.1 $R\left(x_0;\underline{\theta}\right)$	x_0=0.1 *MTBF*	x_0=0.1 *Av*	x_0=0.5 $R\left(x_0;\underline{\theta}\right)$	x_0=0.5 *MTBF*	x_0=0.5 *Av*
2.6	2	1.2	1.5	0.3701	0.8888	0.8481	0.4364	0.2779	0.3905	0.9478
			0.75	0.4249	0.9305	1.3885	0.3060	0.3644	0.4953	0.8578
			0.5	0.4461	0.9449	1.7630	0.2530	0.3989	0.5440	0.8201
		0.9	1.5	0.3378	0.8155	0.4902	0.6891	0.2510	0.3618	0.9337
		0.7		0.3104	0.7415	0.3344	0.9282	0.2327	0.3429	0.9051
	1.5	1.2		0.3147	0.8392	0.5704	0.5516	0.1931	0.3040	1.0000
	1.0			0.2336	0.6940	0.2738	0.8533	0.1077	0.2244	1.0000
1.8	2			0.4126	0.8960	0.9102	0.4533	0.3488	0.4747	0.8691
0.9				0.4934	0.9041	0.9918	0.4975	0.4504	0.6268	0.7872

- For fixed $\alpha = 2.6,\ \beta = 2$ and $c = 1.2$, as k decreases, MTTF, rf and MTBF increase, while Av decreases this is for $x_0 = 0.1$ and 0.5.
- For $x_0 = 0.1$ and 0.5 and for fixed $\alpha = 2.6,\ \beta = 2$ and $k = 1.5$, as c decreases, MTTF, rf, MTBF and Av decrease.
- For $x_0 = 0.1$ and 0.5 and for fixed $\alpha = 2.6,\ c = 1.2$ and $k = 1.5$, as β decreases, MTTF, rf and MTBF decrease, while Av increases.
- For fixed $\beta = 2,\ c = 1.2$ and $k = 1.5$, as α decreases, MTTF, rf, MTBF and Av increase for $x_0 = 0.1$, while for $x_0 = 0.5$, the MTTF, rf, MTBF also increase as α decreases, except the Av decreases.
- For all parameters values, the results of the MTBF when $x_0 = 0.5$ are smaller than the case of $x_0 = 0.1$, whereas the results of the Av for $x_0 = 0.5$ are larger than the case of $x_0 = 0.1$.

vii. Entropy measures

A random variable's uncertainty, randomness, or variation can be measured by entropy. Rényi entropy which was proposed by [34] is considered one of the most significant entropy metrics and is an expansion of Shannon entropy. It is defined by

$$I_\delta(x) = \frac{1}{1-\delta}\ln\int_{-\infty}^{\infty} f^\delta(x)dx, \delta \neq 1, \delta > 0. \tag{31}$$

When X is having the C-BXII distribution, Rényi entropy is given by:

$$I_\delta(x;\underline{\theta}) = \frac{1}{(1-\delta)}\ln\left[\begin{array}{c}\frac{1}{c}\sum_{m=0}^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{\delta}{m}\binom{i}{j}\frac{(-1)^i\alpha^{i+m}j^k\delta^i\beta^m}{i!k!}\\ \times\mathbf{B}\left(\frac{1}{c}[(c-1)(\delta+m)+\beta k+1], \delta(k+1)-m-\frac{1}{c}[(c-1)(\delta+m)+\beta k+1]\right)\end{array}\right], \delta \neq 1, \delta > 0, \tag{32}$$

where $0 < \frac{1}{c}[(c-1)(\delta+m)+\beta k+1] < \delta(k+1) - m$.

As $\delta \to 1$, Rényi entropy tends to Shannon entropy.

Another entropy measure is Tsallis entropy (also called q-entropy) introduced by [35] is defined by:

$$I_q(x) = \frac{1}{(1-q)}\left[1 - \int_{-\infty}^{\infty}(f(x))^q dx\right];\ q \neq 1, q > 0. \tag{33}$$

If X is a random variable which has the C-BXII distribution Tsallis entropy is given by:

$$\mathrm{I}_q(x;\underline{\theta}) = \frac{1}{1-q}\times\ln\left\{1 - \left[\begin{array}{c}\frac{1}{c}\sum_{m=0}^{q}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{q}{m}\binom{i}{j}\frac{(-1)^i\alpha^{i+m}j^k q^i\beta^m}{i!k!}\\ \times\mathbf{B}\left(\frac{1}{c}[(c-1)(q+m)+\beta k+1], q(k+1)-m-\frac{1}{c}[(c-1)(q+m)+\beta k+1]\right)\end{array}\right]\right\}, \quad q \neq 1, q > 0, \tag{34}$$

where $0 < \frac{1}{c}[(c-1)(q+m)+\beta k+1] < q(k+1) - m$.

viii. The order statistics

Given i.i.d. random variables $X_1,\ X_2,\ \ldots,\ X_n$, which have the C-BXII distribution. The associated order statistics are then $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$, and the pdf of the sth order statistic is as follows:

$$f_{s:n}(x;\underline{\theta}) = \sum_{j=0}^{s-1} C_{s,n,j}h(x;\underline{\theta})[R(x;\underline{\theta})]^{j+n-s+1},\ x_{(s)} > 0, \tag{35}$$

where $C_{s,n,j} = \frac{n!(-1)^j}{j!(s-j-1)!(n-s)!}$.

Therefore, the following is the pdf of the C-BXII distribution's sth order statistic:

Substituting (5) and (7) into (35), consequently the following is the pdf of the C-BXII distribution's sth order statistic:

$$f_{s:n}(x;\underline{\theta}) = \sum_{j=0}^{s-1} C_{s,n,j}\left[\left(\alpha\beta x_{(s)}^{\beta-1}e^{x_{(s)}^{\beta}} + \frac{ckx_{(s)}^{c-1}}{1+x_{(s)}^{c}}\right)\right] \times\exp\left\{(j+n-s+1)\left[\alpha\left(1-e^{x_{(s)}^{\beta}}\right) - s\,ln\left(1+x_{(s)}^{c}\right)\right]\right\}, \quad x_{(s)} > 0. \tag{36}$$

Special cases

a. When $s = 1$, the pdf of the smallest order statistics can be derived as

$$f_{1:n}(x;\underline{\theta}) = n\left[\left(\alpha\beta x_{(1)}^{\beta-1}e^{x_{(1)}^{\beta}} + \frac{ckx_{(1)}^{c-1}}{1+x_{(1)}^{c}}\right)\right]\exp\left\{n\left[\alpha\left(1-e^{x_{(1)}^{\beta}}\right) - kln\left(1+x_{(1)}^{c}\right)\right]\right\},\ x_{(1)} > 0. \tag{37}$$

b. If $s = n$, one can obtain the pdf of the largest order statistics as

$$f_{n:n}(x;\underline{\theta}) = \sum_{j=0}^{n-1} C_{n,n,j}\left[\left(\alpha\beta x_{(n)}^{\beta-1}e^{x_{(n)}^{\beta}} + \frac{ckx_{(n)}^{c-1}}{1+x_{(n)}^{c}}\right)\right] \times\exp\left\{(j+1)\left[\alpha\left(1-e^{x_{(n)}^{\beta}}\right) - k\ln\left(1+x_{(n)}^{c}\right)\right]\right\}, x_{(n)} > 0, \tag{38}$$

where $C_{n,j} = \frac{n!(-1)^j}{j!(n-j-1)!}$.

ix. Some sub-models

There are some distributions that can be obtained as sub-models of the C-BXII distribution and are summarized in Table 5.

Table 5. The C-BXII distribution's sub-models.

Parameter	The Resulting Distribution	Pdf
$c = 1$	Chen-compound exponential distribution	$f(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1}e^{x^\beta} + \frac{k}{(1+x)}\right]\left[\frac{e^{\alpha(1-e^{x^\beta})}}{(1+x)^k}\right], x > 0; \underline{\theta} > \underline{0}.$
$c = 2$	Chen-compound Rayleigh distribution	$f(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1}e^{x^\beta} + \frac{2xk}{(1+x^2)}\right]\left[\frac{e^{\alpha(1-e^{x^\beta})}}{(1+x^2)^k}\right], x > 0; \underline{\theta} > \underline{0}.$
$k = 1$	Chen-log logistic	$f(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1}e^{x^\beta} + \frac{cx^{c-1}}{(1+x^c)}\right]\left[\frac{e^{\alpha(1-e^{x^\beta})}}{(1+x^c)}\right], x > 0; \underline{\theta} > \underline{0}.$
$\alpha \to 0^+$	BXII distribution	$f(x;c,k) = \frac{ckx^{c-1}}{(1+x^c)^{k+1}},\quad x > 0; c,k > 0.$
$k \to 0^+$	Chen distribution	$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}e^{x^\beta}e^{\alpha(1-e^{x^\beta})},\quad x > 0; \alpha,\beta > 0.$
$\alpha \to 0^+$ and $c = 1$	Lomax distribution	$f(x;k) = \frac{k}{(1+x)^{k+1}},\quad x > 0; k > 0.$
$\alpha \to 0^+$ and $c = 2$	Compound Rayleigh distribution	$f(x;k) = \frac{2kx}{(1+x^2)^{k+1}},\quad x > 0; k > 0.$
$\alpha \to 0^+$ and $k = 1$	Log logistic distribution	$f(x;c) = \frac{cx^{c-1}}{(1+x^c)^2},\quad x > 0; c > 0.$

5. Maximum Likelihood Estimation

In this subsection, point ML estimators of the parameters, rf, hrf and rhrf are obtained. Furthermore, ACIs of the parameters, rf, hrf and rhrf are derived.

