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Boundedness and Compactness of Weighted Composition Operators from (α, k)-Bloch Spaces to $\mathcal{A}_{(\beta,k)}$ Spaces on Generalized Hua Domains of the Fourth Kind

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Abstract: This paper addresses the weighted composition operators ${}_{\psi}C_{\phi}$ from the (α, k)-Bloch spaces to the $\mathcal{A}_{(\beta,k)}$ spaces of bounded holomorphic functions on W, where W is a generalized Hua domain of the fourth kind. Additionally, we obtain some necessary and sufficient conditions for the boundedness and compactness of these operators.

Keywords: generalized Hua domain of the fourth type; (α, k)-Bloch space; $\mathcal{A}_{(\beta,k)}$ space; weighted composition operators; boundedness and compactness

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1. Introduction

Let Ω be a bounded domain of $\mathbb{C}^n$ and $H(\Omega)$ the class of all holomorphic functions on Ω. For a given holomorphic function (self-map) $\phi : \Omega \to \Omega$ and a function $\psi \in H(\Omega)$, we define the linear operator ${}_{\psi}C_{\phi} : H(\Omega) \to H(\Omega)$ by the following equality:

$$({}_{\psi}C_{\phi} f)(z) = \psi(z) f(\phi(z)) (z \in \Omega).$$

The latter equation is a weighted composition operator for $f \in H(\Omega)$. If $\psi(z) \equiv 1$, it reduces to the composition operator, whereas for $\phi(z) = z$, it becomes the multiplication operator.

In 1930, Cartan [1] was the first to characterize the six types of irreducible bounded symmetric domains. These comprise four bounded symmetric classical domains, also called Cartan domains, and two exceptional domains, whose complex dimensions are 16 and 27, respectively. $\mathfrak{R}_{\mathrm{I}}(m, n)$, $\mathfrak{R}_{\mathrm{II}}(p)$, $\mathfrak{R}_{\mathrm{III}}(q)$, and $\mathfrak{R}_{\mathrm{IV}}(n)$ denote the Cartan domains of the first type, second type, third type, and fourth type, respectively. In addition, Yin introduced the Hua domains [2], which include the Cartan–Hartogs, Cartan–Egg, Hua, generalized Hua domains, and the Hua construction. $\mathrm{GHE}_{\mathrm{I}}, \mathrm{GHE}_{\mathrm{II}}, \mathrm{GHE}_{\mathrm{III}}$, and $\mathrm{GHE}_{\mathrm{IV}}$ denote the generalized Hua domains of the first type, second type, third type, and fourth type, respectively. The fourth type of the generalized Hua domain is defined as follows:

$$\begin{aligned} &\mathrm{GHE}_{\mathrm{IV}}(N_1, N_2, \cdots, N_r; n; q_1, q_2, \cdots, q_r; k) \\ &= \left\{ \zeta_j \in \mathbb{C}^{N_j}, z \in \mathfrak{R}_{\mathrm{IV}}(n) : \sum_{j=1}^{r} |\zeta_j|^{2q_j} < (1 + |zz'|^2 - 2z\bar{z}')^k, j = 1, 2, \cdots, r \right\}, \end{aligned}$$

where

$$\mathfrak{R}_{\mathrm{IV}}(n) := \left\{ z \in \mathbb{C}^n : 1 + |zz'|^2 - 2z\bar{z}' > 0, 1 - |zz'|^2 > 0 \right\}$$

is a Cartan domain of the fourth type. $\zeta_j = (\zeta_{j1}, \cdots, \zeta_{jN_j})$, $j = 1, \cdots, r$; z' denotes the transpose of z; $\bar{z}$ is the conjugate of z; $N_1, \cdots, N_r, n$ are positive integers; and $q_1, \cdots, q_r$ are positive real numbers. Without a loss of generality, it is assumed for $N_j = 1, \zeta_j \in \mathbb{C},\ j = 1, \cdots, r, \zeta = (\zeta_1, \cdots, \zeta_r)$ and $\|\zeta\|_\varphi^2 = \sum_{j=1}^r |\zeta_j|^{2q_j}$. Let

$$\langle \zeta, v\rangle_\varphi = \langle \zeta_1, v_1\rangle^{q_1} + \langle \zeta_2, v_2\rangle^{q_2} + \cdots + \langle \zeta_r, v_r\rangle^{q_r}.$$

