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A Normality Criterion for Sharing a Holomorphic Function

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Abstract: In this paper, we scrutinize a collection of meromorphic functions known as normal families, prove the theorem that normal families share a holomorphic function, and present several illustrative counterexamples.

Keywords: meromorphic function; shared values; normal families

MSC: 30D35; 30D45

1. Introduction

A family $\mathcal{J}$ of meromorphic functions in a subset $S \subset \mathbb{C}$ is considered normal on S if every sequence $\{r_n\} \subset \mathcal{J}$ has a subsequence $\left\{r_{n_j}\right\}$ that converges locally and uniformly with respect to the spherical metric to r on S, where the limit function r may also be equal to ∞. (see [1,2])

It is well known that there is a strong connection between normal families and normal functions, leading us to naturally anticipate the criteria for normal functions aligning with established criteria for normal families. Developed by the French mathematician Paul Montel [3], the concept of normal families of meromorphic functions has played a significant role in complex analysis since its inception in 1912. Montel's theorem establishes that a family of meromorphic functions $\mathcal{J}$ is considered normal on a domain S if there are three distinct points, c, d, e, in the extended complex plane such that each $r(z) \in \mathcal{J}$ omits c, d, e on S. Schiff [4] documented this result as the Fundamental Normality Test (FNT). Subsequently, Carathéodory [5] proved that the omitted values do not need to be fixed and they may depend on the particular function in the family as long as these omitted values are uniformly separated (see [6]). In 2021, Beardon and Minda [7] revealed that Montel had presented an expansion of his three-excluded-values theorem, providing a necessary and sufficient condition for a family of meromorphic functions to be normal: a family of meromorphic functions $\mathcal{J}$ is considered normal on a domain S if there are four ε-separated values in the extended complex plane such that their preimages are equi-separated on compacta. However, this finding was not extensively documented, and there was a minor flaw in Montel's proof that Beardon and Minda addressed (see [7]). The corresponding result for normal functions was presented by Lehto and Virtanen [8], stating that a function meromorphic r on S is deemed normal if there are three distinct points, c, d, e, in the extended complex plane such that $r(z) \neq c, d, e$ on S.

We use the notation $r = \varphi \Rightarrow l = \phi$ to indicate that $l(z) = \phi$ whenever $r(z) = \varphi$. If we write $r = \varphi \Leftrightarrow l = \phi$, it means that $l(z) = \phi$ if and only if $r(z) = \varphi$. We can characterize the sharing of φ on S between functions r and l by stating that they are equivalent when the following relation holds: $r = \varphi \Leftrightarrow l = \varphi$. When both $r - \varphi$ for function r and $l - \varphi$ for function l have identical zeros with the same multiplicity, we express this as saying that the value of φ is shared by functions r and l counting multiplicities (CMs), as denoted by $r = \varphi \rightleftharpoons l = \varphi$.

The concept of normality with respect to shared values was first studied by Schwick [9], who demonstrated the following result:

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Theorem 1. *If every meromorphic function r in a family $\mathcal{J}$ on a domain S shares three separate complex numbers, c, d, and e, that are finite with its derivative r', then we can conclude that $\mathcal{J}$ constitutes a normal family on S.*

In this conclusion, the sharing relationship requires three times. Is it possible to reduce the sharing relationship to two times? The above-mentioned result has been further improved in [10], which yielded the following results:

Theorem 2. *Let $\mathcal{J}$ be a collection of meromorphic functions on a domain S and suppose that the complex numbers a, b, c, and d are mutually distinct, subject to the conditions that $a \neq c$ and $b \neq d$. If each function r in $\mathcal{J}$ satisfies the conditions $r(z) = a \Leftrightarrow r'(z) = b$ and $r(z) = c \Leftrightarrow r'(z) = d$, then we can conclude that $\mathcal{J}$ forms a normal family on S.*

This result mainly expresses the fact that $r(z)$ and $r'(z)$ share the complex numbers; at this point, the most obvious idea is to generalize the first derivative in the conclusion to the k derivative.

In 2001, Fang [11] demonstrated that for a nonzero complex number d representing a finite quantity, $\mathcal{J}$ is a set of meromorphic functions defined on a domain $S \subset \mathbb{C}$, with roots all having at least multiplicity $k+2$, where k is a non-negative integer. If for every $r(z), l(z) \in \mathcal{J}$, such that $r(z) = 0 \Leftrightarrow l(z) = 0$ and $r^{(k)}(z) = d \Leftrightarrow l^{(k)}(z) = d$, then it follows that $\mathcal{J}$ forms a normal family on S.

If we desire to diminish the multiplicity of zeros in the aforementioned conclusion to $k+1$, it suffices to append the stipulation that all functions have multiple poles. In 2001, Zalcman [11] considered the case of sharing one value, changed the form of the function on the left, and proved that for the two finite complex numbers $c \neq 0$, d, $\mathcal{J}$ is a set of meromorphic functions defined on a domain $S \subset \mathbb{C}$, with roots all having at least multiplicity k, where k is a non-negative integer. If for every $r(z) \in \mathcal{J}$, $r(z)r^{(k)}(z) = c \Leftrightarrow r^{(k)}(z) = d$, then $\mathcal{J}$ forms a normal family on S. In 2002, Fang and Zalcman [12] proved that for the two finite distinct complex numbers $c \neq 0$, $d \neq 0$, $\mathcal{J}$ is a set of meromorphic functions defined on the domain $S \subset \mathbb{C}$, with roots all having at least $k+1$ multiplicity, where k is a non-negative integer. If for every $r(z) \in \mathcal{J}$, $r(z) = c \Leftrightarrow r^{(k)}(z) = d$, then $\mathcal{J}$ forms a normal family on S.

In 2008, Zhang [13] considered the form of function $r^n(z)r'(z)$ and proved that if $\mathcal{J}$ is a set of meromorphic functions defined on the domain $S \subset \mathbb{C}$, $n(\geqslant 2)$ is a positive integer. Suppose that for two the functions $r(z), l(z) \in \mathcal{J}$, $r^n(z)r'(z)$ and $l^n(z)l'(z)$ share a nonzero value d, then $\mathcal{J}$ forms a normal family on S.

In 2015, Meng [14] conducted an examination of the scenario involving $r^n r^{(k)} (n \geq 2)$ sharing a function that is holomorphic and proved that, assuming the existence of the three integers $n(\geq 2)$, $k(\geq 1)$, and $m(\geq 0)$, we examine a function $h(z)(\not\equiv 0)$ that is analytic on domain S. This function ensures that all its zeros have a maximum multiplicity of m and are all divisible by $n+1$. Additionally, let $\mathcal{J}$ represent a collection of meromorphic functions on domain S, where each function $r \in \mathcal{J}$ possesses zeros with minimum multiplicity of $k+m$ and possesses poles with minimum multiplicity of $m+1$. If any pair of functions $r, l \in \mathcal{J}$ satisfy the condition that $r^n(z)r^{(k)}(z)$ and $l^n(z)l^{(k)}(z)$ share the same value as $h(z)$(IM) on domain S, it can be concluded that $\mathcal{J}$ forms a normal family on S.

