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Fuzzy $\mathcal{H}$-Quasi-Contraction and Fixed Point Theorems in Tripled Fuzzy Metric Spaces

Yunpeng Zhao [1,2], Fei He [1,*] and Xuan Liu [1]

1 School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China; zyp.12345@163.com (Y.Z.); lxuan97@163.com (X.L.)
2 Teaching Department of Basic Course, Shanxi Vocational University of Engineering Science and Technology, Jinzhong 030619, China
* Correspondence: hefei@imu.edu.cn

Abstract: We consider the concept of fuzzy $\mathcal{H}$-quasi-contraction ($\mathcal{FH}$-$\mathcal{QC}$ for short) initiated by Ćirić in tripled fuzzy metric spaces ($\mathcal{T}$-$\mathcal{FMS}$s for short) and present a new fixed point theorem ($\mathcal{FPT}$ for short) for $\mathcal{FH}$-$\mathcal{QC}$ in complete $\mathcal{T}$-$\mathcal{FMS}$s. As an application, we prove the corresponding results of the previous literature in setting fuzzy metric spaces ($\mathcal{FMS}$s for short). Moreover, we obtain theorems of sufficient and necessary conditions which can be used to demonstrate the existence of fixed points. In addition, we construct relevant examples to illustrate the corresponding results. Finally, we show the existence and uniqueness of solutions for integral equations by applying our new results.

Keywords: fuzzy $\mathcal{H}$-quasi-contraction; fixed point; tripled fuzzy metric space

MSC: 47H10; 54H25

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1. Introduction

In 1965, in [1], Zadeh introduced the theory of fuzzy sets. From then on, many researchers have discussed and developed this theory and applied the results to various different areas, such as mathematical programming, multi-attribute decision making, cybernetics, neural networks, statistics, computational science, and engineering (for example, see [2–7]). In 1975, Kramosil and Michalek [8] first proposed the concept of $\mathcal{FMS}$s. In 1988, Grabiec [9] initiated studying Banach and Edelstein's $\mathcal{FPT}$ in an $\mathcal{FMS}$. In 1994, George and Veeramani [10] slightly modified the conditions of the notion to obtain a Hausdorff topology. The modified definition, called George and Veeramani's type of fuzzy metric space ($\mathcal{GV}$-$\mathcal{FMS}$ for short), is now considered to be the appropriate concept for a fuzzy metric. Since then, many types of $\mathcal{FPT}$s and related results have been presented by different authors (for example, see [11–22]). In 2002, Gregori and Sapena [23] defined fuzzy contraction in an $\mathcal{FMS}$ and obtained a fuzzy Banach contraction theorem. In 2013, Wardowski [24] introduced a new concept of fuzzy $\mathcal{H}$-contraction by mapping η, which is a generalization of fuzzy contractive mapping. Inspired by the notion of quasi-contraction introduced by Ćirić [25], in 2015, Amini-Harandi and Mihet [26] introduced the concept of $\mathcal{FH}$-$\mathcal{QC}$ and obtained $\mathcal{FPT}$s for this mapping in a complete $\mathcal{FMS}$. In 2020, Jing-Feng Tian et al. [27] gave the notion of a $\mathcal{T}$-$\mathcal{FMS}$, which is a new generalization of the $\mathcal{GV}$-$\mathcal{FMS}$, and deduced $\mathcal{FPT}$s for fuzzy ψ-contraction. Moreover, Jing-Feng Tian et al. [27] introduced the concept of a neighborhood into the $\mathcal{T}$-$\mathcal{FMS}$ and obtained a first-countable Hausdorff topology. Recently, many authors have obtained $\mathcal{FPT}$s and other results in the setting of $\mathcal{T}$-$\mathcal{FMS}$s (for example, refer to [28,29]).

Motivated by the above works, we present the notion of $\mathcal{FH}$-$\mathcal{QC}$ which involves ten metrics in a $\mathcal{T}$-$\mathcal{FMS}$. First, we construct examples of $\mathcal{T}$-$\mathcal{FMS}$s. Second, we establish a $\mathcal{FPT}$ for $\mathcal{FH}$-$\mathcal{QC}$ in such a space. Third, as an application, we clarify Amini-Harandi and Mihet's results [26] in the setting of an $\mathcal{FMS}$ using our new results. In addition, we give another form to the theorems of sufficient and necessary conditions which can be used to demonstrate the existence of fixed points. In the meantime, we provide two illustrative examples in support of our new results. Finally, we discuss the existence of a solution for the integral equations formulated in the $\mathcal{T}$-$\mathcal{FMS}$.

2. Preliminaries

Some related concepts and conclusions will be recalled below. Throughout the paper, we always denote sets of real numbers, sets of all non-negative integers and sets of all positive integers as $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}^+$, respectively.

Definition 1 ([30]). *A function $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions satisfy*

(i) $s * l = l * s$ and $s * (l * \tau) = (s * l) * \tau$ for $0 \le s, l, \tau \le 1$;
(ii) $s * 1 = a$, for $0 \le s \le 1$;
(iii) $s * l \le \tau * \lambda$ whenever $s \le \tau$ and $l \le \lambda$ for $0 \le s, l, \tau, \lambda \le 1$;
(iv) $*$ is continuous.

According to Definition 1, we know that for $0 \le s$ and $l \le 1$, $*_m(s,l) = \min\{s,l\}$ and $*_p(s,l) = sl$ are continuous t-norms.

The notion of a $\mathcal{T}$-$\mathcal{FMS}$ was introduced by Tian et al. [27], defined as follows.

Definition 2 ([27]). *A triple $(\mathcal{X}, \mathcal{L}, *)$ is called a $\mathcal{T}$-$\mathcal{FMS}$ if $\mathcal{X}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and $\mathcal{L}$ is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times \mathcal{X} \times (0,\infty)$ such that for all $\alpha, \beta, \gamma, \delta \in \mathcal{X}$ and all $\tau, \sigma > 0$, the following conditions hold:*

($\mathcal{T}$-$\mathcal{FMS}$-1): $\mathcal{L}_{\alpha,\beta,\gamma}(\tau) > 0$;
($\mathcal{T}$-$\mathcal{FMS}$-2): $\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = 1$ if and only if $\alpha = \beta = \gamma$;
($\mathcal{T}$-$\mathcal{FMS}$-3): $\mathcal{L}_{\alpha,\alpha,\beta}(\tau) \ge \mathcal{L}_{\alpha,\beta,\gamma}(\tau)$ for $\gamma \ne \beta$;
($\mathcal{T}$-$\mathcal{FMS}$-4): $\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = \mathcal{L}_{\alpha,\gamma,\beta}(\tau) = \mathcal{L}_{\gamma,\beta,\alpha}(\tau) = ...$;
($\mathcal{T}$-$\mathcal{FMS}$-5): $\mathcal{L}_{\alpha,\beta,\gamma}(\cdot) : (0,\infty) \to (0,1]$ is continuous;
($\mathcal{T}$-$\mathcal{FMS}$-6): $\mathcal{L}_{\alpha,\beta,\gamma}(\tau + \sigma) \ge \mathcal{L}_{\alpha,\delta,\delta}(\tau) * \mathcal{L}_{\delta,\beta,\gamma}(\sigma)$.

We can construct an example of a $\mathcal{T}$-$\mathcal{FMS}$ by an $\mathcal{FMS}$ in the setting of a $\mathcal{GV}$-$\mathcal{FMS}$.

Example 1. *Let $(\mathcal{X}, \mathcal{F}, *)$ be an $\mathcal{FMS}$. For $\alpha, \beta, \gamma \in \mathcal{X}$ and $\tau > 0$, we define*

$$\mathcal{L}_{\alpha,\beta,\gamma,}(\tau) = \min\{\mathcal{F}(\alpha,\beta,\tau), \mathcal{F}(\alpha,\gamma,\tau), \mathcal{F}(\gamma,\beta,\tau)\},$$

and $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$.