5.1. Point Estimation

Given a random sample of size n from the C-BXII distribution with the parameter vector $\underline{\theta} = (\alpha,\beta,c,k)$ and $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$, the likelihood function is as follows:

$$L(\underline{\theta};\underline{x}) = \left[\prod_{i=1}^{n} f\left(x_{(i)};\underline{\theta}\right)\right] = \prod_{i=1}^{n}\left[\alpha\beta\, x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}} + \frac{ckx_{(i)}^{c-1}}{\left(1+x_{(i)}^{c}\right)}\right] e^{\alpha\sum_{i=1}^{n}\left(1-e^{x_{(i)}^{\beta}}\right)} \prod_{i=1}^{n}\left(1+x_{(i)}^{c}\right)^{-k}. \quad (39)$$

Below is the likelihood function's natural logarithm:

$$\ell = lnL(\underline{\theta};\underline{x}) = \sum_{i=1}^{n} \ln\left[\alpha\beta x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}} + \frac{ckx_{(i)}^{c-1}}{\left(1+x_{(i)}^{c}\right)}\right] + \alpha\sum_{i=1}^{n}\left(1-e^{x_{(i)}^{\beta}}\right) - k\sum_{i=1}^{n}\ln\left(1+x_{(i)}^{c}\right). \quad (40)$$

With respect to the parameters α, β, k and c, the natural logarithm likelihood function $\ell = \ln L(\underline{\theta};\underline{x})$ can be differentiated as follows:

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{n} \frac{h_{\alpha}\left(x_{(i)};\underline{\theta}\right)}{h\left(x_{(i)};\underline{\theta}\right)} + \sum_{i=1}^{n}\left(1-e^{x_{(i)}^{\beta}}\right), \quad (41)$$

$$\frac{\partial\ell}{\partial\beta} = \sum_{i=1}^{n} \frac{h_{\beta}\left(x_{(i)};\underline{\theta}\right)}{h\left(x_{(i)};\underline{\theta}\right)} - \alpha\sum_{i=1}^{n} x_{(i)}^{\beta} e^{x_{(i)}^{\beta}} lnx_{(i)}, \quad (42)$$

$$\frac{\partial\ell}{\partial c} = \sum_{i=1}^{n} \frac{h_{c}\left(x_{(i)};\underline{\theta}\right)}{h\left(x_{(i)};\underline{\theta}\right)} - k\sum_{i=1}^{n} \frac{x_{(i)}^{c} lnx_{(i)}}{\left(1+x_{(i)}^{c}\right)}, \quad (43)$$

and

$$\frac{\partial\ell}{\partial k} = \sum_{i=1}^{n} \frac{h_{k}\left(x_{(i)};\underline{\theta}\right)}{h\left(x_{(i)};\underline{\theta}\right)} - \sum_{i=1}^{n} ln\left(1+x_{(i)}^{c}\right), \quad (44)$$

where $h\left(x_{(i)};\underline{\theta}\right)$ is defined in (7),

$$\begin{aligned}
h_{\alpha}\left(x_{(i)};\underline{\theta}\right) &= \tfrac{\partial h\left(x_{(i)};\underline{\theta}\right)}{\partial\alpha} = \beta x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}}, \\
h_{\beta}\left(x_{(i)};\underline{\theta}\right) &= \tfrac{\partial h\left(x_{(i)};\underline{\theta}\right)}{\partial\beta} = \alpha x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}}\left[\beta lnx_{(i)}\left(x_{(i)}^{\beta}+1\right)+1\right], \\
h_{c}\left(x_{(i)};\underline{\theta}\right) &= \tfrac{\partial h\left(x_{(i)};\underline{\theta}\right)}{\partial c} = \tfrac{kx_{(i)}^{c-1}}{1+x_{(i)}^{c}}\left[x_{(i)}^{c} - clnx_{(i)} + 1\right],
\end{aligned}$$

and

$$h_{k}\left(x_{(i)};\underline{\theta}\right) = \frac{\partial h\left(x_{(i)};\underline{\theta}\right)}{\partial k} = \frac{cx_{(i)}^{c-1}}{1+x_{(i)}^{c}}.$$

Equating the nonlinear likelihood Equations (41)–(44) to zero and solving numerically the maximum likelihood estimates can be evaluated.

Utilizing the invariance property of the ML estimators, where the parameters $\underline{\theta} = (\alpha, \beta, c, k)$ can be replaced by their ML estimators, so the ML estimators of $R(x;\underline{\theta})$, $h(x;\underline{\theta})$ and $r(x;\underline{\theta})$ can be derived.

5.2. Asymptotic Confidence Intervals

Since the ML estimators of the parameters, $\underline{\theta} = (\alpha, \beta, c, k)$, cannot be obtained in closed forms, then exact distributions of the ML estimators $\hat{\underline{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{c}, \hat{k}\right)$ cannot be obtained.

Hence, the asymptotic distribution of the ML estimators can be used to obtain the ACIs of the parameters of the C-BXII distribution. The asymptotic variance-covariance matrix is obtained by the inverse of the asymptotic Fisher information matrix in the following manner: The ML estimators are asymptotically normal with mean (α, β, c, k).

$$\tilde{I}^{-1}(\underline{\theta})\Big|_{\hat{\underline{\theta}}} \simeq \begin{pmatrix} \tilde{var}(\hat{\alpha}) & \tilde{cov}(\hat{\alpha},\hat{\beta}) & \tilde{cov}(\hat{\alpha},\hat{c}) & \tilde{cov}(\hat{\alpha},\hat{k}) \\ \tilde{cov}(\hat{\alpha},\hat{\beta}) & \tilde{var}(\hat{\beta}) & \tilde{cov}(\hat{\beta},\hat{c}) & \tilde{cov}(\hat{\beta},\hat{k}) \\ \tilde{cov}(\hat{\alpha},\hat{c}) & \tilde{cov}(\hat{\beta},\hat{c}) & \tilde{var}(\hat{c}) & \tilde{cov}(\hat{c},\hat{k}) \\ \tilde{cov}(\hat{\alpha},\hat{k}) & \tilde{cov}(\hat{\beta},\hat{k}) & \tilde{cov}(\hat{c},\hat{k}) & \tilde{var}(\hat{k}) \end{pmatrix}, \quad (45)$$

where the derivatives of the I_{ij} elements of the asymptotic Fisher information matrix are given in Appendix A.

Consequently, the $(1-\omega)100\%$ bounds of the ACIs of the parameters $\underline{\theta} = (\alpha, \beta, k, c)$ are given by:

$$\hat{\alpha} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{\alpha})}, \hat{\beta} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{\beta})}, \hat{c} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{c})} \text{and} \hat{k} \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{k})}, \quad (46)$$

where $Z_{(1-\frac{\omega}{2})}$ is the $(1-\omega)100\%$ percentage point of the standard normal distribution.

To obtain the ACIs of the rf, hrf and rhrf of the C-BXII distribution, variances of the ML estimators of the rf, hrf and rhrf are needed. Therefore, the delta method discussed in [36] and used by [17,22,37–39] can be used to derive the asymptotic variances of $\hat{R}(x;\hat{\underline{\theta}})$, $\hat{h}(x;\hat{\underline{\theta}})$ and $\hat{r}(x;\hat{\underline{\theta}})$.

The asymptotic variances of $\hat{R}(x;\hat{\underline{\theta}})$, $\hat{h}(x;\hat{\underline{\theta}})$ and $\hat{r}(x;\hat{\underline{\theta}})$ can be obtained, respectively, as:

$$\tilde{var}(\hat{R}(x;\hat{\underline{\theta}})) = \xi' \tilde{I}^{-1}(\underline{\theta})\xi\Big|_{\hat{\underline{\theta}}}, \tilde{var}\Big(\hat{h}(x;\hat{\underline{\theta}})\Big) = \eta' \tilde{I}^{-1}(\underline{\theta})\eta\Big|_{\hat{\underline{\theta}}},$$

and

$$\tilde{var}(\hat{r}(x;\hat{\underline{\theta}})) = \varphi' \tilde{I}^{-1}(\underline{\theta})\varphi\Big|_{\hat{\underline{\theta}}}, \quad (47)$$

where $\xi' = \begin{pmatrix} R_\alpha(x;\underline{\theta}) & R_\beta(x;\underline{\theta}) & R_c(x;\underline{\theta}) & R_k(x;\underline{\theta}) \end{pmatrix}$ is the first partial differentiation of the rf with respect to α, β, c and k, $\eta' = \begin{pmatrix} h_\alpha(x;\underline{\theta}) & h_\beta(x;\underline{\theta}) & h_c(x;\underline{\theta}) & h_k(x;\underline{\theta}) \end{pmatrix}$ is the first partial differentiation of the hrf with respect to α, β, c and k and $\varphi' = \begin{pmatrix} r_\alpha(x;\underline{\theta}) & r_\beta(x;\underline{\theta}) & r_c(x;\underline{\theta}) & r_k(x;\underline{\theta}) \end{pmatrix}$ is the first partial differentiation of the rhrf with respect to α, β, c and k.

Thus, the $(1-\omega)100\%$ bounds of the ACIs of the rf, hrf and rhrf are:

$$\hat{R}(x;\hat{\underline{\theta}}) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{R}(x;\hat{\underline{\theta}}))}, \hat{h}(x;\hat{\underline{\theta}}) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}\Big(\hat{h}(x;\hat{\underline{\theta}})\Big)},$$

and

$$\hat{r}(x;\hat{\underline{\theta}}) \pm Z_{(1-\frac{\omega}{2})}\sqrt{\tilde{var}(\hat{r}(x;\hat{\underline{\theta}}))}. \quad (48)$$

6. Simulation

Through a simulation study, the performance of the ML estimates of the parameters, rf, hrf, and rhrf of the C-BXII distribution is assessed in this section as follows:

(a) Two sets of parameters are used in the simulation study.

$$I : (\alpha = 2.6, \beta = 2, c = 1.2, k = 1.5),$$

and

$$II : (\alpha = 2, \beta = 0.5, c = 0.5, k = 0.05).$$

(b) Random samples are generated from the C-BXII distribution for various sample sizes $(n = 30,\ 60,\ 100,\ 200,\ 500)$

(c) Using Mathematica 11, the simulation study is carried out with a number of replications $NR = 1000$.