In 2019, Deng [15] replaced the requirement that "all its zeros have a maximum multiplicity of m and are divisible by $n+1$ and all poles of f possesses poles with minimum multiplicity of $m+1$", resulting in the derivation of a new theorem, and proved that, given the three integers $n(\geq 2)$, $k(\geq 1)$, and $m(\geq 0)$, $h(z)(\not\equiv 0)$ denotes a function that is analytic on domain S with zeros of multiplicity at most m. Additionally, let $\mathcal{J}$ represent a collection of meromorphic functions on domain S, where each function $r \in \mathcal{J}$ has zeros with minimum multiplicity of $k+m$. If every pair of functions r and l from set $\mathcal{J}$ satisfy the condition that they share $h(z)$ (IM) on domain S under the operations $r^n(z)r^{(k)}(z)$ and $l^n(z)l^{(k)}(z)$, then $\mathcal{J}$ forms a normal family on S.

The starting point of the above conclusion is to change the form of the function, and there are many examples of this (see [16–18]).

In 2007, Liu [19] considered the case of sharing a set and established the following theorem:

Theorem 3. *Let $\mathcal{J}$ be a collection of meromorphic functions on a domain $S \subset \mathbb{C}$ and a, b, c be three distinct finite complex numbers. If for every function $r(z) \in \mathcal{J}$, both the function $r(z)$ and $r'(z)$ share the set $D = \{a, b, c\}$, then it follows that $\mathcal{J}$ forms a normal family on S.*

In 2016, Xu [20] demonstrated that for the two sets $E_1 = \{a_1, a_2\}$ and $E_2 = \{b_1, b_2\}$ in $\mathbb{C}$ with the conditions of $a_1 a_2 \neq 0$ and $\frac{b_1}{b_2} \notin N^- \cup 1/N^-$, N^- denotes the set of all negative integers, and $1/N^-$ stands for the set $\left\{\frac{1}{k}, k \in N^-\right\}$. Furthermore, $\mathcal{J}$ is a collection of meromorphic functions on a domain $S \subset \mathbb{C}$, and for every $r(z) \in \mathcal{J}$ on S satisfies $|r'(z)| \leqslant A$ whenever $r(z) = 0$. If $r(z) \in E_1 \Leftrightarrow r'(z) \in E_2$ on S, then it follows that $\mathcal{J}$ forms a normal family on S. In 2020, Yuan [21] investigated the expression $\left(r^{(k)}(z)\right)^p$ and demonstrated that if $\mathcal{J}$ is a set of meromorphic functions on the domain $S \subset \mathbb{C}$, $E_1 = \{a_1, a_2, a_3\}$, $E_2 = \{b_1, b_2, b_3\}$, with both E_1 and E_2 being made up of finite complex numbers, and given that k is greater than or equal to 2 and p is positive integers, c represents a finite complex number. Assuming that for the functions $r(z) \in \mathcal{J}$, (i) $r(z) = E_1 \Leftrightarrow \left(r^{(k)}(z)\right)^p = E_2$, (ii) both zeros and poles of $r(z) - c$ have multiplicities of at least k, then it follows that $\mathcal{J}$ forms a normal family on S.

In the above background description, most of the conclusions are about one family of meromorphic functions sharing a value.

In 2013, Liu [22] investigated the transitivity of normality between two sets of meromorphic functions under specific assumptions regarding shared values, culminating in the subsequent established findings.

Theorem 4. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of meromorphic functions defined on the domain $S \subset \mathbb{C}$ and the complex numbers a, b, c, and d are mutually distinct. Furthermore, suppose that $\mathcal{L}$ forms a normal family, if for every $r(z) \in \mathcal{J}$, there is $l(z) \in \mathcal{L}$, such that $r(z)$ and $l(z)$ share the values a, b, c, d, then it follows that $\mathcal{J}$ forms a normal family on S.*

Theorem 5. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of holomorphic functions defined on the domain $S \subset \mathbb{C}$, the roots of which all possess a multiplicity of no less than $k+1$, where the value of k is a non-negative whole number. $d(\neq 0)$ represents a complex number that is finite in nature. Furthermore, supposing that $\mathcal{L}$ forms a normal family for any sequence $\{l_n\}$ in $\mathcal{L}$, such that $l_n \Rightarrow l$, it holds that $l \not\equiv \infty$ and $l^{(k)} \not\equiv d$ on S. If there is $l \in \mathcal{L}$ for every $r \in \mathcal{J}$ such that*

a *The equation holds true: $r(z) = 0 \Leftrightarrow l(z) = 0$;*
b *The equation holds true: $r^{(k)}(z) = d \Leftrightarrow l^{(k)}(z) = d$.*

then it follows that $\mathcal{J}$ forms a normal family on S.

Theorem 6. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of meromorphic functions defined on a domain $S \subset \mathbb{C}$, the roots of which all possess a multiplicity of no less than $k+1$, where the value of k is a non-negative whole number. $d(\neq 0)$ represents a complex number that is finite in nature. Furthermore, supposing that $\mathcal{L}$ forms a normal family for any sequence $\{l_n\}$ in $\mathcal{L}$, such that $l_n \Rightarrow l$, it holds that $l \not\equiv \infty$ and $l^{(k)} \not\equiv d$ on S. If there is $l \in \mathcal{L}$ for every $r \in \mathcal{J}$ such that*

a *The equation holds true: $r(z) = 0 \Leftrightarrow l(z) = 0$;*
b *The equation holds true: $r(z) = \infty \Leftrightarrow l(z) = \infty$;*
c *The equation holds true: $r^{(k)}(z) = d \rightleftharpoons l^{(k)}(z) = d$ (CM).*

then it follows that $\mathcal{J}$ forms a normal family on S.

In 2021, Xu [23] suggested replacing the values a, b, c, d with the functions $a(z), b(z), c(z), d(z)$ in Theorem 4. Subsequently, Xu established the following theorem, which improved and generalized Theorem 4.

Theorem 7. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of meromorphic functions defined on the domain $S \subset \mathbb{C}$ and let $a_1(z), a_2(z), a_3(z), a_4(z)$ be four distinct holomorphic functions such that $\#\{a_1(z), a_2(z), a_3(z), a_4(z)\} \geqslant 3$ for $z \in S$, where $\#E$ denotes the number of distinct complex number of the set E. Furthermore, supposing that $\mathcal{L}$ forms a normal family, if for every $r(z) \in \mathcal{J}$, there is $l(z) \in \mathcal{L}$ such that $r(z) = a_i(z) \Rightarrow l(z) = a_i(z)(i = 1, 2, 3, 4)$, then it follows that $\mathcal{J}$ forms a normal family on S.*

According to the above research ideas, an inherent question arises: does Theorem 5 remain valid when the constant b is substituted with $h(z)$? This paper introduces our discoveries, which enhance and extend the scope of Theorem 5.