Proof. It is not difficult to see that $\mathcal{L}$ satisfies ($\mathcal{T}$-$\mathcal{FMS}$-1)-($\mathcal{T}$-$\mathcal{FMS}$-5). Next, we verify that $\mathcal{L}$ satisfies ($\mathcal{T}$-$\mathcal{FMS}$-6). If $\alpha, \beta, \gamma, \delta \in \mathcal{X}$ and $\tau, \theta > 0$, we have

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta,\gamma}(\tau+\theta) &= \min\{\mathcal{F}(\alpha,\beta,\tau+\theta), \mathcal{F}(\alpha,\gamma,\tau+\theta), \mathcal{F}(\beta,\gamma,\tau+\theta)\} \\
&\ge \min\{\mathcal{F}(\alpha,\delta,\tau) * \mathcal{F}(\delta,\beta,\theta), \mathcal{F}(\alpha,\delta,\tau) * \mathcal{F}(\delta,\gamma,\theta), \mathcal{F}(\beta,\gamma,\theta)\} \\
&\ge \min\{\mathcal{F}(\alpha,\delta,\tau) * \mathcal{F}(\delta,\beta,\theta), \mathcal{F}(\alpha,\delta,\tau) * \mathcal{F}(\delta,\gamma,\theta), \mathcal{F}(\alpha,\delta,\tau) * \mathcal{F}(\beta,\gamma,\theta)\} \\
&= \mathcal{F}(\alpha,\delta,\tau) * \min\{\mathcal{F}(\delta,\beta,\theta), \mathcal{F}(\delta,\gamma,\theta), \mathcal{F}(\beta,\gamma,\theta)\} \\
&= \mathcal{L}_{\alpha,\delta,\delta}(\tau) * \mathcal{L}_{\delta,\beta,\gamma}(\theta).
\end{aligned}$$

Therefore, $\mathcal{L}$ satisfies ($\mathcal{T}$-$\mathcal{FMS}$-6), and then $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$. □

Example 2. *Let $\xi, \eta \in \mathbb{N}^+$, $\xi > \eta$, and $* = *_p$; we define*

$$\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = (\frac{\eta+\tau}{\xi+\tau})^{\mu(|\alpha-\beta|+|\beta-\gamma|+|\alpha-\gamma|)} \quad \forall \alpha,\beta,\gamma,\delta \in (-\infty,+\infty), \tau > 0.$$

Then, $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$.

Proof. It is not difficult to ascertain that $\mathcal{L}$ satisfies ($\mathcal{T}$-$\mathcal{FMS}$-1)-($\mathcal{T}$-$\mathcal{FMS}$-5). Next, we verify that $\mathcal{L}$ satisfies the condition ($\mathcal{T}$-$\mathcal{FMS}$-6). In fact, if $\alpha,\beta,\gamma,\delta \in (-\infty,+\infty)$, $\tau > 0$ and $\theta > 0$, we have

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta,\gamma}(\tau+\theta) &= (\frac{\eta+t+s}{\xi+t+s})^{\mu(|\alpha-\beta|+|\beta-\gamma|+|\alpha-\gamma|)} \\
&\geq (\frac{\eta+\tau+\theta}{\xi+\tau+\theta})^{\mu(2|\alpha-\delta|+|\beta-\delta|+|\beta-\gamma|+|\delta-\gamma|)} \\
&= (\frac{\eta+\tau+\theta}{\xi+\tau+\theta})^{2\mu|\alpha-\delta|)} \times (\frac{\eta+\tau+\theta}{\xi+\tau+\theta})^{\mu(|\beta-\alpha|+|\beta-\gamma|+|\delta-\gamma|)} \\
&\geq (\frac{\eta+\tau}{\xi+\tau})^{2\mu|\alpha-\delta|} \times (\frac{\eta+\theta}{\xi+\theta})^{\mu(|\beta-\delta|+|\beta-\gamma|+|\delta-\gamma|)} \\
&= \mathcal{L}_{\alpha,\delta,\delta}(\tau) * \mathcal{L}_{\delta,\beta,\gamma}(\theta).
\end{aligned}$$

Hence, $\mathcal{L}$ satisfies ($\mathcal{T}$-$\mathcal{FMS}$-6), and $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$. □

Tian et al. [27] obtained a Hausdorff topology by defining an open neighborhood,

$$\Re(\alpha_0, r, \tau) = \{\beta \in \mathcal{X} : \mathcal{L}_{\alpha_0,\beta,\beta}(\tau) > 1-r, \mathcal{L}_{\alpha_0,\alpha_0,\beta}(\tau) > 1-r\},$$

and then the concepts of convergence and $\mathcal{L}$-Cauchy sequences ($\mathcal{L}$-$\mathcal{CS}$s for short), related propositions, were given as follows.

Definition 3 ([27]). *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$ and $\{z_n\} \subseteq \mathcal{X}$ be a sequence.*

(1) $z_n \to z_0 \in \mathcal{X}$ $(n \to \infty) \Longleftrightarrow \forall \tau > 0, 0 < r < 1$, $\exists N_{\tau,r} \in \mathbb{N}^*$ s.t. $\forall n > N_{t,r}$ $z_n \in \Re(z_0, r, \tau)$, i.e., $\mathcal{L}_{z_0,z_n,z_n}(\tau) > 1-r$, and $\mathcal{L}_{z_0,z_0,z_n}(\tau) > 1-r$.

(2) $\{z_n\}$ is an $\mathcal{L}$-$\mathcal{CS}$ or a Cauchy sequence ($\mathcal{CS}$ for short). $\Longleftrightarrow \forall \tau > 0, 0 < r < 1$, $\exists N_{\tau,r} \in \mathbb{N}^*$ s.t. $\forall m,n,l > N_{t,r}$, $\mathcal{L}_{z_m,z_n,z_l}(\tau) > 1-r$.

(3) A $\mathcal{T}$-$\mathcal{FMS}$ is $\mathcal{L}$-complete, or complete. $\Longleftrightarrow \forall \mathcal{L}$-$\mathcal{CS}$ $\{z_n\} \subseteq \mathcal{X}$, $\exists\, z_0 \in \mathcal{X}$ s.t. $z_n \to z_0$.

Proposition 1 ([27]). *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$ and $\{z_n\} \subseteq \mathcal{X}$ be a sequence. Then,*

$$\begin{aligned}
(1)\ z_n \to z_0 \in \mathcal{X}. &\Longleftrightarrow \forall \tau > 0, 0 < r < 1, \exists N_{\tau,r} \in \mathbb{N}^* \text{ s.t. } \forall n > N_{t,r},\ \mathcal{L}_{z_0,z_n,z_n}(\tau) > 1-r. \\
&\Longleftrightarrow \forall \tau > 0, 0 < r < 1, \exists N_{\tau,r} \in \mathbb{N}^* \text{ s.t. } \forall n > N_{t,r},\ \mathcal{L}_{z_0,z_0,z_n}(\tau) > 1-r. \\
&\Longleftrightarrow \mathcal{L}_{z_n,z_n,z_0}(t) \to 1 \text{ or } \mathcal{L}_{z_n,z_0,z_0}(t) \to 1 (n \to \infty)\ \forall t > 0.
\end{aligned}$$

$$\begin{aligned}
(2)\{z_n\} \text{ is an } \mathcal{L}-\mathcal{CS} &\Longleftrightarrow \forall \tau > 0, 0 < r < 1, \exists N_{\tau,r} \in \mathbb{N}^* \text{s.t. } \forall m,n > N_{t,r}, \\
&\quad \mathcal{L}_{z_m,z_n,z_n}(\tau) > 1-r. \\
&\Longleftrightarrow \mathcal{L}_{z_n,z_m,z_m}(t) \to 1 (n,m \to \infty)\ \forall t > 0.
\end{aligned}$$

Proposition 2 ([27]). *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$. Then, if α and $\beta \in X$, $\mathcal{L}_{\alpha,\beta,\beta}(\cdot)$ is non-decreasing.*

Proposition 3 ([27]). *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$ and α_0, β_0 and $\gamma_0 \in X$. Let sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be in $\mathcal{X}$. If $\alpha_n \to \alpha_0, \beta_n \to \beta_0$ and $\gamma_n \to \gamma_0$ as $n \to \infty$, then, for any $t > 0$, $\mathcal{L}_{\alpha_n,\beta_n,\gamma_n}(t) \to \mathcal{L}_{\alpha_0,\beta_0,\gamma_0}(t)$ as $n \to \infty$.*

The following notion of fuzzy $\mathcal{H}$-contraction was introduced by Wardowski in [24], as a generalization of the fuzzy contractions of Gregori and Sapena [23].