We also write

$$\begin{aligned} |\langle \zeta, v\rangle_\varphi| &\le |\langle \zeta_1, v_1\rangle^{q_1}| + |\langle \zeta_2, v_2\rangle^{q_2}| + \cdots + |\langle \zeta_r, v_r\rangle^{q_r}| \\ &\le |\zeta_1|^{q_1}|v_1|^{q_1} + \cdots + |\zeta_r|^{q_r}|v_r|^{q_r}. \end{aligned}$$

For convenience, the fourth type of the generalized Hua domain will be referred to as $\mathrm{GHE_{IV}}$.

On $\mathrm{GHE_{IV}}$, the (α, k)-Bloch space $\mathcal{B}^{(\alpha,k)}$ comprises all $f \in H(\mathrm{GHE_{IV}})$, such that

$$\|f\|_{\mathcal{B}^{(\alpha,k)}} := |f(0,0)| + \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\alpha |\nabla f(z,\zeta)| < \infty,$$

where

$$\nabla f(z,\zeta) = \Big(\frac{\partial f(z,\zeta)}{\partial z_1}, \frac{\partial f(z,\zeta)}{\partial z_2}, \cdots, \frac{\partial f(z,\zeta)}{\partial z_n}, \frac{\partial f(z,\zeta)}{\partial \zeta_1}, \cdots, \frac{\partial f(z,\zeta)}{\partial \zeta_r}\Big).$$

It is clear that $\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$ is a Banach space.

On $\mathrm{GHE_{IV}}$, a Bers-type space $\mathcal{A}_{(\beta,k)}$ comprises all $f \in H(\mathrm{GHE_{IV}})$, such that

$$\|f\|_{\mathcal{A}_{(\beta,k)}} := \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |f(z,\zeta)| < \infty.$$

It is evident that $\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is a Banach space with norm $\|\cdot\|_{\mathcal{A}_{(\beta,k)}}$.

In fact, for $\forall\ (z,\zeta) \in \mathrm{GHE_{IV}}$, we have $0 < (1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2 \le 1$; hence, it is easy to prove that $\|\cdot\|_{\mathcal{A}_{(\beta,k)}}$ is a norm using conventional methods.

To show that $\|\cdot\|_{\mathcal{A}_{(\beta,k)}}$ is complete, assume that $\{f_k\}$ is a Cauchy sequence in $\mathcal{A}_{(\beta,k)}$ and for $\forall \varepsilon > 0$ (assume $\varepsilon < 1$), $\exists K > 0$. Whenever $p, l > K$, we have

$$\|f_p - f_l\|_{\mathcal{A}_{(\beta,k)}} = \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |(f_p - f_l)(z,\zeta)| < \varepsilon. \quad (1)$$

For any compact subset F in $\mathrm{GHE_{IV}}$, it must exist $\delta \in (0,1)$, such that

$$(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2 \ge \delta, \qquad \forall (z,\zeta) \in F.$$

From (1), we know that

$$|f_p(z,\zeta) - f_l(z,\zeta)| < \frac{\varepsilon}{\delta^\beta}, \qquad (z,\zeta) \in F.$$