Theorem 8. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of holomorphic functions defined on a domain $S \subset \mathbb{C}$, the roots of which all possess a multiplicity of no less than $k + 1$, where the value of k is a non-negative whole number. $t(z)(\not\equiv 0)$ represents a function that is holomorphic. Furthermore, Suppose that $\mathcal{L}$ forms a normal family, for any sequence $\{l_n\}$ in $\mathcal{L}$, such that $l_n \Rightarrow l$, it holds that $l \not\equiv \infty$, $l^{(k)}(z) \not\equiv t(z)$ on S. If there exists $l \in \mathcal{L}$ for every $r \in \mathcal{J}$ such that*

a *the equation holds true: $r(z) = 0 \Leftrightarrow l(z) = 0$;*

b *the equation holds true: $r^{(k)}(z) = t(z) \Leftrightarrow l^{(k)}(z) = t(z)$.*

then it follows that $\mathcal{J}$ forms a normal family on S.

This considered, we have demonstrated the following properties of meromorphic functions.

Theorem 9. *Suppose $\mathcal{J}$ and $\mathcal{L}$ are two collections of meromorphic functions defined on the domain $S \subset \mathbb{C}$, the roots of which all possess a multiplicity of no less than $k + 1$, where the value of k is a non-negative whole number. $t(z)(\not\equiv 0)$ represents a function that is holomorphic. Furthermore, supposing that $\mathcal{L}$ forms a normal family, for any sequence $\{l_n\}$ in $\mathcal{L}$, such that $l_n \Rightarrow l$, it holds that $l \not\equiv \infty$, $l^{(k)}(z) \not\equiv t(z)$ on S. If there exists $l \in \mathcal{L}$ for every $r \in \mathcal{J}$ such that*

a *the equation holds true: $r(z) = 0 \Leftrightarrow l(z) = 0$;*

b *the equation holds true: $r(z) = \infty \Leftrightarrow l(z) = \infty$;*

c *the equation holds true: $r^{(k)}(z) = t(z) \rightleftharpoons l^{(k)}(z) = t(z)$(CM).*

then it follows that $\mathcal{J}$ forms a normal family on S.

In the forthcoming section, we will present three examples to demonstrate the indispensability of all the conditions in Theorem 9.

Example 1. *Assume that $t(z) = 1 - z$ and k is a non-negative whole number. Consider two families*

$$\mathcal{J} = \left\{ r_n = \frac{nz^{k+1}}{(k+1)!} \;\middle|\; n \in \mathbb{N} \right\} \quad \textit{and} \quad \mathcal{L} = \left\{ l_n = \frac{\left(1 - 2z + \frac{1}{n+1}\right)^{(k+1)}}{(-2)^k (k+1)!} \;\middle|\; n \in \mathbb{N} \right\}$$

on the unit disk Δ. Obviously, the functions r_n and l_n are holomorphic on Δ, which means the condition (b) of Theorem 9 holds for the families $\mathcal{J}$ and $\mathcal{L}$. Since

$${r_n}^{(k)}(z) = nz, \qquad {l_n}^{(k)}(z) = 1 - 2z + \frac{1}{n+1}.$$

It is apparent that ${r_n}^{(k)}(z) = t(z) \rightleftharpoons {l_n}^{(k)}(z) = t(z)$. Thus, the condition (c) of Theorem 9 also holds for the families $\mathcal{J}$ and $\mathcal{L}$.

It is readily apparent that $l_n(z) \Rightarrow \frac{(1-2z)^{k+1}}{(-2)^k(k+1)!}$, *thereby indicating the normality of the family* $\mathcal{L}$ *on* Δ*. However, it is straightforward to verify that the family* $\mathcal{J}$ *does not exhibit normality at the point* 0*. Hence, this example vividly illustrates the indispensable role of condition (a) in Theorem 9.*

Example 2. *Assume that* $t(z) = z$, $k \in \mathbb{N}$*. Consider two families* $\mathcal{J} = \left\{ r_n(z) = \frac{1}{nz} \,\middle|\, n \in \mathbb{N} \right\}$ *and*

$$\mathcal{L} = \left\{ l_n(z) = \frac{z^{2k+2}}{(2k+2)\cdots(k+3)} + \frac{z^{k+1}}{(k+1)!} - \frac{(-1)^k z^k}{n} + (k+3)! + 2 \,\middle|\, n \in \mathbb{N} \right\}$$

on the unit disk Δ*. It follows that* $|l_n(z)| > 1$ *for* $z \in \Delta$ *and* $n \in \mathbb{N}$*, so* r_n *and* r_n *omit* 0 *on* Δ*. Thus, condition (a) of Theorem 9 holds for the families* $\mathcal{J}$ *and* $\mathcal{L}$*. Since*

$$r_n{}^{(k)}(z) = \frac{(-1)^k k!}{nz^{k+1}}, \qquad l_n{}^{(k)}(z) = z^{k+2} + z - \frac{(-1)^k k!}{n},$$

then $r_n{}^{(k)}(z) = z \rightleftharpoons l_n{}^{(k)}(z) = z$*. Then, the condition (c) of Theorem 9 also holds for the families* $\mathcal{J}$ *and* $\mathcal{L}$*.*

Meanwhile, it is readily evident that

$$l_n(z) \Rightarrow \frac{z^{2k+2}}{(2k+2)\cdots(k+3)} + \frac{z^{k+1}}{(k+1)!} + (k+3)! + 2 \not\equiv \infty,$$

which means that $\mathcal{L}$ *is normal on* Δ*. However, it is evident the family* $\mathcal{J}$ *does not exhibit normality at point* 0*. Hence, this example vividly illustrates the indispensable role of condition (b) in Theorem 9.*

Chang [24] provided an example demonstrating the necessity of condition (c) in Theorem 9. In the interest of completeness, we present Chang's example here.

Example 3 ([24])**.** *Let* $\mathcal{J} = \{r_n = \tan(nz)\}$ *and let* $z_{n,1}, z_{n,1}, \cdots, z_{n,k_n}$ *be the zeros of* r_n *in the unit disk* Δ*. It is clear that* $r_n(z)$ *omit the values* i*, and* $-i$ *on* Δ*. Define* $\mathcal{L} = \{l_n\}$*, where*

$$l_n = \frac{1}{2}\prod_{i=1}^{k_n} \frac{z - z_{n,i}}{1 - \overline{z_{n,i}}z}.$$

It is straightforward to verify that $|l_n(z)| \le 1/2$ *for* $z \in \Delta$*. So, the function* $l_n(z)$ *fails to take on the values of* i *and* $-i$ *on the domain* Δ*. Obviously, one has* $r_n(z) = 0 \Leftrightarrow l_n(z) = 0$*. But* $\mathcal{L}$ *conforms to normality on* Δ*, and* $\mathcal{J}$ *fails to satisfy the conditions of normality on* Δ*. Hence, this example vividly illustrates the indispensable role of condition (c) in Theorem 9.*

2. Notation and Preliminary Lemmas

The symbol $n(r, f)$ denotes the count of poles of $f(z)$ within the domain $\Delta(0, r)$ (taking into account their multiplicity), while $n\left(r, \frac{1}{f}\right)$ represents the count of roots of $f(z)$ in the domain $\Delta(0, r)$ (also considering their multiplicity). We say that $f_n \overset{\chi}{\Rightarrow} f$ in the domain D if the sequence f_n converges to f uniformly on compact subsets of D in relation to the metric of a sphere. Furthermore, we state that $f_n \Rightarrow f$ in D if the sequence f_n converges to f under the Euclidean metric.