Definition 4 ([24])**.** *Denote by $\mathcal{H}$ the class of mappings $\eta : (0,1] \to [0,\infty)$ such that η is strictly decreasing, and η transforms $(0,1]$ into $[0,\infty)$.*

Note that if $\eta \in \mathcal{H}$, then η is continuous, and $\eta(1) = 0$. Combining the concepts of a $\mathcal{T}$-$\mathcal{FMS}$ and η, we have the following proposition.

Proposition 4. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$ and $\eta \in \mathcal{H}$. With a sequence $\{\alpha_n\}$ in X, then the following are valid.*

(1) $\{\alpha_n\}$ is an $\mathcal{L}$-$\mathcal{CS}$. $\Longleftrightarrow \lim_{n,m\to\infty} \eta(\mathcal{L}_{\alpha_n,\alpha_m,\alpha_m}(t)) = 0 \; \forall t > 0$;

(2) $\alpha_n \to \alpha \Longleftrightarrow \lim_{n\to\infty} \eta(\mathcal{L}_{\alpha_n,\alpha,\alpha}(t)) = 0$ or $\lim_{n\to\infty} \eta(\mathcal{L}_{\alpha_n,\alpha_n,\alpha}(t)) = 0 \; \forall t > 0$.

Proof. (1) Let any $\epsilon > 0$ be fixed. According to the definition of η, then $\eta^{-1}(\epsilon) \in (0,1)$; for $\epsilon' \in (0, 1 - \eta^{-1}(\epsilon))$, we have $1 - \epsilon' > \eta^{-1}(\epsilon)$, so we deduce that $\eta(1 - \epsilon') < \epsilon$. Remarking that $\{\alpha_n\}$ is an $\mathcal{L}$-Cauchy sequence, for above ϵ' and any $t > 0$, we see $n_0 \in \mathbb{N}^+$ such that $\mathcal{L}_{\alpha_n,\alpha_m,\alpha_m}(t) > 1 - \epsilon'$ for all $n, m > N_0$. Hence, $\eta(\mathcal{L}_{\alpha_n,\alpha_m,\alpha_m}(t)) < \eta(1 - \epsilon') < \epsilon$.

For any $r \in (0,1)$, then $\eta(1 - r) > 0$. Applying this condition, for $\epsilon = \eta(1 - r)$ and any $t > 0$, we see $n_0 \in \mathbb{N}^+$ such that $\eta(M(\alpha_n, \alpha_m, \alpha_m)) < \eta(1 - r)$ for all $m, n > n_0$. Therefore, $M(\alpha_n, \alpha_m, \alpha_m)) > 1 - r$. The proof is completed.

(2) The proof for (2) is analogous. □

3. The Main Results

Now we give the definition of $\mathcal{FH}$-$\mathcal{QC}$ in a $\mathcal{T}$-$\mathcal{FMS}$ below.

Definition 5. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$. A mapping $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ is called $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ if we can find $k \in (0,1)$ such that the following conditions satisfy*

$$\begin{aligned}\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq & k \max\{\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)), \eta(\mathcal{L}_{\alpha,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)), \\ & \eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau), \eta(\mathcal{L}_{\alpha,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \eta(\mathcal{L}_{\gamma,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)), \\ & \eta(\mathcal{L}_{\alpha,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)), \eta(\mathcal{L}_{\gamma,\mathscr{P}\beta,\mathscr{P}\beta}(\tau))\}\end{aligned} \quad (1)$$

for any $\alpha, \beta, \gamma \in \mathcal{X}$ and $\tau > 0$.

Our main theorem is related to $\mathcal{FH}$-$\mathcal{QC}$ in a $\mathcal{T}$-$\mathcal{FMS}$.

Theorem 1. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a complete $\mathcal{T}$-$\mathcal{FMS}$ and let $\mathscr{P} : X \to X$ be $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ such that*

(a) $\alpha \geq \beta * \gamma \Rightarrow \eta(\alpha) \leq \eta(\beta) + \eta(\gamma)$, $\forall\, \alpha, \beta, \gamma \in \{\mathcal{L}_{T^i z, T^j z, T^j z}(\tau) : z \in \mathcal{X}, \tau > 0, i, j \in \mathbb{N}\}$;

(b) for $\forall\, \gamma \in \mathcal{X}$ and each sequence $\{\tau_n\} \subseteq (0,\infty)$, which is decreasing and convergent to 0, $\{\eta(\mathcal{L}_{\gamma,T\gamma,T\gamma}(\tau_i)) : i \in \mathbb{N}\}$ and $\{\eta(\mathcal{L}_{\gamma,\gamma,T\gamma}(\tau_i)) : i \in \mathbb{N}\}$ are bounded.

Then, T has a unique $\mathcal{FP}$ in $\mathcal{X}$.

Proof. For any $\alpha \in X$, take $\alpha := \alpha_0$ and define $\{\alpha_n\}$ in $\mathcal{X}$ by $\alpha_n = \mathscr{P}\alpha_{n-1}, n \in \mathbb{N}^+$. Denote a set $\{(\omega, n) : \omega \in \mathbb{N}, n \in \mathbb{N}^+ \text{ and } \omega < n\}$ by D. For any $\tau > 0$ given, define $P_\tau : D \to [0,\infty)$ by

$$P_\tau(\omega, n) = \max\{\eta(\mathcal{L}_{\alpha_i,\alpha_j,\alpha_j}(\tau))) : \omega \leq i, j \leq n\}.$$

Now, we will show that $\{\alpha_n\}$ is a Cauchy sequence in four steps.

Step 1. We prove that for any $\tau > 0$,

$$P_\tau(\omega+1, n) \leq kP_\tau(\omega, n) \text{ for any } (\omega, n) \in D \text{ with } n > \omega+1. \quad (2)$$

Let $(\omega, n) \in D$ with $n > \omega + 1$ be given. For any $i, j \in \mathbb{N}^+$ with $\omega + 1 \leq i, j \leq n$ and $\tau > 0$, by Equation (1), we have

$$\begin{aligned}
\eta(\mathcal{L}_{\alpha_i,\alpha_j,\alpha_j}(\tau)) &= \eta(\mathcal{L}_{T\alpha_{i-1},T\alpha_{j-1},T\alpha_{j-1}}(\tau)) \\
&\leq k\max\{\eta(\mathcal{L}_{\alpha_{i-1},\alpha_{j-1},\alpha_{j-1}}(\tau)), \eta(\mathcal{L}_{\alpha_{i-1},\alpha_i,\alpha_i}(\tau)), \eta(\mathcal{L}_{\alpha_{j-1},\alpha_j,\alpha_j}(\tau)), \\
&\quad \eta(\mathcal{L}_{\alpha_{j-1},\alpha_j,\alpha_j}(\tau)), \eta(\mathcal{L}_{\alpha_{i-1},\alpha_j,\alpha_j}(\tau)), \eta(\mathcal{L}_{\alpha_{j-1},\alpha_j,\alpha_j}(\tau)), \eta(\mathcal{L}_{\alpha_{j-1},\alpha_i,\alpha_i}(\tau)), \\
&\quad \eta(\mathcal{L}_{\alpha_{i-1},\alpha_j,\alpha_j}(\tau)), \eta(\mathcal{L}_{\alpha_{j-1},\alpha_i,\alpha_i}(\tau)), \eta(\mathcal{L}_{\alpha_{j-1},\alpha_j,\alpha_j}(\tau))\} \\
&\leq kP_\tau(\omega, n), \quad (3)
\end{aligned}$$

which shows that $P_\tau(\omega+1, n) \leq kP_\tau(\omega, n) < P_\tau(\omega, n)$.