Hence, there exists a holomorphic function f in $\mathrm{GHE_{IV}}$, such that

$$\lim_{k\to\infty} f_k(z,\zeta) = f(z,\zeta), \qquad (z,\zeta) \in \mathrm{GHE_{IV}},$$

and $\{f_p\}$ converges uniformly to f on every compact set of $\mathrm{GHE_{IV}}$. In (1), let $l \to \infty$, whenever $p > K$, we obtain

$$\|f_p - f\|_{\mathcal{A}_{(\beta,k)}} = \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |f_p(z,\zeta) - f(z,\zeta)| \le \varepsilon.$$

In particular, $\exists K_0 > 0$, whenever $p > K_0$, we obtain $\|f_p - f\|_{\mathcal{A}_{(\beta,k)}} \leq 1$, hence

$$\|f\|_{\mathcal{A}_{(\beta,k)}} \leq \|f_{K_0+1}\|_{\mathcal{A}_{(\beta,k)}} + \|f_{K_0+1} - f\|_{\mathcal{A}_{(\beta,k)}} \leq \|f_{K_0+1}\|_{\mathcal{A}_{(\beta,k)}} + 1;$$

therefore, $f \in \mathcal{A}_{(\beta,k)}$.

The boundedness and the compactness of the weighted composition operators on (or between) the spaces of the holomorphic functions on various domains have received significant attention. Indeed, the literature has already presented very thorough conclusions on the unit disc [3–6], the unit polydisk [7–11], and the open unit ball [12–18]. In the setting of the infinite dimensional bounded symmetric domains, Zhou and Shi [19] characterized the compactness of the composition operators on the Bloch space using classical bounded symmetric domains. Hamada [20] studied the weighted composition operators from H^∞ to the Bloch space of infinite dimensional bounded symmetric domains. Allen and Colonna [21] investigated the weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain.

Since establishing the Hua domains, many issues have been investigated in these domains. Some examples are the Bergman problem, the convexity problem of the Hua domains and the extreme value problem of the Hua domains. Yin et al. [2] obtained the explicit formula of the Bergman kernel function on Hua domains of four kinds. Although many researchers investigating complex variables have made significant achievements, research on operators in the Hua domains is still limited. For example, Bai [22] investigated the weighted composition operators on Bers-type spaces on Cartan–Hartogs domains of the first kind. Su and Zhang [23] characterized the composition operators from the p-Bloch space to the q-Bloch space on Cartan–Hartogs domains of the fourth kind. Su, Li, and Wang [24] studied the boundedness and compactness of weighted composition operators from the u-Bloch space to the v-Bloch spaces on Hua domains of the first kind. Su and Zhang [25] studied the weighted composition operators from H^∞ to the (α, m)-Bloch space on Cartan–Hartogs domains of the first type. Su and Wang [26] discussed weighted composition operators between Bers-type spaces on generalized Hua–Cartan–Hartogs domains. Jiang and Li [27] studied the boundedness and compactness of weighted composition operators between Bers-type spaces on Hua domains of four kinds. However, there is currently relatively little research on the boundedness and compactness of weighted composition operators on generalized Hua domains. Therefore, the research in this article is of great significance.

Weighted composition operators have widespread applications. For example, R. F. Allen, W. George, and M. A. Pons [28] investigated the properties of the topological space of composition operators on the Banach algebra of bounded functions on an unbounded, locally finite metric space in the operator norm topology and essential norm topology. The authors characterized the compactness of the differences between two such composition operators. Z. Guo [29] studied the boundedness, essential norm, and compactness of the generalized Stevi–Sharma operator from the minimal Mobius invariant space into the Bloch-type space. S. Heidarkhani, S. Moradi, and G. A. Afrouzi [30] characterized the existence of at least one weak solution for a nonlinear Steklov boundary-value problem involving a weighted $p(\cdot)$-Laplacian. Stević and Ueki [31] investigated the boundedness, compactness, and estimated essential norm of a polynomial differentiation composition operator from the Hardy space H^p to the weighted-type spaces of holomorphic functions on the unit ball.