Lemma 1 ([25])**.** *For the positive integer* k*,* $Q(z)(\not\equiv 0)$ *is a polynomial. If* $r(z)$ *exhibits meromorphic, that is, transcendental, traits and has roots of at least* $k+1$ *multiplicity, then the function* $r^{(k)}(z) - Q(z)$ *possesses an infinite number of zeros.*

Lemma 2 ([26])**.** *For the positive integer* k*,* $Q(z)(\not\equiv 0)$ *represents a rational function. If* $r(z)$ *exhibits meromorphic, that is, transcendental, traits and has roots of at least* $k+1$ *multiplicity,* *except possibly a finite number, then the expression $r^{(k)}(z) - Q(z)$ possesses an infinite number of zeros.*

Lemma 3 ([27]). *For the positive integer k, $r(z)$ exhibits nonconstant meromorphic, that is, transcendental, traits and has roots of at least $k+1$ multiplicity. If the value of $r^{(k)}(z)$ is not equal to d on C, where $d \in \mathbb{C}$, and d is not equal to zero, then*

$$r(z) = \frac{d}{k!}\frac{(z-e)^{k+1}}{z-f},$$

for any pair of distinct complex numbers $e, f \in \mathbb{C}$.

Lemma 4 ([27]). *Assume $r(z)$ exhibits nonconstant meromorphic that is transcendental of finite order in $\mathbb{C}$ with multiple zeros. If $r'(z) \neq 1$ for all z, this implies the existence of distinct complex numbers c and d such that the expression of $r(z)$ can be written as $r(z) = \frac{(z-c)^2}{z-d}$.*

Lemma 5 ([27]). *Suppose T is a rational function that is not constant with the property that its derivative $T' \neq 0$ in $\mathbb{C}$. Then, it must be the case that either $T(z) = cz + d$, or $T(z) = \frac{c}{(z+e)^n} + d$, where $n \in \mathbb{N}$, c (not equal to 0), d, and e are complex numbers.*

Lemma 6 ([27]). *Suppose $\mathcal{J}$ is a collection of holomorphic functions defined on $S \subset \mathbb{C}$, each with all their zeros possessing at least multiplicity $k+1$. Supposing that for all $r \in \mathcal{J}$, the function $r^{(k)}$ is not equal to 1, then $\mathcal{J}$ exhibits normal on the domain S.*

Lemma 7 ([28]). *Let us consider a set of meromorphic functions, denoted as $\mathcal{J}$, defined in the domain S. All these functions have zeros with a minimum multiplicity of k. Now, suppose we have a real number, denoted as α, satisfying the condition $-1 < \alpha < k$. If the collection of functions, $\mathcal{J}$, does not exhibit normal at some point a_0 in the domain S, then we can find*

- a *points a_n, approaching a_0;*
- b *functions $r_n \in \mathcal{J}$;*
- c *positive numbers ρ_n, approaching zero.*

such that for each n, $\rho_n^{-\alpha} r_n(a_n + \rho_n\zeta) = r_n(\zeta) \overset{\chi}{\Rightarrow} r(\zeta)$ on $\mathbb{C}$. Here, r represents a nonconstant function that is meromorphic, as defined on complex plane $\mathbb{C}$.

Lemma 8 ([23]). *For any non-negative integer value k, it holds true that*

$$\left(\prod_{i=1}^{p}(z-z_i)^{-\alpha_i}\right)^{(k)} = \sum_{\substack{a_1+\cdots+a_p=\tau+k \\ \alpha_i \leq a_i \leq \alpha_i+k(1\leq i\leq p)}} \frac{A_k}{\prod_{i=1}^{p}(z-z_i)^{a_i}},$$

where the variables α_i and a_i (where $1 \leq i \leq s$) are constrained to be positive integers, while the quantities $z_i (1 \leq i \leq p)$ are defined as complex numbers. Furthermore, the symbol τ is defined as the summation of all α_i from 1 to p, and A_k is a constant that is not zero and varies depending on α_i (for $1 \leq i \leq p$) and k.

3. Proof of Theorem 8

We need to show that $\mathcal{J}$ is normal at the point z_0, where z_0 is in the domain S. Our demonstration will be divided into two separate cases.

Case 1 $t(z_0) \neq 0$.

We may assume $t(z_0) = 1$. The conclusion can be drawn from Theorem 5 that $\mathcal{J}$ exhibits normal at z_0.

Case 2 $t(z_0) = 0$.

There is a positive real number δ such that the set $\overline{\Delta}(0,\delta) = \{z : |z - z_0| \leq \delta\}$ is contained within S, where $t(z)$ does not have any roots but z_0 within the closed disk

$\overline{\Delta}(0,\delta)$. According to Case 1, it can be inferred that $\mathcal{J}$ exhibits normality on $\Delta'(z_0,\delta) = \{z : 0 < |z - z_0| < \delta\}$.

Consider an arbitrary sequence $\{r_n(z)\}$ in the set $\mathcal{J}$. Due to the normality of $\mathcal{J}$ on $\Delta'(z_0,\delta)$, there is a subsequence (denoted by $r_n(z)$ for convenience) that locally uniformly converges to a limit function $r(z)$ on $\Delta'(z_0,\delta)$ in relation to the metric of a sphere. We will now move on to examining the evidence within two specific scenarios.

Subcase 2.1 $r \not\equiv 0$

Subcase 2.1.1 $l(z_0) \neq 0$.

Subsequently, a real number δ' $(\delta' < \delta)$ exists such that $l_n(z) \neq 0$ in $\Delta(z_0,\delta')$ for a value of n that is sufficiently large. According to the requirements specified in Theorem 8, $r(z) = 0 \Leftrightarrow l(z) = 0$, we have $r_n \neq 0$ in $\Delta(z_0,\delta')$. As per the theorem established by Hurwitz, we can deduce that $r \neq 0$ in $\Delta(z_0,\delta')$. Thus, it follows that

$$\min_{0\le\theta\le 2\pi}\left|r\left(z_0 + \frac{\delta'}{2}e^{i\theta}\right)\right| = M > 0,$$

where $M > 0$ is a constant. Therefore, for sufficiently large n, we find

$$\min_{0\le\theta\le 2\pi}\left|r_n\left(z_0 + \frac{\delta'}{2}e^{i\theta}\right)\right| > \frac{M}{2} > 0.$$

Note that $r_n \neq 0$ on $\Delta(z_0,\delta')$. Thus, the function $\frac{1}{r_n}$ is holomorphic function on $\Delta(z_0,\delta')$, and

$$\max_{0\le\theta\le 2\pi}\frac{1}{\left|r_n\left(z_0 + \frac{\delta'}{2}e^{i\theta}\right)\right|} < \frac{M}{2}.$$

In accordance with the principle of maximum modulus, it follows that

$$\max_{|z-z_0|\le\frac{\delta'}{2}}\frac{1}{|r_n(z)|} < \frac{2}{M},$$

then

$$\min_{|z-z_0|\le\frac{\delta'}{2}}|r_n(z)| > \frac{M}{2},$$

and based on the above process, it can be inferred that the sequence $\{r_n\}$ exhibits normality at the point z_0. Consequently, we can confirm that $\mathcal{J}$ also demonstrates normality at z_0 as intended.