(d) The parameters' ACI bounds, rf with their lengths, variances, ML averages *estimated risks* (ER), *relative errors* (RE) and *relative absolute biases* (RAB) are shown in Tables 6 and 7. The estimated risks (ER), relative errors (RE), and relative absolute biases (RAB) are calculated as follows:

Table 6. ML averages, estimated risks, relative errors, Relative absolute biases, variances and 95% ACI bounds and their lengths of the parameters of the C-BXII distribution for different n, $NR = 1000$ and $(\alpha = 2.6,\ \beta = 2,\ c = 1.2,\ k = 1.5)$.

n	$\underline{\theta}$	Average	ER	RE	RAB	Variance	UL	LL	Length
30	α	2.5985	0.4114	0.2467	0.0006	0.4114	3.8557	1.3414	2.5143
	β	1.9906	0.1071	0.1636	0.0047	0.1070	2.6318	1.3494	1.2824
	c	1.2640	0.0537	0.1932	0.0533	0.0496	1.7006	0.8273	0.8733
	k	1.6478	0.3750	0.4083	0.0985	0.3532	2.8126	0.4830	2.3297
60	α	2.5479	0.2846	0.2052	0.0200	0.2819	3.5886	1.5073	2.0813
	β	1.9810	0.0729	0.1350	0.0095	0.0725	2.5089	1.4531	1.0558
	c	1.2511	0.0365	0.1592	0.0426	0.0339	1.6120	0.8903	0.7217
	k	1.6402	0.3603	0.4002	0.0934	0.3407	2.7841	0.4962	2.2879
100	α	2.5596	0.2156	0.1786	0.0156	0.2139	3.4661	1.6530	1.8130
	β	2.0066	0.0598	0.1222	0.0033	0.0597	2.4856	1.5276	0.9579
	c	1.2472	0.0288	0.1414	0.0393	0.0266	1.5667	0.9277	0.6389
	k	1.6297	0.3546	0.3970	0.0865	0.3377	2.7688	0.4907	2.2781
200	α	2.5809	0.1299	0.1386	0.0074	0.1295	3.2862	1.8755	1.4107
	β	2.0138	0.0484	0.1100	0.0070	0.0482	2.4442	1.5835	0.8608
	c	1.2264	0.0177	0.1110	0.0220	0.0170	1.4822	0.9706	0.5116
	k	1.5931	0.3067	0.3692	0.0621	0.2981	2.6632	0.5230	2.1401
500	α	2.5732	0.0642	0.0975	0.0103	0.0635	3.0672	2.0792	0.9880
	β	2.0213	0.0463	0.1076	0.0106	0.0459	2.4410	1.6015	0.8395
	c	1.2165	0.0112	0.0882	0.0137	0.0109	1.4215	1.0115	0.4100
	k	1.5813	0.2689	0.3457	0.0542	0.2623	2.5850	0.5775	2.0075

Table 7. ML averages, estimated risks, relative errors, relative absolute biases, variances and 95% ACIs and their lengths of the parameters of the C-BXII distribution for different n, $NR = 1000$ and $(\alpha = 2,\ \beta = 0.5,\ c = 0.5,\ k = 0.05)$.

n	$\underline{\theta}$	Average	ER	RE	RAB	Variance	UL	LL	Length
30	α	2.0696	0.1523	0.1951	0.0348	0.1474	2.8222	1.3171	1.5051
	β	0.5074	0.0061	0.1567	0.0148	0.0061	0.6603	0.3545	0.3059
	c	0.5100	0.0149	0.2440	0.0201	0.0148	0.7483	0.2717	0.4766
	k	0.0233	0.0155	2.4870	0.5349	0.0148	0.2613	0	0.2613
60	α	2.0389	0.0865	0.1471	0.0194	0.0850	2.6103	1.4675	1.1428
	β	0.5062	0.0031	0.1111	0.0123	0.0031	0.6144	0.3979	0.2165
	c	0.5067	0.0082	0.1809	0.0134	0.0081	0.6835	0.3299	0.3637
	k	0.0242	0.0066	1.6297	0.5167	0.0060	0.1756	0	0.1756
100	α	2.0346	0.0478	0.1093	0.0173	0.0466	2.4578	1.6114	0.8464
	β	0.5034	0.0019	0.0865	0.0068	0.0019	0.5880	0.4189	0.1691
	c	0.5073	0.0072	0.1693	0.0145	0.0071	0.6726	0.3419	0.3307
	k	0.0262	0.0090	1.8974	0.4762	0.0084	0.2062	0	0.2062
200	α	2.0261	0.0273	0.0827	0.0130	0.0267	2.3461	1.7061	0.6400
	β	0.5007	0.0009	0.0605	0.0014	0.0009	0.5600	0.4414	0.1186
	c	0.5022	0.0041	0.1284	0.0044	0.0041	0.6280	0.3764	0.2516
	k	0.0333	0.0062	1.5804	0.3350	0.0060	0.1846	0	0.1846
500	α	2.0071	0.0107	0.0518	0.0036	0.0107	2.2095	1.8048	0.4048
	β	0.4997	0.0004	0.0386	0.0007	0.0004	0.5375	0.4618	0.0757
	c	0.5042	0.0035	0.1179	0.0083	0.0035	0.6194	0.3889	0.2304
	k	0.0333	0.0028	1.0631	0.3333	0.0026	0.1323	0	0.1979

$$ER(\hat{\theta}) = \frac{\sum_{i=1}^{NP} \left(\hat{\theta}_i - \theta\right)^2}{NR},$$

$$RE(\hat{\theta}) = \frac{\sqrt{ER}}{\theta},$$

and

$$RAB(\hat{\theta}) = \frac{\left|bias(\hat{\theta})\right|}{\theta},$$

where

$$bias(\hat{\theta}) = \hat{\theta}_i - \theta.$$

(e) The ML averages, ERs, REs, RABs variances of the rf, hrf and rhrf as well as with their ACI bounds and the lengths of the ACIs at time $x_0 = 0.5$ are shown in Tables 8 and 9.

(f) Tables 6–9 are graphically shown in Figures 5–8.

Concluding remarks

Table 8. ML averages, estimated risks, relative errors, Relative absolute biases, variances and 95% ACI bounds of the rf, hrf and rhrf of the C-BXII distribution for different n, $NR = 1000$, $(\alpha = 2.6,\ \beta = 2,\ c = 1.2,\ k = 1.5)$ and $x_0 = 0.5$.

n	*Function*	Average	ER	RE	RAB	Variance	UL	LL	Length
30	$R(x_0;\underline{\theta})$	0.2738	0.0030	0.0905	0.0146	0.0029	0.3802	0.1675	0.2127
	$h(x_0;\underline{\theta})$	4.5181	0.4940	0.0374	0.0198	0.4863	5.8849	3.1513	2.7336
	$r(x_0;\underline{\theta})$	1.6864	0.1080	0.4123	0.0109	0.1076	2.3293	1.0434	1.2859
60	$R(x_0;\underline{\theta})$	0.2736	0.0015	0.1939	0.0156	0.0015	0.3494	0.1977	0.1517
	$h(x_0;\underline{\theta})$	4.4843	0.2962	0.1578	0.0122	0.2933	5.5457	3.4229	2.1228
	$r(x_0;\underline{\theta})$	1.6777	0.0546	0.3192	0.0160	0.0538	2.1324	1.2230	0.9094
100	$R(x_0;\underline{\theta})$	0.2760	0.0011	0.1401	0.0069	0.0011	0.3403	0.2117	0.1286
	$h(x_0;\underline{\theta})$	4.4768	0.2072	0.1228	0.0105	0.2051	5.3643	3.5892	1.7751
	$r(x_0;\underline{\theta})$	1.6973	0.0341	0.2670	0.0045	0.0340	2.0587	1.3359	0.7228
200	$R(x_0;\underline{\theta})$	0.2750	0.0006	0.1182	0.0106	0.0005	0.3207	0.2293	0.0914
	$h(x_0;\underline{\theta})$	4.4727	0.1079	0.1028	0.0096	0.1061	5.1111	3.8342	1.2769
	$r(x_0;\underline{\theta})$	1.6924	0.0207	0.1927	0.0074	0.0205	1.9729	1.4118	0.5611
500	$R(x_0;\underline{\theta})$	0.2759	0.0002	0.0555	0.0073	0.0002	0.3059	0.2459	0.0600
	$h(x_0;\underline{\theta})$	4.4492	0.0457	0.0482	0.0043	0.0453	4.8664	4.0320	0.8344
	$r(x_0;\underline{\theta})$	1.6932	0.0079	0.1254	0.0069	0.0077	1.8653	1.5211	0.3442

Table 9. ML averages, estimated risks, relative errors, relative absolute biases, variances and 95% ACI bounds and their lengths of the rf, hrf and rhrf of the C-BXII distribution for different n, $NR = 1000$, $(\alpha = 2,\ \beta = 0.5,\ c = 0.5,\ k = 0.05)$ and $x_0 = 0.5$.