Recently, we studied the boundedness and compactness of weighted composition operators from the α-Bloch spaces to the Bers-type spaces on generalized Hua domains of the first kind [32]. Motivated by [32], we characterized the generalized Hua's inequalities on the generalized Hua domains of the fourth kind. These inequalities are used to study the boundedness and the compactness of weighted composition operators from the (α, k)-Bloch spaces $\mathcal{B}^{(\alpha,k)}$ to the $\mathcal{A}_{(\beta,k)}$ spaces built on generalized Hua domains of the fourth kind and we obtain some necessary and sufficient conditions.

Notes: We investigated the boundedness and the compactness of the weighted composition operators from α-Bloch to $\mathcal{A}_\beta$ on generalized Hua domains of the first kind in [32]. We also discuss these issues in a similar way on generalized Hua domains of the second kind, excluding the discussion presented herein. We must use new basic knowledge and skills to discuss these issues on generalized Hua domains of the fourth kind. Regarding generalized Hua domains of the third kind, we cannot discuss these issues yet since we cannot prove that our results are similar to Lemmas 2 and 4; this is an open question. We speculate that similar results regarding the boundedness and the compactness of weighted composition operators from α-Bloch to Bers on generalized Hua domains of the third kind are also valid.

2. Preliminaries

Lemma 1 ([32]). *Let*

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

be an $m \times n$ matrix $(m \le n)$. Then, there exists an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V, such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_m & 0 & \dots & 0 \end{pmatrix} V \ (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0)$$

and

$$Z\overline{Z}' = U \begin{pmatrix} {\lambda_1}^2 & 0 & \dots & 0 \\ 0 & {\lambda_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {\lambda_m}^2 \end{pmatrix} \overline{U}',$$

where ${\lambda_1}^2, \cdots, {\lambda_m}^2$ are the characteristic values of $Z\overline{Z}'$. $I - Z\overline{Z}' > 0 \Longleftrightarrow \lambda_1 < 1$.

Lemma 2 ([32]). *Let p_i $(i = 1, 2, \cdots, r)$ be positive integers, $0 < km \leqslant 1$ and $t \in [0, 1]$, then,*

$$1 - \det(I - t^2 Z\overline{Z}')^k + \|t\xi\|_p^2 \le t^2 \left[1 - \det(I - Z\overline{Z}')^k + \|\xi\|_p^2 \right],$$

for $(Z, \xi) \in \mathrm{GHE_I}$.

Lemma 3 ([32]). *Let p_j $(j = 1, 2, \cdots, r)$ be positive integers, $0 < km \leqslant 1$, $t \in [0, 1]$, $(Z, \zeta) \in$ $\mathrm{GHE_I}$, $q = \max\{p_1, p_2, \cdots, p_r\}$. Then, the following inequality holds:*

$$|\overline{(Z, \xi)}| \le M \sqrt{1 - \det(I - Z\overline{Z}')^{\frac{k}{q}} + \|\xi\|_p^{\frac{2}{q}}},$$

where $M = \max\{\sqrt{\frac{q}{k}}, \sqrt{r^{1-\frac{1}{q}}}\}$.

Lemma 4 ([32]). *Let $(Z, \xi), (S, t) \in \mathrm{GHE_I}$, and if $0 < km \leqslant 1$, then*

$$\det(I_m - Z\overline{Z}')^k + \det(I_m - S\overline{S}')^k \le 2|\det(I_m - Z\overline{S}')^k| \tag{2}$$

and "=" holds if and only if $(Z, \xi) = (S, t)$. *If* $km > 1$, *then*

$$\det(I_m - Z\overline{Z}')^k + \det(I_m - S\overline{S}')^k \leq 2^{mk}|\det(I_m - Z\overline{S}')^k|. \tag{3}$$

Lemma 5 ([32]). *Assume* $(Z, \xi), (S, t) \in \mathrm{GHE_I}$ *and* $0 < km \leq 1$, *then*

$$[\det(I_m - Z\overline{Z}')^k - \|\xi\|_p^2] + [\det(I_m - S\overline{S}')^k - \|t\|_p^2] \leq 2||\det(I_m - Z\overline{S}')^k| - \|\xi\|_p\|t\|_p|, \tag{4}$$

with equality that holds if and only if $(Z, \xi) = (S, t)$.