Subcase 2.1.2 $l(z_0) = 0$.

If l is identically zero, then $l^{(k)}(z_0) = 0$. If l is not identically zero, noting that all the roots of the function $l_n(z)$ exhibit a multiplicity of no less than $k+1$, and that the sequence of functions $\{l_n(z)\}$ exhibits convergence to $l(z)$ on the closed disk $\overline{\Delta}(z_0,\delta)$, then it can be deduced that $l^{(k)}(z_0) = 0$.

Hence, irrespective of the situation, we have $l^{(k)}(z_0) = 0$. Let us suppose that $\mathcal{J}$ fails to satisfy the condition of normality at z_0. As per Zalcman's lemma, there are points z_m tending to z_0, a sequence of positive numbers $\rho_m \to 0^+$, in turn a subset of functions $r_{n_m}(z) \subseteq r_n(z)$, such that

$$T_m(\zeta) = \frac{r_{n_m}(z_m + \rho_m\zeta)}{\rho_m^k} \Rightarrow T(\zeta)$$

on $\mathbb{C}$, $T(\zeta)$ is a function that is not constant and is holomorphic everywhere. All its zeros have at least a multiplicity of $k+1$.

We assert that $T^{(k)}(\zeta)$ is not equal to 1, for ζ belongs to the set of complex numbers.

Noting that the roots of $T(\zeta)$ possess at least a multiplicity of $k+1$, it can be deduced that $T^{(k)}(\zeta) \not\equiv 1$.

Let $\zeta_0 \in \mathbb{C}$ with $T^{(k)}(\zeta_0) = 1$. According to Hurwitz' theorem, there are complex numbers $\zeta_{m,0} \in \mathbb{C}$, such that $T_m^{(k)}(\zeta_{m,0}) = 1$, that is, $r_{n_m}^{(k)}(z_m + \rho_m \zeta_{m,0}) = 1$. According to what Theorem 8 assumes, $r^{(k)}(z) = t(z) \Leftrightarrow l^{(k)}(z) = t(z)$, we have $l_{n_m}^{(k)}(z_m + \rho_m \zeta_{m,0}) = 1$. Let $m \to \infty$, so we find $l^{(k)}(z_0) = 1$, which is in contrast to the condition $l^{(k)}(z_0) = 0$.

Therefore, we can deduce that $T^{(k)}(\zeta) \neq 1$. By virtue of lemma 1, it follows that $T(\zeta)$ is a rational function. Considering the holomorphic property of $T(\zeta)$, it can be inferred that $T(\zeta)$ can be expressed as a polynomial. Assuming that the degree of $T(\zeta)$ is denoted by s,

$$T(\zeta) = a_0 + a_1\zeta + \cdots + a_s\zeta^s, (a_s \neq 0),$$

if $s \geq k$, then

$$T^{(k)}(\zeta) = a_s s(s-1)\cdots(s-k+1)\zeta^{s-k},$$

which contradicts $T^{(k)}(\zeta) \neq 1$. Then, $s < k$, which indicates that the roots of the function $T(\zeta)$ must possess a minimum multiplicity of $k+1$. Therefore, the sequence $\{r_n(z)\}$ exhibits normality at point z_0.

Subcase 2.2 $r \equiv 0$

In this case, $\{r_n(z)\}$ converges to 0 in $\Delta'(z_0, \delta')$. Since $\{r_n(z)\}$ is holomorphic in $\Delta\left(z_0, \frac{\delta'}{2}\right)$, $\{r_n(z)\}$ thus converges to 0 in $\Delta\left(z_0, \frac{\delta'}{2}\right)$. Hence, $\{r_n(z)\}$ is normal at z_0.

4. Proof of Theorem 9

We simply need to demonstrate the normality of $\mathcal{J}$ at the point z_0, where z_0 in the domain S. We will divide our demonstration into two separate situations.

Case 1 $t(z_0) \neq 0$.

We may assume $t(z_0) = 1$. The conclusion can be drawn from Theorem 6 that $\mathcal{J}$ exhibits normality at z_0.

Case 2 $t(z_0) = 0$.

There exists a positive real number δ such that the set $\overline{\Delta}(0, \delta) = \{z : |z - z_0| \leq \delta\}$ is entirely contained within S, as $t(z)$ has no roots other than z_0 within the closed disk $\overline{\Delta}(0, \delta)$. Based on the conditions in Case 1, it can be concluded that $\mathcal{J}$ demonstrates normality on $\Delta'(z_0, \delta) = \{z : 0 < |z - z_0| < \delta\}$.

Consider an arbitrary sequence $\{r_n(z)\}$ in the set $\mathcal{J}$. Due to the normality of $\mathcal{J}$ on $\Delta'(z_0, \delta)$, there is a subsequence (denoted by $r_n(z)$ for convenience) that locally uniformly converges to a limit function $r(z)$ on $\Delta'(z_0, \delta)$ in relation to the metric of a sphere. We will now move on and examine the evidence within three specific scenarios.

Subcase 2.1 $r \not\equiv 0$ and $r \not\equiv \infty$.

Given the normality of $\mathcal{L}$ on S, it is reasonable to presume that the corresponding sequence $\{l_n(z)\}$ satisfies the convergence property $l_n \rightrightarrows l$ on the closed disk $\overline{\Delta}(z_0, \delta)$. Based on the premise of Theorem 9, it follows that $l \not\equiv \infty$, $l^{(k)} \not\equiv t(z)$.

Subcase 2.1.1 $l(z_0) \neq 0, \infty$.

Subsequently, a real number δ' ($\delta' < \delta$) exists, such that $l_n(z) \neq 0, \infty$ in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. By virtue of the conditions (a) and (b) outlined in Theorem 9, it follows that $r_n \neq 0, \infty$ in $\Delta(z_0, \delta')$. As per the theorem established by Hurwitz, we can deduce that $r \neq 0, \infty$ in $\Delta(z_0, \delta') \backslash \{z_0\}$. Thus, it follows that

$$\min_{0 \leq \theta \leq 2\pi} \left| r\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right| = M > 0,$$

where $M > 0$ is a constant. Consequently, for sufficiently large value of n, we obtain

$$\min_{0 \leq \theta \leq 2\pi} \left| r_n\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right| > \frac{M}{2} > 0. \quad (1)$$

Note that $r_n \neq 0$ on $\Delta(z_0, \delta')$. Thus, the function $\frac{1}{r_n}$ is holomorphic function on $\Delta(z_0, \delta')$, and

$$\max_{0 \leq \theta \leq 2\pi} \frac{1}{\left| r_n\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right|} < \frac{M}{2}.$$

In accordance with the principle of maximum modulus, it follows that

$$\max_{|z - z_0| \leq \frac{\delta'}{2}} \frac{1}{|r_n(z)|} < \frac{2}{M},$$

then

$$\min_{|z - z_0| \leq \frac{\delta'}{2}} |r_n(z)| > \frac{M}{2},$$

and based on the above process, it can be inferred that the sequence $\{r_n\}$ exhibits normality at the point z_0. Consequently, we can confirm that $\mathcal{J}$ also demonstrates normality at z_0 as intended.