n	*Function*	Average	ER	RE	RAB	Variance	UL	LL	Length
30	$R(x_0;\underline{\theta})$	0.1282	0.0024	0.3940	0.0036	0.0290	0.0024	0.2241	0.0323
	$h(x_0;\underline{\theta})$	2.9844	0.4430	0.2304	0.0955	0.0331	0.4339	4.2755	1.6933
	$r(x_0;\underline{\theta})$	0.4156	0.0119	1.6193	0.0045	0.0110	0.0119	0.6290	0.2021
60	$R(x_0;\underline{\theta})$	0.1278	0.0013	0.2861	0.0033	0.0262	0.0013	0.1974	0.0583
	$h(x_0;\underline{\theta})$	2.9457	0.2527	0.1740	0.0568	0.0197	0.2495	3.9247	1.9666
	$r(x_0;\underline{\theta})$	0.4179	0.0058	1.2231	0.0068	0.0166	0.0057	0.5663	0.2695
100	$R(x_0;\underline{\theta})$	0.1253	0.0006	0.2685	0.0007	0.0055	0.0006	0.1746	0.0759
	$h(x_0;\underline{\theta})$	2.9347	0.1366	0.1663	0.0458	0.0159	0.1345	3.6536	2.2159
	$r(x_0;\underline{\theta})$	0.4134	0.0035	0.8993	0.0023	0.0057	0.0035	0.5290	0.2977
200	$R(x_0;\underline{\theta})$	0.1240	0.0003	0.2022	−0.0006	0.0045	0.0003	0.1592	0.0888
	$h(x_0;\underline{\theta})$	2.9205	0.0771	0.1280	0.0316	0.0110	0.0761	3.4612	2.3800
	$r(x_0;\underline{\theta})$	0.4094	0.0017	0.6755	−0.0016	0.0039	0.0017	0.4893	0.3296
500	$R(x_0;\underline{\theta})$	0.1253	0.0001	0.1443	0.0007	0.0058	0.0001	0.1475	0.1030
	$h(x_0;\underline{\theta})$	2.8916	0.0308	0.0961	0.0027	0.0009	0.0308	3.2353	2.5479
	$r(x_0;\underline{\theta})$	0.4125	0.0006	0.4267	0.0015	0.0036	0.0006	0.4619	0.3632

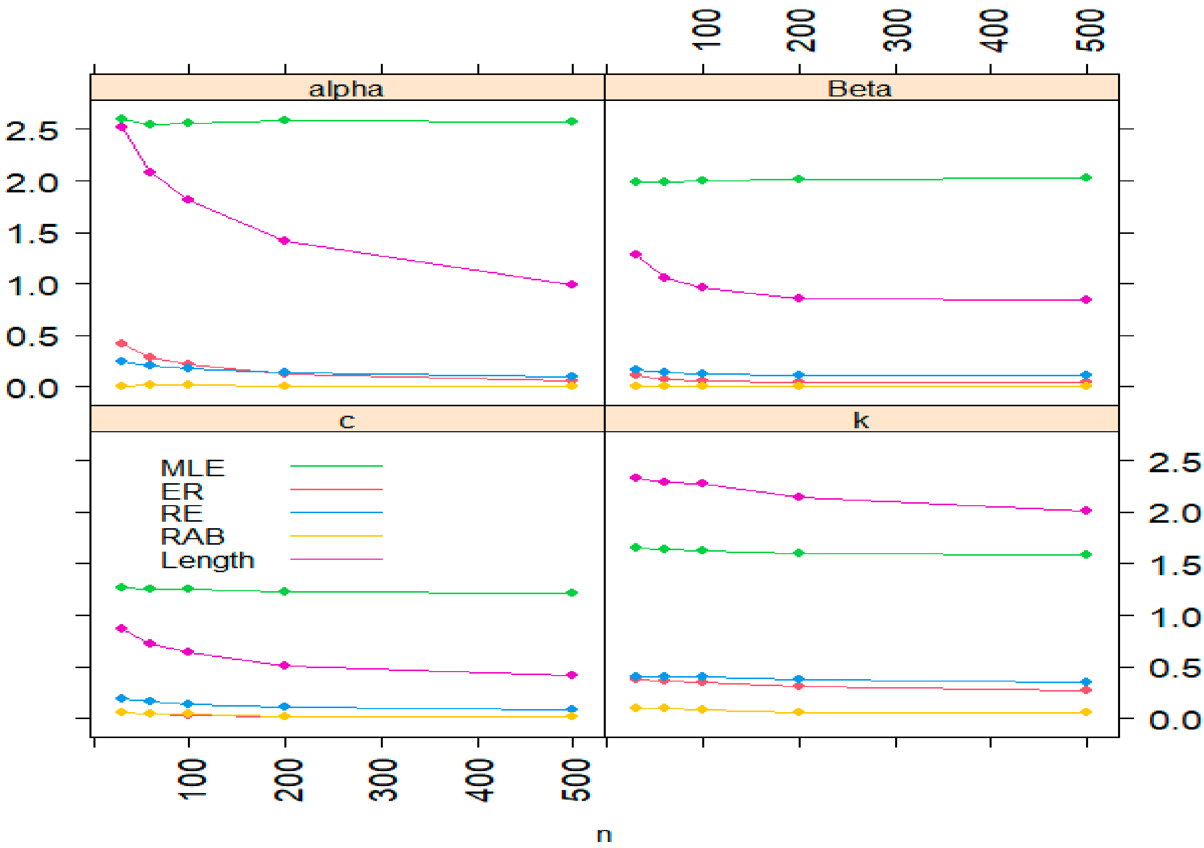


Figure 5. Plots of the ML averages, estimated risks, relative errors, Relative absolute biases, variances and the ACI lengths of the parameters, for $(\alpha = 2.6, \beta = 2, c = 1.2, k = 1.5)$ versus n.

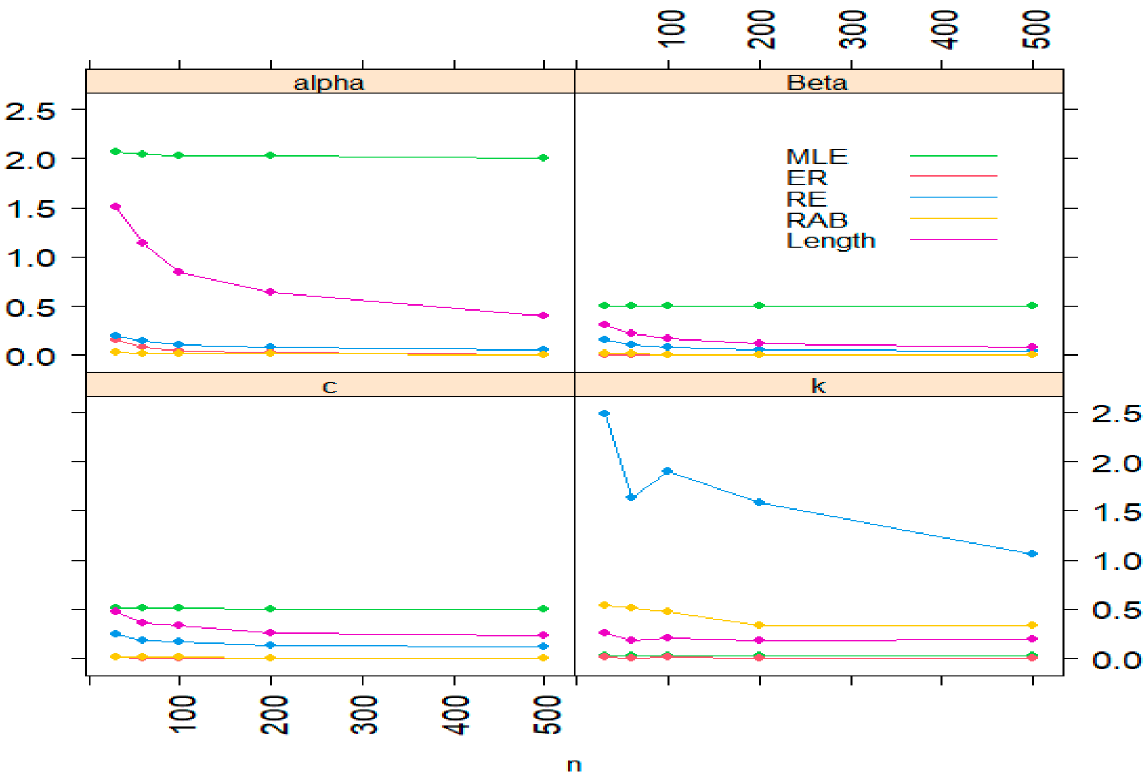


Figure 6. Plots of the ML averages, estimated risks, relative errors, Relative absolute biases, variances and the ACI lengths of the parameters, for $(\alpha = 2, \beta = 0.5, c = 0.5, k = 0.05)$ versus n.