Lemma 6. *Assume* $A, B \in \mathbb{C}^{m\times n}$ *and if* $I - A\overline{A}' > 0,\ I - B\overline{B}' > 0, 0 < km \leq 1$, *then*

$$2^{m(1-k)} \geq \det(I - A\overline{A}')^{1-k}|\det(I - A\overline{B}')|^{k-1}.$$

Proof. By [25], we know

$$2|\det(I - A\overline{B}')|^{\frac{1}{m}} \geq \det(I - A\overline{A}')^{\frac{1}{m}} + \det(I - B\overline{B}')^{\frac{1}{m}}.$$

The inequality is obtained on both sides to the power of $m(1-k)$and we obtain

$$\begin{aligned} 2^{m(1-k)}|\det(I - A\overline{B}')|^{1-k} &\geq [\det(I - A\overline{A}')^{\frac{1}{m}} + \det(I - B\overline{B}')^{\frac{1}{m}}]^{m(1-k)} \\ &\geq [\det(I - A\overline{A}')^{\frac{1}{m}}]^{m(1-k)} \\ &\geq \det(I - A\overline{A}')^{1-k}. \end{aligned}$$

□

Lemma 7 ([26]). *Let* $z = (z_1, z_2, z_3, z_4) \in \Re_{\mathrm{IV}}(4)$. *Hence,*

$$1 + |z_2^2 + z_2^2 + z_3^2 + z_4^2|^2 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) > 0,$$

$$1 - |z_1|^2 - |z_2|^2 - |z_3|^2 - |z_4|^2 > 0.$$

There exists a type of linear mapping, where

$$a_1 = z_1 + iz_2, a_2 = iz_3 - z_4,$$

$$a_3 = iz_3 + z_4, a_4 = z_1 - iz_2.$$

These are mapped one by one to a domain $\Re_{\mathrm{I}}(2,2)$, *where*

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

and

$$1 + |z_2^2 + z_2^2 + z_3^2 + z_4^2|^2 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) = \det(I - A\overline{A}').$$

Lemma 8. *Let* $q_j (j = 1, 2, \cdots, r)$ *be positive integers.* $q = \max\{q_1, q_2, \cdots, q_r\}$, $(z, \zeta) \in \mathrm{GHE_{IV}}$, $t \in [0,1]$, $0 < k \leq \frac{1}{2}$. *Then,*

(1) $1 - (1 + t^4|zz'|^2 - 2t^2|z|^2)^k + \|t\zeta\|_\varphi^2 \leq t^2(1 - (1 + |zz'|^2 - 2|z|^2)^k + \|\zeta\|_\varphi^2)$.

(2) $|\overline{(z, \zeta)}| \leq M\sqrt{1 - (1 + |zz'|^2 - 2|z|^2)^{\frac{k}{q}} + \|\zeta\|_\varphi^{\frac{2}{q}}}$,

where $M = \max\{\sqrt{\frac{q}{k}}, \sqrt{r^{1-\frac{1}{q}}}\}$.