Subcase 2.1.2 $l(z_0) = 0$.

If l is identically zero, then $l^{(k)}(z_0) = 0$. If l is not identically zero, given that all the roots of the function $l_n(z)$ exhibit a multiplicity of no less than $k + 1$, and that the sequence of functions $\{l_n(z)\}$ exhibits convergence to $l(z)$ on the closed disk $\overline{\Delta}(z_0, \delta)$, then it can be deduced that $l^{(k)}(z_0) = 0$.

Hence, irrespective of the situation, we have $g^{(k)}(z_0) = 0$. Let us suppose that $\mathcal{J}$ fails to satisfy the condition of normality at z_0. As per Zalcman's lemma, there is point z_m tending to z_0, a sequence of positive numbers $\rho_m \to 0^+$, and a subset of functions $r_{n_m}(z) \subseteq r_n(z)$, such that

$$T_m(\zeta) = \frac{r_{n_m}(z_m + \rho_m \zeta)}{\rho_m^k} \Rightarrow T(\zeta)$$

on $\mathbb{C}$, $T(\zeta)$ is a function that is not constant and is meromorphic everywhere. All its roots have at least a multiplicity of $k + 1$.

We assert that $T^{(k)}(\zeta)$ is not equal to 1, for ζ belongs to the set of complex numbers.

Noting that the roots of $T(\zeta)$ possess at least a multiplicity of $k + 1$, it can be deduced that $T^{(k)}(\zeta) \not\equiv 1$.

Let $\zeta_0 \in \mathbb{C}$ with $T^{(k)}(\zeta_0) = 1$. According to Hurwitz' theorem, there are complex numbers $\zeta_{m,0} \in \mathbb{C}$, $\zeta_{m,0} \to \zeta_0$, such that $T_m^{(k)}(\zeta_{m,0}) = 1$, that is, $r_{n_m}^{(k)}(z_m + \rho_m \zeta_{m,0}) = 1$. By the assumptions of Theorem 9, $r^{(k)}(z) = t(z) \Leftrightarrow l^{(k)}(z) = t(z)$, we have $l_{n_m}^{(k)}(z_m + \rho_m \zeta_{m,0}) = 1$. Let $m \to \infty$, so we find $l^{(k)}(z_0) = 1$, which contradicts the idea that $l^{(k)}(z_0) = 0$.

Thus, we have $T^{(k)}(\zeta) \neq 1$. According to the findings in Lemma 3, it can be inferred that

$$T(\zeta) = \frac{1}{k!} \frac{(\zeta - \zeta_0)^{k+1}}{\zeta - \zeta_1} \quad \text{for some } \zeta_0 \neq \zeta_1.$$

As per the theorem established by Hurwitz, there is $\zeta_{m,1} \in \mathbb{C}$, $\zeta_{m,1} \to \zeta_1$, such that

$$T_m(\zeta_{m,1}) = \frac{r_{n_m}(z_m + \rho_m \zeta_{m,1})}{\rho_m^k} = \infty,$$

that is, $r_{n_m}(z_m + \rho_m \zeta_{m,1}) = \infty$. By the assumptions of Theorem 9, $r(z) = \infty \Leftrightarrow l(z) = \infty$, so we have

$$l_{n_m}(z_m + \rho_m \zeta_{m,1}) = \infty.$$

Let $m \to \infty$, so we find $l(z_0) = \infty$, which contradicts $l(z_0) = 0$.

Therefore, the sequence $\{r_n(z)\}$ exhibits normality at point z_0.

Subcase 2.1.3 $l(z_0) = \infty$.

Drawing from the postulation presented in Theorem 9, it is evident that $l \not\equiv \infty$. Subsequently, a real number δ' ($\delta' < \delta$) exists, such that $l_n(z) \neq 0$ in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. By virtue of the conditions (a) outlined in Theorem 9, it follows that $r_n \neq 0, \infty$ in $\Delta(z_0, \delta')$. As per the theorem established by Hurwitz, we can deduce that $r \neq 0, \infty$ in $\Delta(z_0, \delta') \backslash \{z_0\}$.

Thus, it follows that

$$\min_{0 \leq \theta \leq 2\pi} \left| r\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right| = M > 0,$$

where $M > 0$ is a constant.

Note that $r_n \neq 0$ on $\Delta(z_0, \delta')$. Thus, the functions $\frac{1}{r_n}$ is a holomorphic function on $\Delta(z_0, \delta')$. By using the same arguments as in subcase 2.1.1, we can confirm that $\mathcal{J}$ also demonstrates normality at z_0 as intended.

Subcase 2.2 $r \equiv \infty$.

In this case, $\{r_n\}$ converges to ∞ in $\Delta'(z_0, \delta)$ and $\{r_n\}$ converges to ∞ in $\{z : |z - z_0| = \delta'\}$; then, for an arbitrarily large number and for every large enough n, it follows that

$$\min_{0 \leq \theta \leq 2\pi} \left| r\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right| = M > 0.$$

Subcase 2.2.1 $l(z_0) \neq 0, \infty$.

We can choose a real number δ' ($\delta' < \delta$), such that $l_n(z) \neq 0, \infty$ in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. Moreover, due to the conditions (a) and (b) of Theorem 9, it also follows that $r_n \neq 0, \infty$ in $\Delta(z_0, \delta')$. Applying Hurwitz's theorem, we conclude that $r \neq 0, \infty$ in $\Delta(z_0, \delta') \backslash \{z_0\}$.

By using the same arguments as in subcase 2.1.1, we can confirm that $\mathcal{F}$ also demonstrates normality at z_0 as intended.

Subcase 2.2.2 $l(z_0) = \infty$.

Given that l is not identically equal to infinity, it is possible to identify a neighborhood $U(z_0) \subset \Delta(z_0, \delta')$ of z_0, such that $l(z) \neq 0$ in $U(z_0)$. According to Hurwitz' theorem, for sufficiently large values of n, $l_n(z)$ also does not equal 0 in $U(z_0)$.

In accordance with the postulate outlined in Theorem 9, $r(z) = 0 \Leftrightarrow l(z) = 0$, it is evident that $r_n(z) \neq 0$ in $U(z_0)$. Therefore, for a sufficiently large value of n, we have

$$\min_{0 \leq \theta \leq 2\pi} \left| r_n\left(z_0 + \frac{\delta'}{2} e^{i\theta}\right) \right| > \frac{M}{2} > 0.$$

Note that $r_n \neq 0$ in a neighborhood $V(z_0) \subset U(z_0)$ of z_0. Thus, $\frac{1}{r_n}$ is holomorphic on $V(z_0)$. By using the same arguments as in subcase 2.1.1, we can confirm that $\mathcal{J}$ also demonstrates normality at z_0 as intended.

Subcase 2.2.3 $l(z_0) = 0$.

Subsequently, a real number δ' ($\delta' < \delta$) exists, such that $l_n(z) \neq \infty$ in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. By virtue of the condition (b) outlined in Theorem 9, it follows that $r_n \neq \infty$ in $\Delta(z_0, \delta')$.