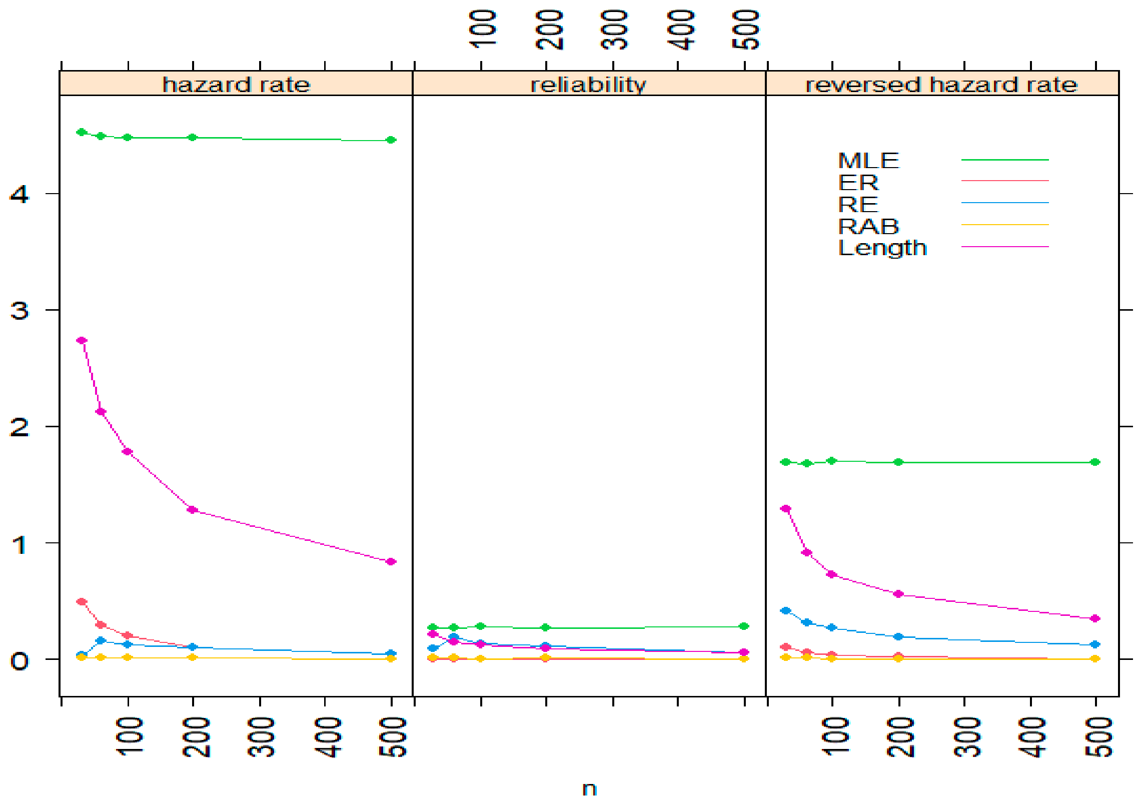


Figure 7. Plots of the ML averages, estimated risks, relative errors, Relative absolute biases, variances and the ACI lengths of the re, hrf and rhrf, for $(\alpha = 2.6, \beta = 2, c = 1.2, k = 1.5)$ versus n.

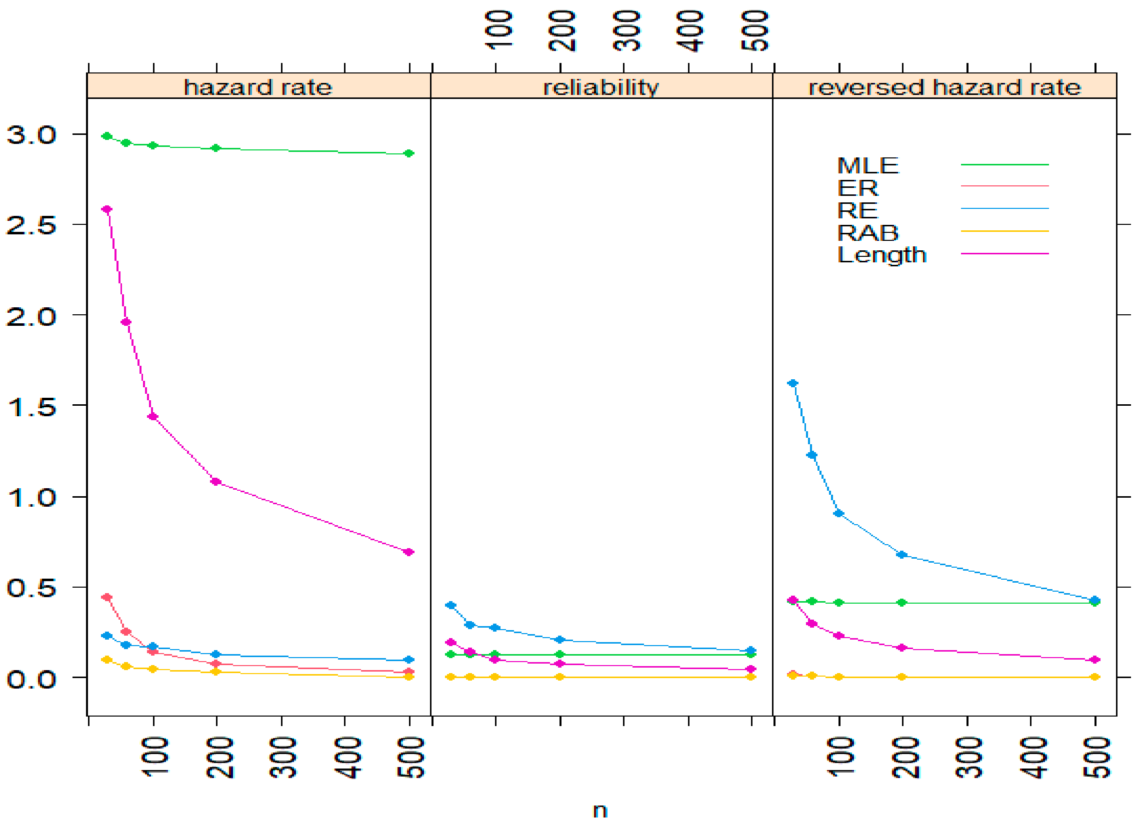


Figure 8. Plots of the ML averages, estimated risks, relative errors, Relative absolute biases, variances and the ACI lengths of the re, hrf and rhrf, for $(\alpha = 2, \beta = 0.5, c = 0.5, k = 0.05)$ versus n.

The simulation tables and figures lead to the following conclusions:

- As the sample size n increases, the ML averages of the estimates for the parameters of the C-BXII distribution stabilize.
- As the sample size increases, the ERs and REs of the ML estimates of the parameters $\underline{\theta} = (\alpha, \beta, c, k)$, rf, hrf and rhrf decrease.
- As the sample size increases, the RABs of the ML estimates of the parameters $\underline{\theta} = (\alpha, \beta, c, k)$, rf, hrf and rhrf decrease.
- The variances of the parameters, rf, hrf and rhrf decrease, as the sample size increases.
- As the sample size increases, the lengths of the 95% ACIs of the parameters, rf, hrf and rhrf decrease.

7. Applications

This section is devoted to show the applicability and flexibility of the C-BXII distribution for data modeling. Three applications are used to demonstrate the superiority of the C-BXII distribution over some known distributions namely, AC-P, W-C, BXII-MW, AW, AC, ABXII, Chen and BXII distributions.

The ML estimates of the parameters, rf, hrf and rhrf and their standard errors (SE), Kolmogorov-Smirnov (K-S), Anderson Darling (A-D), Cramér-Von Mises (C-V) statistics and their corresponding p-values, the $-2\log$ likelihood statistic $(-2\mathcal{L})$, Akaike information criterion (AIC), Bayesian information criterion (BIC) and corrected Akaike information criterion (CAIC), are used to compare the fit of the competitor distributions, where

$$AIC = 2m - 2\mathcal{L},$$
$$BIC = m\ln(n) - 2\mathcal{L},$$

and

$$CAIC = AIC + 2\left(\frac{m(m+1)}{n-m-1}\right),$$

where $\mathcal{L}$ is the natural logarithm of the value of the likelihood function evaluated at the ML estimates, n is the number of the observations and m is the number of the estimated parameters.

The best distribution corresponds to the lowest values of AIC, BIC and CAIC, also the highest p-values.

7.1. Application 1

This application is given by [40]. In this application the survival times of patients suffering from the COVID-19 epidemic in China is considered. The data represents the survival times of patients from the time admitted to the hospital until death. Among them, a group of 53 COVID-19 patients were found in critical condition in hospital from January to February 2020. Among them, 37 patients (70%) were men and 16 women (30%), 40 patients (75%) were diagnosed with chronic diseases, especially including high blood pressure, and diabetes, 47 patients (88%) had common clinical symptoms of the flu, 42 patients (81%) were coughing, 37 (69%) were short of breath, and 28 patients (53%) had fatigue. 50 (95%) patients had bilateral pneumonia showed by chest computed tomographic scans.

The data are: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Figure 9 presents the plot of the empirical scaled TTT-transform of COVID-19 data of China, this plot indicates that these data have a bathtub hazard function, boxplot and the histogram of the data show that these data are right-skewed. The P-P plot, Q-Q plot and the fitted pdf of the C-BXII distribution plots indicate that the C-BXII distribution provides good fit for these data.