Step 2. We verify that for each $\tau > 0$,

$$P_\tau(\omega, n) = \max\{\eta(\mathcal{L}_{\alpha_\omega,\alpha_{p_1},\alpha_{p_1}}(\tau)), \eta(\mathcal{L}_{\alpha_\omega,\alpha_\omega,\alpha_{p_2}}(\tau)) : \omega < p_1, p_2 \leq n\}. \quad (4)$$

Let $(\omega, n) \in D$ be given. If $n - \omega = 1$, then $n = \omega + 1$, and hence

$$\begin{aligned}
P_\tau(\omega, n) &= \max\{\eta(\mathcal{L}_{\alpha_\omega,\alpha_n,\alpha_n}(\tau)), \eta(\mathcal{L}_{\alpha_\omega,\alpha_\omega,\alpha_n}(\tau))\} \\
&= \max\{\eta(\mathcal{L}_{\alpha_\omega,\alpha_{p_1},\alpha_{p_1}}(\tau)), \eta(\mathcal{L}_{\alpha_{p_2},\alpha_\omega,\alpha_\omega}(\tau)) : \omega < p_1, p_2 \leq n\}.
\end{aligned}$$

We now suppose that $n - \omega > 1$. For any $i, j \in \mathbb{N}^+$ with $\omega + 1 \leq i, j \leq n$, by Equation (3), we obtain $\eta(\mathcal{L}_{\alpha_i,\alpha_j,\alpha_j}(\tau)) \leq P_\tau(\omega+1, n) \leq kP_\tau(\omega, n) < P_\tau(\omega, n)$. Thus,

$$P_\tau(\omega, n) = \max\{\eta(\mathcal{L}_{\alpha_\omega,\alpha_{p_1},\alpha_{p_1}}(\tau)), \eta(\mathcal{L}_{\alpha_\omega,\alpha_\omega,\alpha_{p_2}}(\tau)) : \omega < p_1, p_2 \leq n\}.$$

Step 3. We shall show that for every $\tau > 0$, we can find $M > 0$ satisfies

$$P_\tau(0, n) \leq M \quad \forall n \in \mathbb{N}^+.$$

We find a sequence $\{b_l\}$ which is positive, strictly decreasing and $\sum_{l=1}^{\infty} b_l = 1$. By Equation (4), we obtain $P_\tau(0, n) = \max\{\eta(\mathcal{L}_{\alpha_0,\alpha_{p_1},\alpha_{p_1}}(\tau)), \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_{p_2}}(\tau)) : 0 < p_1, p_2 \leq n\}$.

Let us consider the following two cases.

Case 1. We can find that the positive integer $p_1 \leq n$ satisfies

$$\begin{aligned}
P_\tau(0, n) &= \eta(\mathcal{L}_{\alpha_0,\alpha_{p_1},\alpha_{p_1}}(\tau)) \\
&= \eta(\mathcal{L}_{\alpha_0,\alpha_{p_1},\alpha_{p_1}}(\sum_{l=1}^{\infty} b_l\tau)) \\
&\leq \eta(\mathcal{L}_{\alpha_0,\alpha_1,\alpha_1}(\sum_{l=q+1}^{\infty} b_l t)) + \eta(\mathcal{L}_{\alpha_1,\alpha_{p_1},\alpha_{p_1}}(\sum_{l=1}^{q} b_l t)), \forall j
\end{aligned}$$

According to condition (b) and the continuity of η and $\mathcal{L}_{\alpha,\beta,\gamma}(\cdot)$, we have

$$\begin{aligned}
P_\tau(0, n) &\leq \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_1,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau)) + \eta(\mathcal{L}_{\alpha_1,\alpha_{p_1},\alpha_{p_1}}(\tau)) \\
&\leq \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_1,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau)) + kP_\tau(0, n).
\end{aligned}$$

Hence,

$$P_\tau(0,n) \leq \frac{1}{1-k} \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_1,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau).$$

Case 2. We can choose for the positive integer $p_2 \leq n$ to satisfy

$$\begin{aligned} P_\tau(0,n) &= \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_{p_1}}(\tau)) \\ &= \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_{p_1}}(\sum_{l=1}^{\infty} b_l\tau)) \\ &\leq \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau)) + \eta(\mathcal{L}_{\alpha_1,\alpha_1,\alpha_{p_1}}(\sum_{l=1}^{q} b_l\tau)). \end{aligned}$$

Similarly,

$$P_\tau(0,n) \leq \frac{1}{1-k} \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau)).$$

Let

$$M =: \max\{\frac{1}{1-k} \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_1,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau)), \frac{1}{1-k} \limsup_{q\to\infty} \eta(\mathcal{L}_{\alpha_0,\alpha_0,\alpha_1}(\sum_{l=q+1}^{\infty} b_l\tau))\}.$$

We conclude that

$$P_t(0,n) \leq M \quad \forall n \in \mathbb{N}^+.$$

Step 3. We shall prove that $\{\alpha_n\}$ is an $\mathcal{L}$-$\mathcal{CS}$.

For each $\tau > 0$ and $\omega, n \in \mathbb{N}^+$ and $\omega < n$, by applying Equation (2), we have

$$\begin{aligned} \eta(\mathcal{L}_{\alpha_\omega,\alpha_n,\alpha_n}(\tau)) &\leq P_\tau(\omega,n) \\ &\leq kP_\tau(\omega-1,n) \\ &\leq \ldots \\ &\leq k^\omega P_\tau(0,n) \\ &\leq k^\omega M \end{aligned}$$

Therefore, $\eta(\mathcal{L}_{\alpha_\omega,\alpha_n,\alpha_n}(\tau)) \to 0$ as $\omega, n \to \infty$. We know that $\{\alpha_n\}$ is an $\mathcal{L}$-$\mathcal{CS}$ from Proposition 4. Next, we shall show that $\alpha' \in \mathcal{X}$ is the $\mathcal{FP}$ of $\mathscr{P}$.

Since $(\mathcal{X}, \mathcal{L}, *)$ is complete, $x' \in \mathcal{X}$ such that $x_n \to x' \in \mathcal{X}(n \to \infty)$. By Equation (1), we have

$$\begin{aligned} \eta(\mathcal{L}_{\alpha_{n+1},T\alpha',T\alpha'}(\tau)) \leq& k\max\{\eta(\mathcal{L}_{\alpha_n,\alpha',\alpha'}(\tau)), \eta(\mathcal{L}_{\alpha_n,\alpha_{n+1},\alpha_{n+1}}(\tau)), \eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)), \\ &\eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)), \eta(\mathcal{L}_{\alpha_n,T\alpha',T\alpha'}(\tau)), \eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)), \\ &\eta(\mathcal{L}_{\alpha',\alpha_{n+1},\alpha_{n+1}}(\tau)), \eta(\mathcal{L}_{\alpha_n,T\alpha',T\alpha'}(\tau)), \\ &\eta(\mathcal{L}_{\alpha',\alpha_{n+1},\alpha_{n+1}}(\tau)), \eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau))\} \\ =& k\max\{\eta(\mathcal{L}_{\alpha_n,\alpha',\alpha'}(\tau)), \eta(\mathcal{L}_{\alpha_n,\alpha_{n+1},\alpha_{n+1}}(\tau)), \eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)),) \\ &\eta(\mathcal{L}_{\alpha_n,T\alpha',T\alpha'}(\tau)), \eta(\mathcal{L}_{\alpha',\alpha_{n+1},\alpha_{n+1}}(\tau))\}. \end{aligned}$$

Letting $n \to \infty$ in the previous inequality, according to Proposition 3 and the continuity of η, we have $\eta(\mathcal{L}_{\alpha,T\alpha',T\alpha'}(\tau)) \leq k\eta(\mathcal{L}_{\alpha,T\alpha',T\alpha'}(\tau))$ for $\forall\ \tau > 0$; therefore, $\eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)) = 0$ for $\forall\ \tau > 0$, and it follows that $\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau) = 1$ for $\forall\ \tau > 0$, and so $\alpha' = T\alpha'$.

Finally, we shall verify that α' is the unique $\mathcal{FP}$ of $\mathscr{P}$. If β' is also an $\mathcal{FP}$ of T, then for $\forall\ \tau > 0$, by Equation (1), we see

$$\begin{aligned}\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau)) =& \eta(\mathcal{L}_{T\alpha',T\beta',T\beta'}(\tau)) \\ \le& k\max\{\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',T\alpha',T\alpha'}(\tau)),\eta(\mathcal{L}_{\beta',T\beta',T\beta'}(\tau)), \\ & \eta(\mathcal{L}_{\beta',T\beta',T\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',T\beta',T\beta'}(\tau)),\eta(\mathcal{L}_{\beta',T\beta',T\beta'}(\tau)),\eta(\mathcal{L}_{\beta',T\alpha',T\alpha'}(\tau)), \\ & \eta(\mathcal{L}_{\alpha',T\beta',T\beta'}(\tau)),\eta(\mathcal{L}_{\beta',T\alpha',T\alpha'}(\tau)),\eta(\mathcal{L}_{\beta',T\beta',T\beta'}(\tau))\} \\ =& k\max\{\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau))\}.\end{aligned}$$

Similarly, $\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau)) \le k\max\{\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau))\}$

Hence, $\forall\ \tau > 0$

$$\max\{\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau))\} \le k\max\{\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau))\}$$

$\max\{\eta(\mathcal{L}_{\alpha',\alpha',\beta'}(\tau)),\eta(\mathcal{L}_{\alpha',\beta',\beta'}(\tau))\} = 0$. Therefore, $\alpha' = \beta'$. □

Note that $\mathscr{P}$ is not required to be continuous in Theorem 1; now, we construct the following example to illustrate this.