If $l(z)$ is identically zero, then $l^{(k)}(z_0) = 0$. Conversely, if $l(z)$ is not identically zero, according to the argument principle, each zero of $l(z)$ must possess at least a multiplicity of $k+1$, since the sequence of functions $\{l_n(z)\}$ converges to $l(z)$ on the closed disk $\overline{\Delta}(z_0, \delta)$ and each root of $\{l_n(z)\}$ possesses at least a multiplicity of $k+1$. Consequently, it can be inferred that $l^{(k)}(z_0) = 0$.

In either case, we have $l^{(k)}(z_0) = 0 \neq 1$. As per the theorem established by Hurwitz, for a sufficiently large value of n, it is guaranteed that $l_n^{(k)}(z) \neq 1$ in $\Delta(z_0, \delta')$. Considering condition (c) as stipulated in Theorem 9, $r^{(k)}(z) = t(z) \rightleftharpoons l^{(k)}(z) = t(z)$, we can establish that $r_n^{(k)}(z) \neq 1$. Leveraging lemma 7, we can derive the conclusion that $\mathcal{J}$ also demonstrates normality at z_0 as intended.

Subcase 2.3 $r \equiv 0$.

In this scenario, the sequence $\{r_n\}$ uniformly converges to 0 in relation to the metric of a sphere within $\Delta'(z_0, \delta)$, and $\left\{ {r_n}^{(k)} \right\}$ and $\left\{ {r_n}^{(k+1)} \right\}$ also converge to 0.

Subcase 2.3.1 $l(z_0) = 0$.

For a given real number $\delta'(\delta' < \delta)$, it can be established that for sufficiently large n, the functions $l_n(z)$ exhibit holomorphic properties within $\Delta(z_0, \delta')$. Additionally, based on conditions (a) and (b) of Theorem 9, it can be concluded that the functions $r_n(z)$ are also holomorphic in $\Delta(z_0, \delta')$. Considering that $r_n(z)$ tends towards zero in $\Delta'(z_0, \delta)$, it can be inferred that sequence $r_n(z)$ tends towards zero in $\Delta(z_0, \frac{\delta'}{2})$. Consequently, the normality of $r_n(z)$ at z_0 implies the desired normality of $\mathcal{J}$ at z_0.

Subcase 2.3.2 $l(z_0) \neq 0, \neq \infty$.

We can ascertain the existence of a real number $\delta'(\delta' < \delta)$, such that the functions $l_n(z)$ demonstrate holomorphic properties in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. By virtue of conditions (a) and (b) stipulated in Theorem 9, it is evident that the functions $r_n(z)$ also exhibit holomorphic in $\Delta(z_0, \delta')$. Employing analogous reasoning as previously expounded, we are able to identify a subsequence $r_{n_j}(z)$ converging towards 0 within $\Delta(z_0, \frac{\delta'}{2})$. Consequently, it follows that $r_n(z)$ manifests normality at z_0, thereby establishing the normality of $\mathcal{J}$ at z_0.

Subcase 2.3.3 $l(z_0) = \infty$.

We know that $l \not\equiv \infty$, so we can ascertain the existence of a real number $\delta'(\delta' < \delta)$, such that the functions $l_n(z) \neq 0$ in $\Delta(z_0, \delta')$, for a value of n that is sufficiently large. By virtue of condition (a) stipulated in Theorem 9, it is evident that the function $r_n(z) \neq 0$ in $\Delta(z_0, \delta')$.

Suppose that $\mathcal{J}$ does not satisfy the condition of normality at z_0. As per Zalcman's lemma, there are points z_m approaching z_0, a sequence of positive numbers $\rho_m \to 0^+$, and a subsequence of function $r_{n_m}(z) \subseteq r_n(z)$, such that

$$T_m(\zeta) = \frac{r_{n_m}(z_m + \rho_m \zeta)}{\rho_m^k} \Rightarrow T(\zeta)$$

on $\mathbb{C}$, $T(\zeta)$ is a function that is not constant and is meromorphic everywhere. All its zeros have at least a multiplicity of $k+1$.

We assert that $T(\zeta)$ is not equal to 0, for ζ belongs to the set of complex numbers.

Let $\zeta_0 \in \mathbb{C}$ with $T(\zeta_0) = 0$, since $T(\zeta) \not\equiv 0$, as per the theorem established by Hurwitz, and there is $\zeta_{m,0} \in \mathbb{C}$, $\zeta_{m,0} \to \zeta_0$, such that $T_m(\zeta_{m,0}) = \frac{r_{n_m}(z_m + \rho_m \zeta_{m,0})}{\rho_m^k} = 0$, that is, $r_{n_m}(z_m + \rho_m \zeta_{m,0}) = 0$. By the condition (a) of Theorem 9, we have $l_{n_m}(z_m + \rho_m \zeta_{m,0}) = 0$, so let $m \to \infty$, then $l(z_0) = 0$, which contradicts the idea that $l(z_0) = \infty$.

By the principle of argument, we have

$$\frac{1}{2\pi i} \int_{|z-z_0|=\delta'} \frac{r_n^{(k+1)}(z) - t'(z)}{r_n^{(k)}(z) - t(z)} dz \to \frac{1}{2\pi i} \int_{|z-z_0|=\delta'} \frac{t'(z)}{t(z)} dz.$$

This indicates

$$n\left(\Delta(z_0, \delta'), \frac{1}{r_n^{(k)}(z) - t(z)}\right) - n\left(\Delta(z_0, \delta'), r_n^{(k)}(z)\right) = n\left(\Delta(z_0, \delta'), \frac{1}{t(z)}\right). \tag{2}$$

Obviously,

$$n\left(\Delta(z_0, \delta'), \frac{1}{t(z)}\right) \geq 1, \tag{3}$$

and according to the requirements specified in Theorem 9, it can be inferred that

$$n\left(\Delta(z_0, \delta'), \frac{1}{r_n^{(k)}(z) - t(z)}\right) = n\left(\Delta(z_0, \delta'), \frac{1}{l_n^{(k)}(z) - t(z)}\right). \tag{4}$$

Since $l(z_0)=\infty$, it is reasonable to infer that

$$l(z)=\frac{L(z)}{(z-z_0)^{\tau}}, z\in\Delta'(z_0,\delta'),$$

$L(z_0)\neq 0$, and $L(z)$ is holomorphic on $\Delta(z_0,\delta')$, and τ is a non-negative whole number. Since $l_n\overset{\chi}{\Rightarrow} l$, we have

$$l_n(z)=\frac{L_n(z)}{(z-z_{n,1})^{\tau_{n,1}}\cdots(z-z_{n,s_n})^{\tau_{n,s_n}}},$$

where $L_n(z)$ is holomorphic on $\Delta(z_0,\delta')$, $L_n(z_{n,i})\neq 0$, $\tau_{n,i}\geq 1$ are integers, and $i=1,2,\cdots s_n$ and $\sum_{i=1}^{s_n}\tau_{n,i}=\tau$. Then, we have

$$l_n^{(k)}(z)=\frac{H_n(z)}{\prod_{i=1}^{s_n}(z-z_{n,i})^{\tau_{n,i}+k}},$$

where

$$H_n(z)=\sum_{i=0}^{k}\binom{k}{i}L_n^{(k-i)}Q_{n,i}(z)\prod_{j=1}^{s_n}(z-z_{n,j})^{k-i},$$

and

$$Q_{n,i}(z)=\left(\prod_{j=1}^{s_n}(z-z_{n,j})^{-\tau_{n,j}}\right)^{(i)}\prod_{j=1}^{s_n}(z-z_{n,j})^{\tau_{n,j}+i}.$$