Proof. For $z \in \Re_{\mathrm{IV}}(n)$, there exists a real orthogonal matrix Γ, such that

$$z = (z_1^*, z_2^*, z_3^*, z_4^*, 0, \ldots, 0)\Gamma.$$

Let

$$z^* = (z_1^*, z_2^*, z_3^*, z_4^*),$$
$$A = \begin{pmatrix} z_1^* + iz_2^* & iz_3^* - z_4^* \\ iz_3^* + z_4^* & z_1^* - iz_2^* \end{pmatrix}.$$

Since $1 + |z^* z^{*'}|^2 - 2|z^*|^2 = 1 + |zz'| - 2|z|^2 > 0, 1 - |z^* z^{*'}|^2 > 0$, one has $z^* \in \Re_{\mathrm{IV}}(4)$. From Lemma 7, we obtain $A \in \Re_{\mathrm{I}}(2,2)$ and for all $z \in \Re_{\mathrm{IV}}(n)$, we have

$$1 + |zz'| - 2|z|^2 = 1 + |z^* z^{*'}|^2 - 2|z^*|^2 = \det(I - A\overline{A}').$$

For $t \in [0,1]$, $tz \in \Re_{\mathrm{IV}}(n)$ we obtain

$$1 + t^4 |zz'|^2 - 2t^2 |z|^2 = \det(I - t^2 A\overline{A}').$$

According to Lemma 2,

$$\begin{aligned} 1 - (1 + t^4|zz'|^2 - 2t^2|z|^2)^k + \|t\zeta\|_\varphi^2 &= 1 - \det(I - t^2 A\overline{A}')^k + \|t\zeta\|_\varphi^2 \\ &\le t^2[1 - \det(I - A\overline{A}')^k + \|\zeta\|_\varphi^2] \\ &= t^2[1 - (1 + |zz'|^2 - 2|z|^2)^k + \|\zeta\|_\varphi^2]. \end{aligned}$$

According to Lemma 3 and

$$\begin{aligned} |A|^2 &= |z_1^* + z_2^*|^2 + |z_1^* - z_2^*|^2 + |z_3^* + z_4^*|^2 + |z_3^* - z_4^*|^2 \\ &= 2(|z_1^*|^2 + |z_2^*|^2 + |z_3^*|^2 + |z_4^*|^2) \\ &= 2z\overline{z}' \\ &= 2|z|^2, \end{aligned}$$

we obtain

$$\begin{aligned} |z|^2 &= \frac{|A|^2}{2} \\ &\le \frac{q}{2k}[1 - \det(I - A\overline{A}')^{\frac{k}{q}}] \\ &= \frac{q}{2k}[1 - (1 + |zz'|^2 - 2|z|^2)^{\frac{k}{q}}] \\ &\le \frac{q}{k}[1 - (1 + |zz'|^2 - 2|z|^2)^{\frac{k}{q}}]. \end{aligned} \tag{5}$$

If $0 < p < 1$, then

$$\sum_{k=1}^{n} |a_k|^p \ge \left[\sum_{k=1}^{n} |a_k|\right]^p \ge n^{p-1} \sum_{k=1}^{n} |a_k|^p. \tag{6}$$

One has

$$\begin{aligned} \|\zeta\|_\varphi^{\frac{2}{q}} &= (|\zeta_1|^{2q_1} + |\zeta_2|^{2q_2} + \cdots + |\zeta_r|^{2q_r})^{\frac{1}{q}} \\ &\ge r^{\frac{1}{q}-1}(|\zeta_1|^{\frac{2q_1}{q}} + |\zeta_2|^{\frac{2q_2}{q}} + \cdots + |\zeta_r|^{\frac{2q_r}{q}}) \\ &\ge r^{\frac{1}{q}-1}(|\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_r|^2) \\ &= r^{\frac{1}{q}-1}|\zeta|^2, \end{aligned}$$

and then

$$|\zeta|^2 \leq r^{1-\frac{1}{q}} \|\zeta\|_{\varphi}^{\frac{2}{q}}. \tag{7}$$

Therefore, by combining (6) and (7), we obtain

$$\begin{aligned}|\overline{(z,\zeta)}| &= \sqrt{|z|^2+|\zeta|^2} \\ &\leq \sqrt{\frac{q}{k}[1-(1+|zz'|^2-2|z|^2)^{\frac{k}{q}}] + r^{1-\frac{1}{q}}\|\zeta\|_{\varphi}^{\frac{2}{q}}} \\ &\leq M\sqrt{[1-(1+|zz'|^2-2|z|^2)^{\frac{k}{q}}] + \|\zeta\|_{\varphi}^{\frac{2}{q}}},\end{aligned}$$