Example 3. *Let* $\mathcal{X} = [0,2]$ *and* $* = *_p$. *Define*

$$\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = \left(\frac{3+\tau}{4+\tau}\right)^{|\alpha-\beta|+|\gamma-\beta|+|\gamma-\alpha|} \text{ for } \alpha,\beta,\gamma \in \mathcal{X} \text{ and } \tau > 0.$$

From Example 2, we see that $(\mathcal{X},\mathcal{L},*)$ is a $\mathcal{T}$-$\mathcal{FMS}$; furthermore, $(\mathcal{X},\mathcal{L},*)$ is $\mathcal{L}$-complete. Now, we consider the following mapping:

$$\mathscr{P}\delta = \begin{cases} \frac{1}{8}, & \delta = 0, \\ \frac{1}{4}, & \delta \in (0,2]. \end{cases}$$

For $\eta(t) = ln\frac{1}{t}, t \in (0,1]$, obviously, $\eta(t) \in \mathcal{H}$. Then, the following holds:

(1) $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ is $F\mathcal{H}$-QC related to $\eta = ln\frac{1}{t} \in \mathcal{H}$;

(2) $\mathscr{P}$ is not continuous on $\mathcal{X}$, and $\mathscr{P}$ allows for a unique $\mathcal{FP}$ in $\mathcal{X}$.

Proof. (1) It is sufficient to prove that $\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) \le \frac{1}{2}\mathcal{M}(\alpha,\beta,\gamma)$ for any $\alpha,\beta,\gamma \in \mathcal{X}$ and $\tau > 0$, where

$$\begin{aligned}\mathcal{M}(\alpha,\beta,\gamma) =& \max\{\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)),\eta(\mathcal{L}_{\alpha,T\alpha,T\alpha}(\tau)),\eta(\mathcal{L}_{\beta,T\beta,T\beta}(\tau)), \\ & \eta(\mathcal{L}_{\gamma,T\gamma,T\gamma}(\tau)),\eta(\mathcal{L}_{\alpha,T\beta,T\beta}(\tau)),\eta(\mathcal{L}_{\beta,T\gamma,T\gamma}(\tau)),\eta(\mathcal{L}_{\gamma,T\alpha,T\alpha}(\tau)), \\ & \eta(\mathcal{L}_{\alpha,T\gamma,T\gamma}(\tau)),\eta(\mathcal{L}_{\beta,T\alpha,T\alpha}(\tau)),\eta(\mathcal{L}_{\gamma,T\beta,T\beta}(\tau))\}.\end{aligned}$$

Let us discuss three cases:

Case 1. If $\alpha = 0, \beta,\gamma \in (0,2]$, then for any $\tau > 0$,

$$\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau) = \mathcal{L}_{\frac{1}{8},\frac{1}{4},\frac{1}{4}}(\tau) = \left(\frac{\tau+3}{\tau+4}\right)^{\frac{1}{4}} = (\mathcal{L}_{0,\frac{1}{4},\frac{1}{4}}(\tau))^{\frac{1}{2}} = (\mathcal{L}_{\alpha,T\beta,T\beta}(\tau))^{\frac{1}{2}}.$$

Hence, $\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) = -\frac{1}{2}\ln(\mathcal{L}_{\alpha,T\beta,T\beta}(\tau)) = \frac{1}{2}\eta(\mathcal{L}_{\alpha,T\beta,T\beta}(\tau))$. Therefore, for any $t > 0$,

$$\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) \le \frac{1}{2}\mathcal{M}(\alpha,\beta,\gamma).$$

Case 2. If $\alpha = \beta = 0, \gamma \in (0, 2]$, then for any $\tau > 0$,

$$\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau) = \mathcal{L}_{\frac{1}{8},\frac{1}{8},\frac{1}{4}}(\tau) = (\frac{t+3}{t+4})^{\frac{1}{4}} = (\mathcal{L}_{0,\frac{1}{4},\frac{1}{4}}(\tau))^{\frac{1}{2}} = (\mathcal{L}_{\beta,T\gamma,T\gamma}(t))^{\frac{1}{2}}.$$

Similarly, for any $\tau > 0$,

$$\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) \leq \frac{1}{2}\mathcal{M}(\alpha, \beta, \gamma).$$

Case 3. If $\alpha = \beta = \gamma = 0$ or $\alpha = \beta = \gamma \in (0, 2]$, then $\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) = 0$, and apparently, for any $\tau > 0$,

$$\eta(\mathcal{L}_{T\alpha,T\beta,T\gamma}(\tau)) \leq \frac{1}{2}\mathcal{M}(\alpha, \beta, \gamma).$$

Note that $\eta(t) = ln\frac{1}{t}(t \in (0, 1])$ and $*_p$ satisfy (a),(b) of Theorem 1; hence, all the conditions of Theorem 1 are fulfilled.

(2) $\mathscr{P}$ is not continuous at $\delta = 0$. In fact, for any $\tau > 0$, $\mathcal{L}_{\frac{1}{n},\frac{1}{n},0}(\tau) = (\frac{3+\tau}{4+\tau})^{\frac{2}{n}} \to 1$ as $n \to \infty$; hence, $\frac{1}{n} \to 0$ as $n \to \infty$. However, $\mathscr{P}(\frac{1}{n}) \equiv \frac{1}{4} \neq \mathscr{P}(0) = \frac{1}{8}$. Therefore, $\mathscr{P}$ is not continuous on $\mathcal{X}$.

Obviously, $\delta = \frac{1}{4}$ is a unique $\mathcal{FP}$ of $\mathscr{P}$. □

In 2015, Amini-Harandi, A. and Mihet, D. considered the concept of $\mathcal{FH}$-$\mathcal{QC}$ in an $\mathcal{FMS}$-$(\mathcal{X}, \mathcal{F}, *)$ as follows.

The mapping $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ is called $\mathcal{FH}$-$\mathcal{QC}$ related to $\eta \in \mathcal{H}$ if $k \in (0, 1)$ satisfies

$$\begin{aligned}\eta(\mathcal{F}(\mathscr{P}\alpha, \mathscr{P}\beta, \tau)) \leq &\lambda \max\{\eta(\mathcal{F}(\alpha, \beta, \tau)), \eta(\mathcal{F}(\alpha, \mathscr{P}\alpha, \tau)), \eta(\mathcal{F}(\beta, \mathscr{P}\beta, \tau)),\\ &\eta(\mathcal{F}(\alpha, \mathscr{P}\beta, \tau)), \eta(\mathcal{F}(\mathscr{P}\alpha, \beta, \tau))\} \text{ for all } \alpha, \beta \in \mathcal{X} \text{ and } \tau > 0.\end{aligned}$$

We will clarify their results in [26] using Theorem 1 as a consequence of our theorem shortly.

Corollary 1 (See Theorem 2.3 of [26])**.** *Let $(\mathcal{X}, \mathcal{F}, *)$ be a complete $\mathcal{FMS}$ and let $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ be $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ such that*

(a) $\alpha \geq \beta * \gamma \Rightarrow \eta(\alpha) \leq \eta(\beta) + \eta(\gamma), \forall\, \alpha, \beta, \gamma \in \{\mathcal{F}(\mathscr{P}^i z, \mathscr{P}^j z, t) : z \in \mathcal{X}, \tau > 0, i, j \in \mathbb{N}\}$;

(b) for $\forall\, \gamma \in \mathcal{X}$ and each sequence $\{\tau_n\} \subseteq (0, \infty)$ which is decreasing and convergent to 0, $\{\eta(\mathcal{F}(\gamma, \mathscr{P}\gamma, \tau_i)) : i \in \mathbb{N}\}$ is bounded.