Hence, we find

$$l_n^{(k)}(z)-t(z)=K_n(z)\prod_{i=1}^{s_n}(z-z_{n,i})^{-\tau_{n,i}-k}, \tag{5}$$

where

$$K_n(z)=H_n(z)-t(z)\prod_{i=1}^{s_n}(z-z_{n,i})^{\tau_{n,i}+k}.$$

By lemma 8, one has

$$\left(\prod_{j=1}^{s_n}(z-z_{n,j})^{-\tau_{n,j}}\right)^{(i)}=\sum_{\substack{a_1+\cdots+a_{s_n}=\tau+i\\ \tau_{n,j}\leq a_j\leq\tau_{n,j}+i(1\leq j\leq s_n)}}\frac{A_i}{\prod_{j=1}^{s_n}(z-z_{n,j})^{a_j}};$$

therefore, we know that

$$Q_{n,i}(z)=\sum_{\substack{a_1+\cdots+a_{s_n}=\tau+i\\ \tau_{n,j}\leq a_j\leq\tau_{n,j}+i(1\leq j\leq s_n)}}A_i\prod_{j=1}^{s_n}(z-z_{n,j})^{\tau_{n,j}+i-a_j}, \tag{6}$$

and

$$\begin{aligned}Q_{n,i}(z)&\Rightarrow\left((z-z_0)^{-\tau}\right)^{(i)}(z-z_0)^{\tau+s_ni}\\&=(-1)^i\tau(\tau+1)\cdots(\tau+i-1)(z-z_0)^{(s_n-1)i}.\end{aligned}$$

Letting $D_i=(-1)^i\tau(\tau+1)\cdots(\tau+i-1)\binom{k}{i}$, we have

$$
\begin{aligned}
K_n(z) &\Rightarrow \sum_{i=0}^{k} D_i L^{(k-i)}(z)(z-z_0)^{(s_n-1)i}(z-z_0)^{s_n k - s_n i} - h(z)(z-z_0)^{s_n k+\tau} \\
&= (z-z_0)^{k(s_n-1)}\left(\sum_{i=0}^{k} D_i L^{(k-i)}(z)(z-z_0)^{k-i} - h(z)(z-z_0)^{k+\tau}\right).
\end{aligned} \tag{7}
$$

Then,

$$
\sum_{i=0}^{k} D_i L^{(k-i)}(z)(z-z_0)^{k-i} - t(z)(z-z_0)^{k+\tau}|_{z=z_0} = D_k L(z_0) \neq 0,
$$

which indicates that $T_n(z)$ possesses a maximum of $k(s_n-1)$ distinct roots in the domain $\Delta(z_0, \delta')$. By utilizing Equations (5) and (7), we deduce

$$
n\left(\Delta\big(z_0, \delta'\big), \frac{1}{l_n^{(k)}(z) - t(z)}\right) = (s_n - 1)k, \tag{8}
$$

which implies that $l_n^{(k)}(z) - t(z)$ possesses a finite number of roots in $\Delta(z_0, \delta')$. Consequently, the function $r_n^{(k)}(z) - t(z)$ also exhibits a limited number of roots in $\Delta(z_0, \delta')$, implying that the function $T^{(k)}(z) - t(z)$ has a limited number of roots across the entire complex plane. According to Lemma 2, the function $T(\zeta)$ can be represented as a function that is rational. Note that $T(\zeta) \neq 0$, so we can express it as $T(\zeta) = \frac{1}{M(\zeta)}$, where $M(\zeta)$ is a polynomial that is not constant. Hence, for every zero of $M(\zeta)$, by the conditions of Theorem 9, the sequence r_n has s_n distinct poles $z_{n,i}$ with at least a multiplicity of 1 in the domain $\Delta(z_0, \delta')$. Therefore,

$$
n\Big(\Delta\big(z_0, \delta'\big), r_n^{(k)}(z)\Big) \geq s_n k. \tag{9}
$$

Accordingly, we procure

$$
\begin{aligned}
(s_n - 1)k &= n\left(\Delta\big(z_0, \delta'\big), \frac{1}{l_n^{(k)}(z) - t(z)}\right) && (since(8)) \\
&= n\left(\Delta\big(z_0, \delta'\big), \frac{1}{r_n^{(k)}(z) - t(z)}\right) && (since(4)) \\
&= n\Big(\Delta\big(z_0, \delta'\big), r_n^{(k)}(z)\Big) + n\left(\Delta\big(z_0, \delta'\big), \frac{1}{t(z)}\right) && (since(2)) \\
&\geq s_n k + 1, && (since(3) and (9))
\end{aligned}
$$

a contradiction. Consequently, $\mathcal{F}$ demonstrates normality at z_0.

5. Conclusions

Combining the normal family of meromorphic functions with shared values or shared functions constitutes a pivotal focus within the realm of meromorphic function theory. Currently, two primary avenues of research prevail in this domain: one involves altering the structure of meromorphic functions, while the other pertains to modifying the configuration of sharing functions. For instance, if $r^{(k)}(z) = d \Leftrightarrow l^{(k)}(z) = d$, we may explore modifying the function representation. One approach is to substitute $r^{(k)}(z)$ with $r'(z) - cr^{(k)}(z)(c(\neq 0)$ is a finite complex number) or a differential polynomial. Another possibility is to consider altering the form of the right-hand-side value by replacing the constant d with a polynomial, holomorphic function or meromorphic function.

In this paper, we have replaced the constant d with holomorphic functions; however, our aim is to extend this to meromorphic functions. We are currently seeking an appropriate method for doing so.

In 2023, Arpita Kundu and Abhijit Banerjee [29] investigated the uniqueness problem of Selberg class L-functions; in particular, they focused on a class of arbitrary meromorphic functions. Similarly, if we replace $r(z)$ in Theorem 9 with the L function, does the conclusion hold? In 2024, Sayantan Maity [30] demonstrated that if $r(z)$ and $l(z)$ are two transcendental or admissible meromorphic functions in Ω, where $a_i \in E(r) \cap E(l), i = 1, 2, \cdots, 5$ represent five distinct small functions and k is a positive integer or $+\infty$. Assuming that $\widetilde{D}_{\Omega}(a_i, k; r(z)) = \widetilde{D}_{\Omega}(a_i, k; l(z))$ for $i = 1, 2, \cdots, 5$ and $k \geqslant 14$, then $r(z) \equiv l(z)$. This article involves sharing small functions, which provides us with a research direction. If we replace $a_1(z), a_2(z), a_3(z), a_4(z)$ in Theorem 7 with small functions, is the conclusion also valid? We can even consider the case where L-functions share small functions(more examples, see [31,32]). All these provide a good reference for our follow-up research.

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