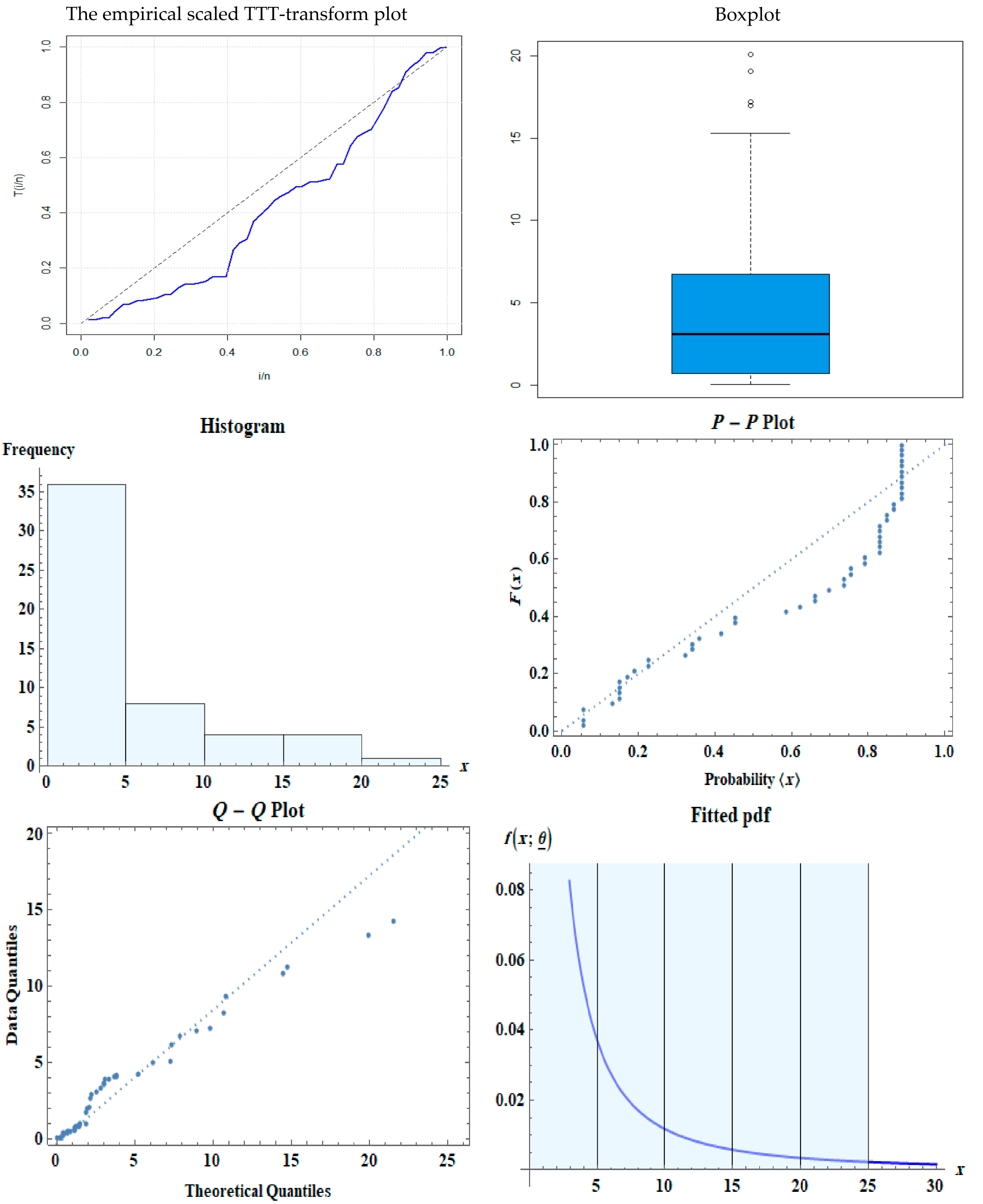


Figure 9. The empirical scaled TTT-transform plot, Boxplot, P-P plot, Q-Q plot and the histogram and the fitted pdf for COVID-19 data of China.

Table 10 displays the K-S, A-D and C-V statistics and their corresponding p-values, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 11 presents the ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution along with their SEs for COVID-19 data of China.

Table 10. K-S, A-D and C-V statistics and the corresponding p-values (in square brackets), $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data of China.

Model	K-S (p-Value)	A-D (p-Value)	C-V (p-Value)	$-2\mathcal{L}$	AIC	BIC	CAIC
C-BXII	0.0943 (0.9747)	0.4040 (0.8444)	0.0462 (0.8990)	278.116	286.116	293.997	286.95
AC-P	0.1698 (0.4327)	2.1492 (0.0763)	0.3178 (0.1205)	294.082	302.082	309.963	302.915
W-C	0.2076 (0.2045)	3.3996 (0.0173)	0.5385 (0.0318)	283.565	291.565	299.446	292.399
BXII-MW	0.2264 (0.1317)	2.3973 (0.0562)	0.3595 (0.0927)	975.042	985.042	994.894	986.319
AW	0.1132 (0.8898)	0.6363 (0.5700)	0.0533 (0.8560)	313.350	321.350	329.232	322.184
AC	0.1698 (0.4308)	1.6355 (0.1473)	0.1939 (0.2795)	357.847	365.847	373.728	366.68
ABXII	0.1321 (0.7471)	0.7992 (0.4811)	0.1007 (0.5817)	281.693	293.693	305.515	295.519
C	0.1887 (0.3042)	2.2943 (0.0638)	0.2833 (0.1508)	314.931	318.931	322.871	319.171
BXII	0.3019 (0.0155)	3.9657 (0.0091)	0.7486 (0.0097)	288.079	292.079	296.019	292.319

Table 11. ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution and their relevant SEs for COVID-19 data of China when $x_0 = 0.5$.

$\underline{\theta}$, rf, hrf and rhrf	MLE	SE
α	0.0412	0.0002
β	0.2562	0.0001
c	1.4104	0.0017
k	0.5354	0.0007
$R(x_0;\underline{\theta})$	0.7986	0.0001
$h(x_0;\underline{\theta})$	0.4536	0.0003
$r(x_0;\underline{\theta})$	1.7989	0.0003

7.2. Application 2

This application is given by [41]. The application represents COVID-19 data which belongs to the United Kingdom of 76 days, from 15 April to 30 June 2020. The data are formed of drought mortality rates.

The data are: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019 and 11.4584.

Figure 10 exhibits the plot of the empirical scaled TTT-transform of COVID-19 data of the United Kingdom, which implies that these data have a modified bathtub hazard function, boxplot and the histogram of the data. One can notice that these data are right-skewed. P-P plot, Q-Q plot and the fitted pdf of the C-BXII distribution plots indicate that the C-BXII distribution provides a good fit to these data.

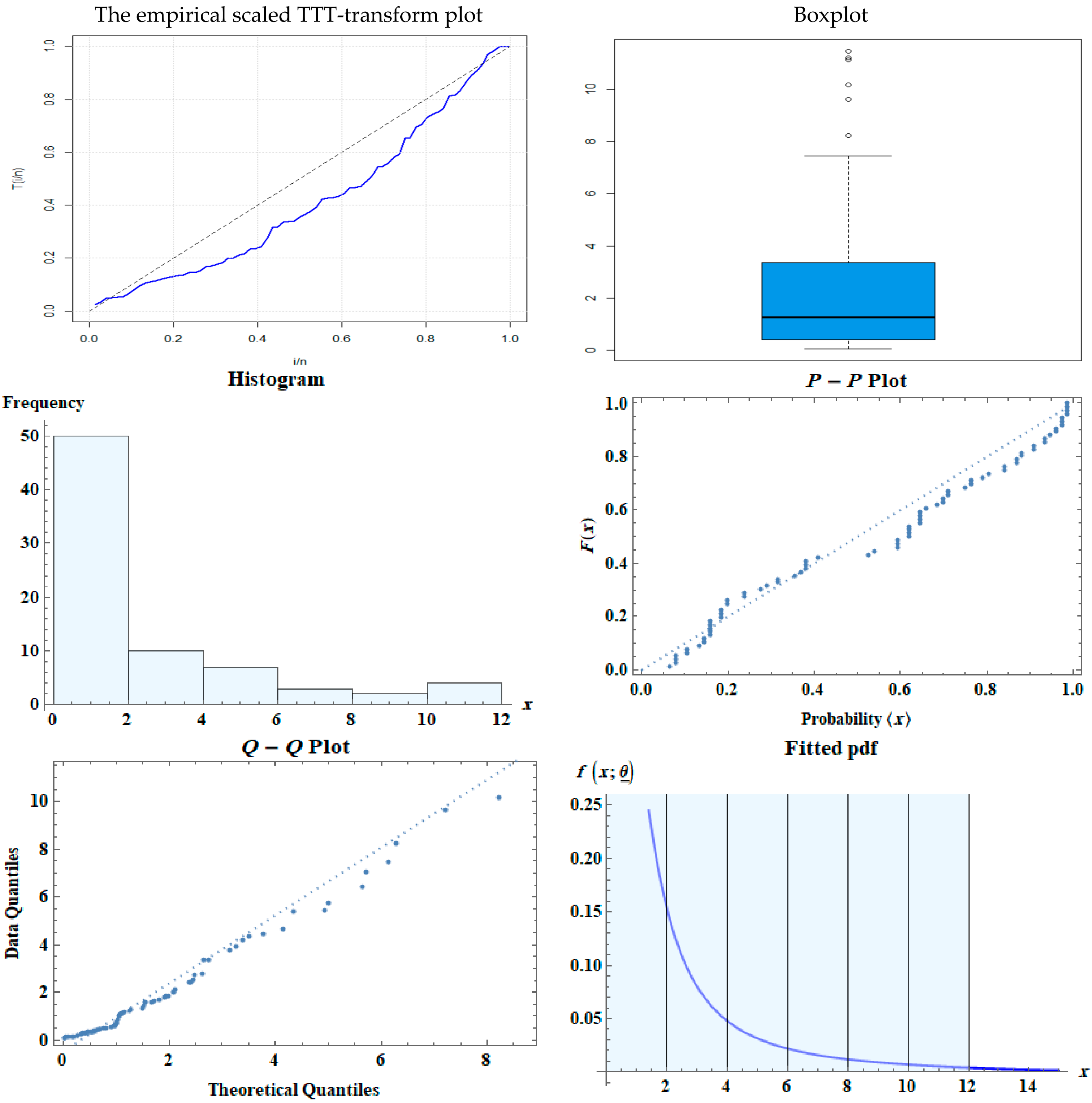


Figure 10. The empirical scaled TTT-transform plot, boxplot, histogram, P-P plot, Q-Q plot and the fitted pdf for COVID-19 data of the United Kingdom.

Table 12 displays the K-S, A-D and C-V statistics and their corresponding p-values, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 13 lists the ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution along with their SEs for COVID-19 data of the United Kingdom.

Table 12. K-S, A-D and C-V statistics and the corresponding p-values (in square brackets), $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data of the United Kingdom.