Then, $\mathscr{P}$ has a unique $\mathcal{FP}$ in $\mathcal{X}$.

Proof. For all $\alpha, \beta, \gamma, \in \mathcal{X}$ and $\tau > 0$, define

$$\mathcal{L}_{\alpha,\beta,\gamma,}(t) = \min\{\mathcal{F}(\alpha, \beta, \tau), \mathcal{F}(\alpha, \gamma, \tau), \mathcal{F}(\gamma, \beta, \tau)\}.$$

By Example 1, we know that $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$. Moreover, it is obvious that $(\mathcal{X}, \mathcal{L}, *)$ is $\mathcal{L}$-complete due to the completeness of $(\mathcal{X}, M, *)$. Since

$$\begin{aligned}\mathcal{L}_{\mathscr{P}^i\alpha,\mathscr{P}^j\alpha,\mathscr{P}^j\alpha}(\tau) =& \min\{\mathcal{F}(\mathscr{P}^i\alpha, \mathscr{P}^j\alpha, \tau), \mathcal{F}(\mathscr{P}^j\alpha, \mathscr{P}^i\alpha, \tau), \mathcal{F}(T^j\alpha, \mathscr{P}^j\alpha, \tau)\}\\ =& \mathcal{F}(\mathscr{P}^i\alpha, T^j\alpha, \tau)\end{aligned}$$

and $\mathcal{L}_{\alpha,T\alpha,T\alpha}(\tau) = \mathcal{L}_{\alpha,T\alpha,T\alpha}(\tau) = M(\alpha, T\alpha, \tau)$, it follows that T satisfies (a), (b) of Theorem 1 in the $\mathcal{T}$-$\mathcal{FMS}$ (X, $\mathcal{L}$, $*$).

Next, we will prove that $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ is $F\mathcal{H}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ in $(\mathcal{X}, \mathcal{L}, *)$. In fact, for any $\alpha, \beta, \gamma \in \mathcal{X}$ and $\tau > 0$, according to the definition of $\mathcal{L}_{\alpha,\beta,\gamma}(\tau)$ and Equation (1), we obtain

$$
\begin{aligned}
\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) &= \eta(\min\{\mathcal{F}(\mathscr{P}\alpha,\mathscr{P}\beta,\tau),\mathcal{F}(\mathscr{P}\alpha,\mathscr{P}\gamma,t),\mathcal{F}(\mathscr{P}\gamma,\mathscr{P}\beta,t)\}) \\
&= \max\{\eta(\mathcal{F}(\mathscr{P}\alpha,\mathscr{P}\beta,\tau)),\eta(\mathcal{F}(\mathscr{P}\alpha,\mathscr{P}\gamma,\tau)),\eta(\mathcal{F}(\mathscr{P}\gamma,\mathscr{P}\beta,\tau))\} \\
&\leq k\max\{\eta(\mathcal{F}(\alpha,\beta,\tau)),\eta(\mathcal{F}(\alpha,\mathscr{P}\alpha,\tau)),\eta(\mathcal{F}(\beta,\mathscr{P}\beta,\tau)),\eta(\mathcal{F}(\alpha,\mathscr{P}\beta,t)), \\
&\quad \eta(\mathcal{F}(\mathscr{P}\alpha,\beta,\tau)),\eta(\mathcal{F}(\alpha,\gamma,\tau)),\eta(\mathcal{F}(\alpha,\mathscr{P}\alpha,\tau)),\eta(\mathcal{F}(\gamma,\mathscr{P}\gamma,\tau)), \\
&\quad \eta(\mathcal{F}(\alpha,T\gamma,\tau)),\eta(\mathcal{F}(\gamma,T\alpha,\tau)),\eta(\mathcal{F}(\gamma,\beta,\tau)),\eta(\mathcal{F}(\gamma,T\gamma,\tau)), \\
&\quad \eta(\mathcal{F}(\beta,\mathscr{P}\beta,\tau)),\eta(\mathcal{F}(\gamma,\mathscr{P}\beta,\tau)),\eta(\mathcal{F}(\beta,\mathscr{P}\gamma,\tau))\} \\
&= k\eta\min\{\mathcal{L}_{\alpha,\beta,\gamma}(\tau),\mathcal{L}_{\alpha,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau),\mathcal{L}_{\beta,\mathscr{P}\beta,\mathscr{P}\beta}(\tau),\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(t),\mathcal{L}_{\alpha,\mathscr{P}\beta,\mathscr{P}\beta}(\tau), \\
&\quad \mathcal{L}_{\beta,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau),\mathcal{L}_{\gamma,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau),\mathcal{L}_{\alpha,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau),\mathcal{L}_{\beta,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau),\mathcal{L}_{\gamma,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)\} \\
&= k\max\{\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)),\eta(\mathcal{L}_{\alpha,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)),\eta(\mathcal{L}_{y,\mathscr{P}y,\mathscr{P}y}(t\tau)),\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau), \\
&\quad \eta(\mathcal{L}_{\alpha,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)),\eta(\mathcal{L}_{\beta,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)),\eta(\mathcal{L}_{\gamma,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)),\eta(\mathcal{L}_{\alpha,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \\
&\quad \eta(\mathcal{L}_{\beta,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)),\eta(\mathcal{L}_{\gamma,\mathscr{P}\beta,\mathscr{P}\beta}(\tau))\}.
\end{aligned}
$$

Therefore, applying Theorem 1, we find that $\mathscr{P}$ has an $\mathcal{FP}$ $x' \in \mathcal{X}$ in the context of the $\mathcal{T}$-$\mathcal{FMS}$ $(\mathcal{X},\mathcal{L},*)$.

This shows that T has a unique $\mathcal{FP}$ in the context of the $\mathcal{FMS}$ $(\mathcal{X},\mathcal{F},*)$. □

In the next proposition, we will give an equivalent form of condition (b) in Theorem 1.

Proposition 5. *Let $(\mathcal{X},\mathcal{L},*)$ be a $\mathcal{T}$-$\mathcal{FMS}$, $\eta \in \mathcal{H}$ and $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ a mapping. Given $\gamma \in \mathcal{X}$, the following statements are equivalent:*

(1) $\bigwedge_{\tau>0}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) > 0$.

(2) For each sequence $\{\tau_n\} \subseteq (0,\infty)$ which is decreasing and convergent to 0, $\{\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i)) : i \in \mathbb{N}\}$ is bounded.

Proof. According to Proposition 2, we see $\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\cdot)$ is non-decreasing. Hence,

$$\bigwedge_{\tau>0}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) = \lim_{t\to 0^+}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau).$$

Suppose $\bigwedge_{\tau>0}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) > 0$, and let $\bigwedge_{\tau>0}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) = a > 0$; then, $a \in (0,1]$ and $\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i) \geq a$ for every $i \in \mathbb{N}$. Remarking that η is strictly decreasing, it is obvious that

$$\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i)) \leq \eta(a) \quad \forall\, i \in \mathbb{N}.$$

For any sequence $\{\tau_n\} \subseteq (0,\infty)$, $\tau_n \downarrow 0$, we can find $M > 0$ such that

$$\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i)) \leq M \text{ for every } i \in \mathbb{N}.$$

Since η is strictly decreasing, we have

$$\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i) = \eta^{-1}(\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i))) \geq \eta^{-1}(M) > 0 \text{ for every } i \in \mathbb{N}.$$

Therefore,

$$\bigwedge_{t>0}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) = \liminf_{i}\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau_i) \geq \eta^{-1}(M) > 0.$$

□

Similarly, we can deduce the following result.

Proposition 6. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a $\mathcal{T}$-$\mathcal{FMS}$, $\eta \in \mathcal{H}$ and $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ a mapping. Given $\gamma \in \mathcal{X}$, the following statements are equivalent:*

(1) $\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\gamma,\mathscr{P}\gamma}(\tau) > 0$.

(2) For each sequence $\{\tau_n\} \subseteq (0, \infty)$ which is decreasing and convergent to 0, $\{\eta(\mathcal{L}_{\gamma,\gamma,\mathscr{P}\gamma}(\tau_i)) : i \in \mathbb{N}\}$ is bounded.