Model	K-S (p-Value)	A-D (p-Value)	C-V (p-Value)	$-2\mathcal{L}$	AIC	BIC	CAIC
C-BXII	0.0921 (0.9068)	0.5252 (0.7209)	0. 0748 (0.7223)	283.318	291.318	300.640	291.881
AC-P	0.1184 (0.6643)	1.2924 (0.2346)	0.4306 (0.1373)	294.285	302.285	311.608	302.848
W-C	0.1447 (0.4057)	1.4633 (0.1855)	0.3961 (0.1479)	299.544	307.544	316.867	308.107
BXII-MW	0.1053 (0.7963)	0.8546 (0.4429)	0.0854 (0.6613)	376.489	386.489	398.143	387.346
AW	0.1316 (0.5291)	1.7241 (0.1310)	0.1734 (0.3257)	374.345	382.345	391.668	382.908
AC	0.1711 (0.2170)	2.3739 (0.0578)	0.3072 (0.1290)	516.645	524.645	533.968	525.208
ABXII	0.1579 (0.3012)	2.2556 (0.0668)	0.3096 (0.1270)	287.540	299.540	313.525	300.758
C	0.1974 (0.1036)	2.1916 (0.0723)	0.4073 (0.0691)	295.386	299.386	303.048	299.551
BXII	0.2105 (0.0687)	4.6664 (0.0042)	0.7933 (0.0076)	293.421	297.421	302.083	297.586

Table 13. ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution and their relevant SEs for COVID-19 data of the United Kingdom when $x_0 = 0.5$.

$\underline{\theta}$, rf, hrf and rhrf	MLE	SE
α	0.0779	0.0004
β	0.4452	0.0007
c	1.5611	0.0008
k	0.6595	0.0012
$R(x_0;\underline{\theta})$	0.7581	1.5022×10^{-5}
$h(x_0;\underline{\theta})$	0.6274	0.0004
$r(x_0;\underline{\theta})$	1.9660	0.0012

7.3. Application 3

This application is given by [42,43] and used by [44]. The following 101 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed.

The failure times in hours are: 0.01, 0.01, 0.02, 0.02, 0.02,0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18,0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60,0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92,0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18,1.20, 1.29, 1.31, 1.33, 1.34,1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81,2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69 and 7.89.

Figure 11 displays the plot of the empirical scaled TTT-transform of the kevlar 49/epoxy strands data, which implies that these data have a decreasing-unimodal hazard function, boxplot and the histogram of the data. It is obvious that these data are right-skewed. P-P plot, Q-Q plot and the fitted pdf of the C-BXII distribution plots indicate that the C-BXII distribution presents a good fit to these data.

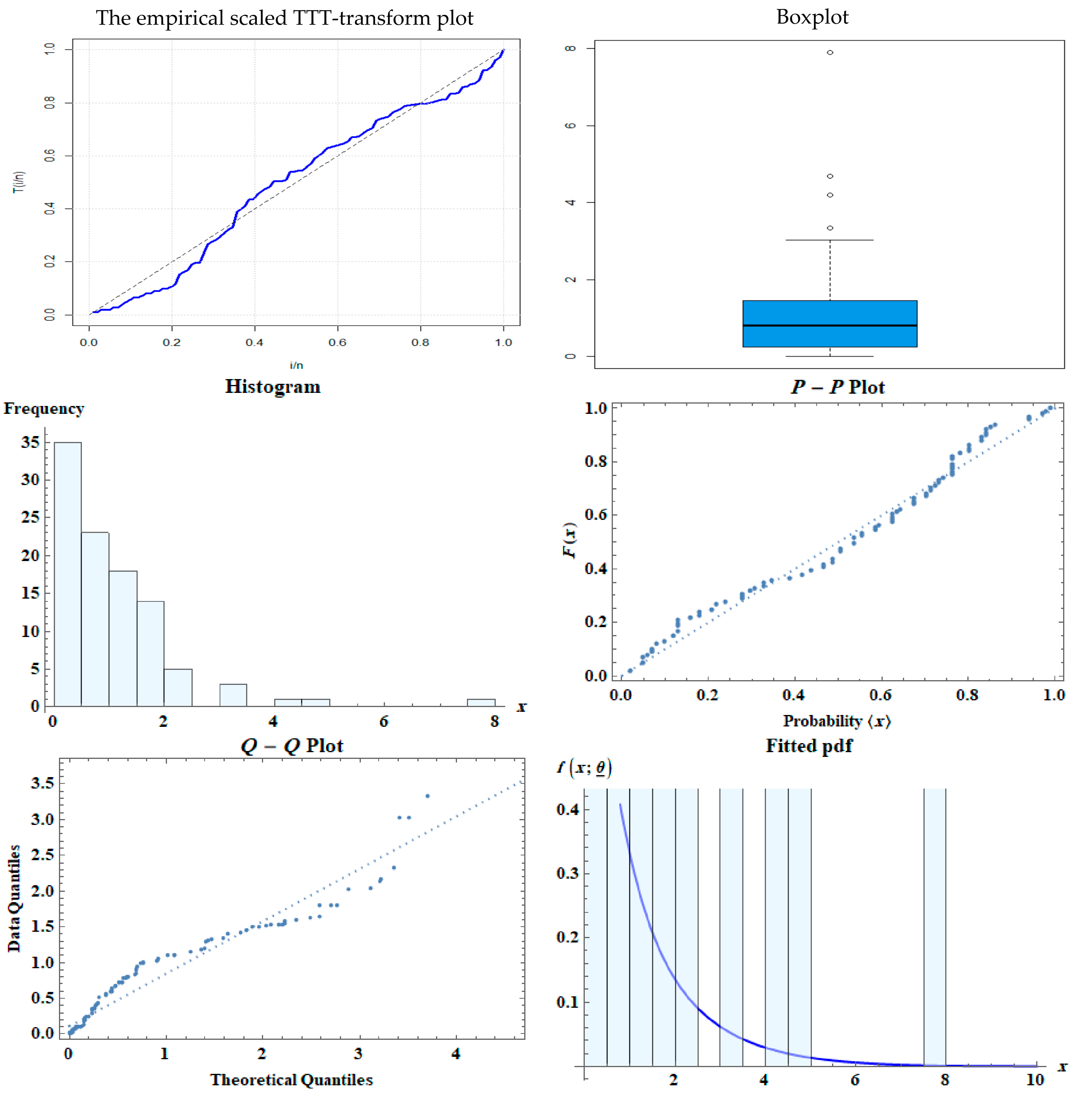


Figure 11. The empirical scaled TTT-transform plot, boxplot, histogram, P-P plot, Q-Q plot and the fitted pdf for kevlar 49/epoxy strands data.

Table 14 provides the K-S, A-D and C-V statistics and their corresponding p-values, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 15 lists the ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution along with their SEs for kevlar 49/epoxy strands data.

Table 14. K-S, A-D and C-V statistics and the corresponding p-values (in square brackets), $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data for kevlar 49/epoxy strands data.

Model	K-S (p-Value)	A-D (p-Value)	C-V (p-Value)	$-2\mathcal{L}$	AIC	BIC	CAIC
C-BXII	0.0792 (0.9103)	0.5774 (0.6694)	0.0835 (0.6719)	210.869	218.869	229.330	219.286
AC-P	0.1584 (0.1560)	2.1414 (0.0770)	0.3696 (0.0871)	212.157	220.157	230.618	220.574
W-C	0.1188 (0.4735)	1.1331 (0.2942)	0.1677 (0.3399)	215.749	223.749	234.209	224.165
BXII-MW	0.0990 (0.7042)	0.6346 (0.6156)	0.0712 (0.7446)	250.885	260.885	273.961	261.517
AW	0.1287 (0.3692)	1.1576 (0.2840)	0.2012 (0.2651)	266.897	274.897	285.357	275.314
AC	0.1089 (0.5860)	1.1622 (0.2822)	0.1877 (0.2926)	226.157	234.157	244.618	234.574
ABXII	0.1683 (0.1131)	2.8641 (0.0322)	0.4060 (0.0697)	218.453	230.453	246.144	231.347
C	0.1485 (0.2134)	1.2900 (0.2355)	0.2641 (0.1713)	267.435	271.435	276.665	271.557
BXII	0.1782 (0.0801)	4.9159 (0.0032)	0.6365 (0.0182)	223.879	227.879	233.109	228.002

Table 15. ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution and their relevant SEs for kevlar 49/epoxy strands data when $x_0 = 0.5$.

$\underline{\boldsymbol{\theta}}$, rf, hrf and rhrf	MLE	SE
α	0.3696	0.0002
β	0.5234	0.0002
c	1.2435	0.0026
k	0.4516	0.0005
$R(x_0;\underline{\theta})$	0.5882	0.0001
$h(x_0;\underline{\theta})$	0.8733	0.0002
$r(x_0;\underline{\theta})$	1.2475	0.0009

7.4. Concluding Remarks

- The C-BXII distribution has the lowest K-S, A-D and C-V values and the highest p-values for the three applications. Thus, it provides the best fit for this data compared to the other competitor distributions.
- Moreover, the C-BXII distribution has the smallest values of the $-2\mathcal{L}$ statistic, AIC, BIC and CAIC, which imply that the proposed model is the best among the other competitor distributions (AC-P, W-C, BXII-MW, AW, AC, ABXII, Chen and BXII).
- The ML estimates of the parameters, rf, hrf and rhrf of the C-BXII distribution have small SEs for the three applications.

8. Conclusions

In this paper, a new four-parameter competing risks model, called the C-BXII distribution is introduced by combining Chen and BXII distributions in a series system with two components functioning independently. The proposed distribution has high flexibility and diversity in the shapes of the pdf as well as the hrf. The pdf displays unimodal and bimodal shapes, whereas the hrf exhibits important shapes: bathtub, modified bathtub and decreasing-unimodal shapes. These shapes increase the applicability of the proposed distribution for lifetime data modeling. Moreover, the proposed distribution has some new additive models as special cases these models have not been introduced in the statistical literature. Also, it has some well-known models as special cases. Several statistical properties of the proposed model are derived. The ML estimators of the parameters, rf, hrf and rhrf are presented. Moreover, The ACIs of the parameters, rf, hrf and rhrf are obtained. The performance the ML estimates is evaluated through a simulation study. Furthermore, three real applications are used to demonstrate the applicability of the C-BXII distribution over some existing distributions. The C-BXII distribution provides the best fitting compared with the used competitor distributions.

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Appendix A

Appendix A.1. The Mode

The mode of the C-BXII distribution can be obtained by differentiating the pdf in (8) with respect to x and equating to zero as follows:

$$\overset{\prime}{f}(x_0;\underline{\theta}) = 0.$$

Since

$$f(x;\underline{\theta}) = h(x;\underline{\theta})R(x;\underline{\theta}) = h(x;\underline{\theta})e^{-H(x;\underline{\theta})},$$

where $H(x;\underline{\theta})$ is the chrf defined in (10), then,

$$\overset{\prime}{f}(x;\underline{\theta}) = h(x;\underline{\theta})\left\{\left[-\frac{\partial}{\partial x}H(x;\underline{\theta})\right]R(x;\underline{\theta})\right\} + \left[\frac{\partial}{\partial x}h(x;\underline{\theta})\right]R(x;\underline{\theta}), \tag{A1}$$

where

$$\frac{\partial}{\partial x}H(x;\underline{\theta}) = h(x;\underline{\theta}),$$

and

$$\frac{\partial}{\partial x}h(x;\underline{\theta}) = \overset{\prime}{h}(x;\underline{\theta}).$$

Hence, (A1) can be written as

$$\overset{\prime}{f}(x;\underline{\theta}) = \left[\overset{\prime}{h}(x;\underline{\theta}) - h^2(x;\underline{\theta})\right]R(x;\underline{\theta}), \tag{A2}$$

where

$$\overset{\prime}{h}(x;\underline{\theta}) = \alpha\beta x^{\beta-2}e^{x^\beta}\left[\beta x^\beta + \beta - 1\right] + \tfrac{ckx^{c-2}}{(1+x^c)^2}[(c-1)(1+x^c) - cx^c],$$
$$h^2(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1}e^{x^\beta} + \tfrac{ckx^{c-1}}{(1+x^c)}\right]^2.$$

Therefore, Equation (A2) to zero, one can obtain the following nonlinear equation

$$\begin{aligned}\left\{\alpha\beta x_0^{\beta-2}e^{x_0^\beta}\left[\beta x_0^\beta+\beta-1\right]+\frac{ckx_0^{c-2}}{\left(1+x_0^c\right)^2}\left[(c-1)\left(1+x_0^c\right)-cx_0^c\right]\right\} \\ -\left[\alpha\beta x_0^{\beta-1}e^{x_0^\beta}+\frac{ckx_0^{c-1}}{\left(1+x_0^c\right)}\right]^2\frac{e^{\alpha(1-e^{x_0^\beta})}}{\left(1+x_0^c\right)^k}=0.\end{aligned} \tag{A3}$$

Equation (A3) is a nonlinear equation, which can be solved numerically to obtain the mode of the C-BXII distribution.

Appendix A.2. The rth Non-Central Moment

Since

$$\mu_r'=\int_0^\infty x^r f(x;\underline{\theta})dx=-\int_0^\infty x^r dR(x;\underline{\theta}).$$

Integration by parts can be used to get the following equation:

$$\mu_r'=\int_0^\infty rx^{r-1}R(x;\underline{\theta})dx=\int_0^\infty rx^{r-1}\frac{e^{\alpha(1-e^{x^\beta})}}{(1+x^c)^k}dx.$$

Since the power series expansion of $e^{\alpha(1-e^{x^\beta})}$ is as follows:

$$e^{\alpha(1-e^{x^\beta})}=\sum_{i=0}^\infty\frac{(-1)^i\alpha^i}{i!}\left(1-e^{x^\beta}\right)^i,$$

and by using the binomial expansion of $\left(1-e^{x^\beta}\right)^i$

$$\left(1-e^{x^\beta}\right)^i=\sum_j^i\binom{i}{j}(-1)^j e^{jx^\beta},$$

then

$$\mu_r'=\sum_{i=0}^\infty\sum_{j=0}^\infty\binom{i}{j}\frac{(-1)^{i+j}\alpha^i}{i!}\int_0^\infty rx^{r-1}\frac{e^{jx^\beta}}{(1+x^c)^k}dx,$$

Using the power series expansion of e^{jx^β},

$$\mu_r'=\sum_{i=0}^\infty\sum_{j=0}^{\mathrm{i}}\sum_{k=0}^\infty\frac{(-1)^{i+j}j^k\alpha^i r}{i!k!}\int_0^\infty\frac{x^{\beta k+r-1}}{(1+x^c)^k}dx.$$

Using integration by substitution, we get

$$\mu_r'=\frac{r}{c}\sum_{i=0}^\infty\sum_{j=0}^i\sum_{k=0}^\infty\binom{i}{j}(-1)^j\left[\frac{\alpha^i j^k}{i!k!}\mathbf{B}\left(\frac{\beta k+r}{c},k-\frac{\beta k+r}{c}\right)\right],r=1,2,\ldots,$$

where

$$0<\frac{\beta k+r}{c}<k.$$

Appendix A.3. Asymptotic Fisher Information Matrix

For the C-BXII distribution the asymptotic Fisher information is given by:

$$\widetilde{I}(\underline{\theta})\simeq\left[I_{ij}\right],i,j=1,2,3,4, \tag{A4}$$

where

$$I_{11} = -\frac{\partial^2 \ell}{\partial \alpha^2} = \sum_{i=1}^{n} \frac{h_\alpha^2\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

$$I_{12} = -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \left(\frac{h\left(x_{(i)};\underline{\theta}\right) h_{\alpha\beta}\left(x_{(i)};\underline{\theta}\right) - h_\alpha\left(x_{(i)};\underline{\psi}\right) h_\beta\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)} \right) + \sum_{i=1}^{n} x_{(i)}^{\beta} e^{x_{(i)}^{\beta}} lnx_{(i)},$$

$$I_{13} = -\frac{\partial^2 \ell}{\partial \alpha \partial c} = \sum_{i=1}^{n} \frac{h_\alpha\left(x_{(i)};\underline{\theta}\right) h_c\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

$$I_{14} = -\frac{\partial^2 \ell}{\partial \alpha \partial k} = \sum_{i=1}^{n} \frac{h_\alpha\left(x_{(i)};\underline{\theta}\right) h_k\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

$$I_{22} = -\frac{\partial^2 \ell}{\partial \beta^2} = -\sum_{i=1}^{n} \left(\frac{h\left(x_{(i)};\underline{\theta}\right) h_{\beta\beta}\left(x_{(i)};\underline{\theta}\right) - h_\beta^2\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)} \right) + \alpha \sum_{i=1}^{n} \left(lnx_{(i)}\right)^2 x_{(i)}^{\beta} e^{x_{(i)}^{\beta}} \left(x_{(i)}^{\beta} + 1\right),$$

$$I_{23} = -\frac{\partial^2 \ell}{\partial \beta \partial c} = \sum_{i=1}^{n} \frac{h_\beta\left(x_{(i)};\underline{\theta}\right) h_c\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

$$I_{24} = -\frac{\partial^2 \ell}{\partial \beta \partial k} = \sum_{i=1}^{n} \frac{h_\beta\left(x_{(i)};\underline{\theta}\right) h_k\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

$$I_{33} = -\frac{\partial^2 \ell}{\partial c^2} = -\sum_{i=1}^{n} \left(\frac{h\left(x_{(i)};\underline{\theta}\right) h_{cc}\left(x_{(i)};\underline{\theta}\right) - h_c^2\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)} \right) + \mathrm{k} \sum_{i=1}^{n} \frac{x_{(i)}^{c}\left(lnx_{(i)}\right)^2}{\left(1 + x_{(i)}^{c}\right)^2},$$

$$I_{34} = -\frac{\partial^2 \ell}{\partial c \partial k} = -\sum_{i=1}^{n} \left(\frac{h\left(x_{(i)};\underline{\theta}\right) h_{kk}\left(x_{(i)};\underline{\theta}\right) - h_k^2\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)} \right) + \sum_{i=1}^{n} \frac{x_{(i)}^{c} lnx_{(i)}}{\left(1 + x_{(i)}^{c}\right)},$$

and

$$I_{44} = -\frac{\partial^2 \ell}{\partial k^2} = \sum_{i=1}^{n} \frac{h_k^2\left(x_{(i)};\underline{\theta}\right)}{h^2\left(x_{(i)};\underline{\theta}\right)},$$

where

$$h_{\alpha\beta}\left(x_{(i)};\underline{\theta}\right) = \frac{\partial h_\alpha\left(x_{(i)};\underline{\theta}\right)}{\partial \beta} = x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}} \left[\beta lnx_{(i)}\left(x_{(i)}^{\beta} + 1\right) + 1\right],$$

$$\begin{aligned} h_{\beta\beta}\left(x_{(i)};\underline{\theta}\right) &= \frac{\partial^2 h\left(x_{(i)};\underline{\theta}\right)}{\partial \beta^2} = \frac{\partial h_\beta\left(x_{(i)};\underline{\theta}\right)}{\partial \beta} \\ &= \alpha lnx_{(i)} x_{(i)}^{\beta-1} e^{x_{(i)}^{\beta}} \Big[\beta x_{(i)}^{\beta} lnx_{(i)} + x_{(i)}^{\beta} + 1 \\ &\quad + \left(\beta lnx_{(i)}\left(x_{(i)}^{\beta} + 1\right) + 1\right)\left(x_{(i)}^{\beta} + 1\right)\Big], \end{aligned}$$

and

$$h_{cc}\left(x_{(i)};\underline{\theta}\right) = \frac{\partial^2 h\left(x_{(i)};\underline{\theta}\right)}{\partial c^2} = \frac{\partial h_\lambda\left(x_{(i)};\underline{\theta}\right)}{\partial c} = kx_{(i)}^{c-1} lnx_{(i)}.$$

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