Theorem 1 can be written in a more elegant way, as follows:

Theorem 2. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a complete $\mathcal{T}$-$\mathcal{FMS}$ and let $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ be $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ such that*

(a) $\alpha \geq \beta * \gamma \Rightarrow \eta(\alpha) \leq \eta(\beta) + \eta(\gamma), \forall\, \alpha, \beta, \gamma \in \{\mathcal{L}_{\mathscr{P}^i z, \mathscr{P}^j z, \mathscr{P}^j z'}(\tau) : z \in \mathcal{X}, \tau > 0, i, j \in \mathbb{N}\}$;

(b) $\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\gamma,\mathscr{P}\gamma}(\tau) > 0$ and $\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) > 0$ for all $\gamma \in \mathcal{X}$.

Then, $\mathscr{P}$ has a unique $\mathcal{FP}$ in $\mathcal{X}$.

Remark 1. *From the proof of Theorem 1, we know that for any $\alpha \in \mathcal{X}$, sequence $\{T^n\alpha\}$ is convergent to the $\mathcal{FP}$. We will give another idea of the theorem of sufficient and necessary conditions which are more widely used for the existence of $\mathcal{FP}$s.*

Theorem 3. *Let $(\mathcal{X}, \mathcal{L}, *)$ be a complete $\mathcal{T}$-$\mathcal{FMS}$ and let $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ be $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ such that $\alpha \geq \beta * \gamma \Rightarrow \eta(\alpha) \leq \eta(\beta) + \eta(\gamma)$, $\forall\, \alpha, \beta, \gamma \in \{\mathcal{L}_{\mathscr{P}^i z, \mathscr{P}^j z, \mathscr{P}^j z'}(\tau) : z \in \mathcal{X}, \tau > 0, i, j \in \mathbb{N}\}$; then, T has a unique $\mathcal{FP}$ in $\mathcal{X}$ if and only if $\gamma \in \mathcal{X}$ such that $\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\gamma,\mathscr{P}\gamma}(\tau) > 0$ and $\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) > 0$.*

The following example is constructed to illustrate that Theorem 3 has wider applications in the existence of $\mathcal{FP}$s to some extent.

Example 4. *Let $\mathcal{X} = [0, 4]$, and define $\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = [e^{\frac{|\beta-\alpha|+|\gamma-\beta|+|\gamma-\alpha|}{\tau}}]^{-1}$ for any α, β and $\gamma \in \mathcal{X}$ and $\tau > 0$; then, $(\mathcal{X}, \mathcal{L}, *)$ is a $\mathcal{T}$-$\mathcal{FMS}$ (refer to [Example 2.8] of [27]). Furthermore, $(X, \mathcal{L}, *)$ is $\mathcal{L}$-complete. Consider $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ as follows.*

$$\mathscr{P}\delta = \begin{cases} \frac{2}{3}\delta, & \delta \in [0, 4) \\ 1, & \delta = 4. \end{cases}$$

Then, the following holds:

(1) $\mathscr{P}$ is not continuous on $\mathcal{X}$;

(2) $\mathscr{P}$ is $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta = ln\frac{1}{t}, t \in (0, 1] \in \mathcal{H}$;

(3) Condition (b) of Theorem 1 is not fulfilled;

(4) For $\eta(t) = ln\frac{1}{t}$, T satisfies all the conditions of Theorem 3, and $\mathscr{P}$ has a unique $\mathcal{FP}$.

Proof. (1) It is not difficult to prove that $\mathscr{P}$ is not continuous at $\delta = 4$. Hence, $\mathscr{P}$ is not continuous on $\mathcal{X}$.

(2) Now, we will prove that for $\eta = ln\frac{1}{t} \in \mathcal{H}$, $k = \frac{2}{3} \in (0, 1)$, satisfying the following condition:

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha, \beta, \gamma)$$

for any $\alpha, \beta, \gamma \in \mathcal{X}$ and $\tau > 0$, where

$$\begin{aligned} M(\alpha, \beta, \gamma) = \max\{&\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)), \eta(\mathcal{L}_{\alpha,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)), \\ &\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau), \eta(\mathcal{L}_{\alpha,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \\ &\eta(\mathcal{L}_{\alpha,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)), \eta(\mathcal{L}_{\beta,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau)), \eta(\mathcal{L}_{\gamma,\mathscr{P}\beta,\mathscr{P}\beta}(\tau))\}. \end{aligned}$$

In fact, for $\eta = ln\frac{1}{t} \in \mathcal{H}$ and $\alpha, \beta, \gamma \in \mathcal{X}$, we have

$$\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)) = \frac{|\beta - \alpha| + |\gamma - \beta| + |\gamma - \alpha|}{\tau}.$$

We consider the following two cases:

Case 1. Suppose $\alpha \in [0, 4)$.

If $\beta \in [0, 4), \gamma \in [0, 4)$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \frac{\frac{2}{3}|\beta - \alpha| + \frac{2}{3}|\gamma - \beta| + \frac{2}{3}|\gamma - \alpha|}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)).$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha, \beta, \gamma).$$

If $\beta \in [0, 4), \gamma = 4$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \eta(\mathcal{L}_{\frac{2}{3}\alpha,\frac{2}{3}\beta,1}(\tau)) = \frac{\frac{2}{3}|\beta - \alpha| + |\frac{2}{3}\beta - 1| + |\frac{2}{3}\alpha - 1|}{\tau}.$$

If $\alpha \in [0, \frac{3}{2}], \beta \in [0, \frac{3}{2}]$, without a loss of generality, we assume $\alpha \geq \beta$; then,

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \frac{\frac{2}{3}(\beta - \alpha) + (1 - \frac{2}{3}\beta) + (1 - \frac{2}{3}\alpha)}{\tau} = \frac{2 - \frac{4}{3}\beta}{\tau},$$

$$\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)) = \eta(\mathcal{L}_{4,1,1}(\tau)) = \frac{6}{\tau}.$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{\tau} \leq \frac{4}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha, \beta, \gamma).$$

If $\alpha \in [\frac{3}{2}, 4], \beta \in [\frac{3}{2}, 4]$, without a loss of generality, we assume $\alpha \geq \beta$; then,

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(t)) = \frac{\frac{2}{3}(\beta - \alpha) + (\frac{2}{3}\beta - 1) + (\frac{2}{3}\alpha - 1)}{\tau} = \frac{\frac{4}{3}\alpha - 2}{\tau}.$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{\frac{4}{3} \times 4 - 2}{\tau} = \frac{\frac{10}{3}}{\tau} \leq \frac{\frac{12}{3}}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha, \beta, \gamma).$$

If $\alpha \in [\frac{3}{2}, 4], \beta \in [0, \frac{3}{2}]$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \frac{\frac{2}{3}(\beta - \alpha) + (1 - \frac{2}{3}\beta) + (\frac{2}{3}\alpha - 1)}{\tau} = \frac{\frac{4}{3}(\beta - \alpha)}{\tau},$$

$$\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)) = \eta(\mathcal{L}_{\alpha,\beta,4}(\tau)) = \frac{8 - 2\beta}{\tau}.$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{\frac{4}{3}(4 - \beta)}{\tau} = \frac{\frac{2}{3}(8 - 2\beta)}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\alpha,\beta,\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha, \beta, \gamma).$$

If $\beta = 4, \gamma = 4$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \eta(\mathcal{L}_{\frac{2}{3}\alpha,1,1}(\tau)) = \frac{2|\frac{2}{3}\alpha - 1|}{\tau}.$$

If $\alpha \in [0, \frac{3}{2})$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \frac{2 - \frac{4}{3}\alpha}{\tau},$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{\tau} \leq \frac{4}{\tau} = \frac{2}{3} \times \frac{6}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\beta,\mathscr{P}\beta,\mathscr{P}\beta}(\tau)) \leq \frac{2}{3}M(\alpha,\beta,\gamma).$$

If $\alpha \in [\frac{3}{2}, 4)$, then

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) = \frac{\frac{4}{3}\alpha - 2}{\tau}.$$

Thus, for any $\tau > 0$, we have

$$\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{\frac{4}{3} \times 4 - 2}{\tau} = \frac{\frac{10}{3}}{\tau} \leq \frac{\frac{12}{3}}{\tau} = \frac{2}{3} \times \frac{6}{\tau} = \frac{2}{3}\eta(\mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}\mathcal{M}(\alpha,\beta,\gamma).$$

Case 2. Suppose $\alpha = 4$.

If $\beta \in [0,4), \gamma \in [0,4)$, then we can conclude that $\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}M(\alpha,\beta,\gamma)$ from a similar argument in Case 1.

If $\beta \in [0,4), \gamma = 4$, then we can prove that $\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}M(\alpha,\beta,\gamma)$, as in the proof for Case 1.

If $\beta = 4, \gamma = 4$, then $\eta(\mathcal{L}_{\mathscr{P}\alpha,\mathscr{P}\beta,\mathscr{P}\gamma}(\tau)) \leq \frac{2}{3}M(\alpha,\beta,\gamma)$ for any $\tau > 0$ apparently.

(3) In fact, for $\alpha = 4$, we see $\bigwedge_{t>0} \mathcal{L}_{\alpha,\alpha,\mathscr{P}\alpha}(\tau) = \bigwedge_{\tau>0} \mathcal{L}_{4,4,1}(\tau) = \bigwedge_{\tau>0}[e^{\frac{6}{\tau}}]^{-1} = 0$, and $\bigwedge_{\tau>0} \mathcal{L}_{\alpha,\mathscr{P}\alpha,\mathscr{P}\alpha}(\tau) = 0$. Hence, condition (b) of Theorem 1 is not fulfilled.

(4) For $\eta(t) = ln\frac{1}{t}$, condition (a) of Theorem 3 is clearly fulfilled. In addition, $\gamma = 0$ such that

$$\bigwedge_{\tau>0} \mathcal{L}_{\gamma,\gamma,\mathscr{P}\gamma}(\tau) = \bigwedge_{\tau>0} \mathcal{L}_{\gamma,\mathscr{P}\gamma,\mathscr{P}\gamma}(\tau) = 1 > 0,$$

and thus, $\mathscr{P}$ and η meet all the conditions of Theorem 3 and T has a unique $\mathcal{FP}$. Indeed, that is $\gamma = 0$. □

Similarly, Corollary 1 (or Theorem 2.3 in [26]) can be written as follows.

Theorem 4. *Let $(\mathcal{X}, M, *)$ be a complete $\mathcal{FMS}$ and let $\mathscr{P} : \mathcal{X} \to \mathcal{X}$ be $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$ such that $\alpha \geq \beta * \gamma \Rightarrow \eta(\alpha) \leq \eta(\beta) + \eta(\gamma)$, for all $\alpha, \beta, \gamma \in M(T^i z, T^j z, t) : z \in \mathcal{X}, \tau > 0$, $i, j \in \mathbb{N}$; then, T has a unique $\mathcal{FP}$ in $\mathcal{X}$ if and only if $\gamma \in \mathcal{X}$ such that $\bigwedge_{\tau>0} M(\gamma, \mathscr{P}\gamma, \tau) > 0$.*

4. Application to the Existence of Solutions to Integral Equations

In this section, by using Theorem 2, we discuss the existence of solutions to the following integral equations:

$$x(t) = \mu \int_a^b G(t,v) f(v, x(v)) dv, \tag{5}$$

where $G : [a,b] \times [a,b] \to \mathbb{R}$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\mathcal{X} = C[a,b]$ be a set of all real continuous functions on [a, b], and $* = *_p$. Define $\mathcal{L} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to (0,1]$ by

$$\mathcal{L}_{\alpha,\beta,\gamma}(\tau) = e^{-\frac{\max\{\|\alpha-\beta\|,\|\beta-\gamma\|,\|\alpha-\gamma\|\}}{\tau}} \text{ for all } \alpha,\beta,\gamma \in \mathcal{X}, \tau > 0,$$

where $\|\alpha - \beta\| = \max\limits_{a \le t \le b} |\alpha(t) - \beta(t)|$, $\|\beta - \gamma\| = \max\limits_{a \le t \le b} |\beta(t) - \gamma(t)|$, $\|\alpha - \gamma\| = \max\limits_{a \le t \le b} |\alpha(t) - \gamma(t)|$. Apparently, $(\mathcal{X}, \mathcal{L}, *)$ is a complete $\mathcal{T}$-$\mathcal{FMS}$.

Consider the self-mapping $T : \mathcal{X} \to \mathcal{X}$ defined by

$$Tx(t) = \mu \int_a^b G(t, v) f(v, x(v)) dv$$

Clearly, $x(t)$ is a solution of Equation (5) if and only if x is a fixed point of T. Suppose the following conditions are satisfied:

(1) $|\mu| < 1$;

(2) $\max\limits_{a \le t \le b} \int_a^b G(t, v) dv \le 1$;

(3) $\max\limits_{a \le t \le b} |f(t, \alpha) - f(t, \beta)| \le \|\alpha - \beta\|$.

Hence, we have

$$\begin{aligned} |T\alpha - T\beta| =& |\mu \int_a^b G(t, v)[f(v, \alpha(v)) - f(v, \beta(v)) dv]| \\ \le& |\mu| \int_a^b G(t, v) |f(v, \alpha(v)) - f(v, \beta(v)| dv \\ \le& |\mu| \int_a^b G(t, v) dv \cdot \|\alpha - \beta\| \\ \le& |\mu| \|\alpha - \beta\|, \end{aligned}$$

and then, we obtain $\|T\alpha - T\beta\| \le |\mu| \|\alpha - \beta\|$ for all α and $\beta \in \mathcal{X}$. Similarly, $\|T\beta - T\gamma\| \le |\mu| \|\beta - \gamma\|$ for all β and $\gamma \in \mathcal{X}$ and $\|T\alpha - T\gamma\| \le |\mu| \|\alpha - \gamma\|$ for all $\alpha, \beta, \gamma \in \mathcal{X}$. Therefore,

$$\max\{\|T\alpha - T\beta\|, \|T\beta - T\gamma\|, \|T\alpha - T\gamma\|\} \le |\mu| \max\{\|\alpha - \beta\|, \|\beta - \gamma\|, \|\alpha - \gamma\|\},$$

for all $\alpha, \beta, \gamma \in \mathcal{X}$. We show that $\mathcal{L}_{T\alpha, T\beta, T\gamma}(\tau) \ge |\mu| \mathcal{L}_{\alpha, \beta, \gamma}(\tau)$ for all $\alpha, \beta, \gamma \in \mathcal{X}$.

For $\eta = -lnt$, $\eta(\mathcal{L}_{T\alpha, T\beta, T\gamma}(\tau)) \le |\mu| \eta(\mathcal{L}_{\alpha, \beta, \gamma}(\tau)) \le |\mu| \mathcal{M}(\alpha, \beta, \gamma)$, for all $\alpha, \beta, \gamma \in \mathcal{X}$, we know that T is $\mathcal{FH}$-$\mathcal{QC}$ relating to $\eta \in \mathcal{H}$, and condition (a) of Theorem 2 is satisfied; if condition (b) of Theorem 2 is also satisfied, then we can conclude that T has a unique fixed point in $\mathcal{X}$ using Theorem 2, and then Equation (5) has a unique solution, $x(t) \in \mathcal{X}$.

5. Conclusions

In this paper, we present the notion of $\mathcal{FH}$-$\mathcal{QC}$ in a $\mathcal{T}$-$\mathcal{FMS}$ and derive $\mathcal{FPT}$s for this contraction. Satisfyingly, we can obtain Amini-Harandi and Mihet's results using our theorem in the setting of a $\mathcal{GV}$-$\mathcal{FMS}$. We propose the conditional equivalence of the theorem and give another form of the theorem which is more widely used. Moreover, we construct interesting examples to illustrate our results. As an application, we show the existence of solutions to integral equations in a $\mathcal{T}$-$\mathcal{FMS}$.

In addition, because we took only one type of function η, the examples we constructed in this paper lack variety. Whether richer examples exist is worthy of further investigation.

As future research direction, we point out the following:

1. To study the relationship between the fixed point theorems in $\mathcal{T}$-$\mathcal{FMS}$s and $\mathcal{GV}$-$\mathcal{FMS}$s and whether all fixed point results in $\mathcal{GV}$-$\mathcal{FMS}$s can be derived from the corresponding results in $\mathcal{T}$-$\mathcal{FMS}$s.

2. To study more applications of fixed point theorems in $\mathcal{T}$-$\mathcal{FMS}$s, especially numerical examples with the help of real-life applications.

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