where $M = \max\{\sqrt{\frac{q}{k}}, \sqrt{r^{1-\frac{1}{q}}}\}$. □

Lemma 9. *Let* $(z,\zeta) \in \mathrm{GHE_{IV}}$, $(\omega, v) \in \mathrm{GHE_{IV}}$, $0 < k \leq \frac{1}{2}$, *then*
(i) $[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_{\varphi}^2] + [(1+|\omega\omega'|^2-2|\omega|^2)^k - \|v\|_{\varphi}^2]$
$\leq 2||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k| - \|\zeta\|_{\varphi}\|v\|_{\varphi}|$.
(ii) $(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')|^{k-1} \leq 4^{1-k}$.

Proof. For $z, \omega \in \Re_{\mathrm{IV}}(n)$, there exists a real orthogonal matrix Γ, such that

$$z = (z_1^*, z_2^*, z_3^*, z_4^*, 0, \ldots, 0)\Gamma,$$

$$s = (\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*, 0, \ldots, 0)\Gamma.$$

Let

$$A = \begin{pmatrix} z_1^* + iz_2^* & iz_3^* - z_4^* \\ iz_3^* + z_4^* & z_1^* - iz_2^* \end{pmatrix},$$

$$B = \begin{pmatrix} \omega_1^* + i\omega_2^* & \omega_3^* - \omega_4^* \\ i\omega_3^* + \omega_4^* & \omega_1^* - i\omega_2^* \end{pmatrix}.$$

According to Lemma 7, we know that $A, B \in \Re_{\mathrm{I}}(2,2)$, $1+|zz'|^2-2|z|^2 = \det(I - A\overline{A}')$, $1+|\omega\omega'|^2-2|\omega|^2 = \det(I - B\overline{B}')$ and

$$\begin{aligned}1+zz'\overline{\omega\omega'}-2z\overline{\omega}' &= 1+z^*z^{*\prime}\overline{\omega\omega'}-2z^*\overline{\omega^*}' \\ &= 1+[(z_1^*)^2+(z_2^*)^2+(z_3^*)^2+(z_4^*)^2]\overline{(\omega_1^*)^2+(\omega_2^*)^2+(\omega_3^*)^2+(\omega_4^*)^2} \\ &\quad -(z_1^*\omega_1^*+z_2^*\omega_2^*+z_3^*\omega_3^*+z_4^*\omega_4^*) \\ &= \det(I - A\overline{B}').\end{aligned}$$

From Lemma 5, we have

$$\begin{aligned}&[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_{\varphi}^2] + [(1+|\omega\omega'|^2-2|\omega|^2)^k - \|v\|_{\varphi}^2] \\ &\leq 2||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k| - \|\zeta\|_{\varphi}\|v\|_{\varphi}|.\end{aligned}$$

From Lemma 6, we obtain

$$(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')|^{k-1} \leq 4^{1-k}.$$

□

Lemma 10. *Let q_j $(j = 1, 2, \cdots, r)$ be positive integers, $q = \max\{q_1, q_2, \cdots, q_r\}$, $0 < k \leq \frac{1}{2}$ and $f \in \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$. Then, there exists a constant C, such that*

$$|f(z,\zeta)| \leq \begin{cases} C\|f\|_{\mathcal{B}^{(\alpha,k)}} & 0<\alpha<1 \\ C\|f\|_{\mathcal{B}^{(\alpha,k)}} \ln \dfrac{2q}{(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2} & \alpha = 1 \\ C\|f\|_{\mathcal{B}^{(\alpha,k)}} \dfrac{1}{[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^{\alpha-1}} & \alpha > 1 \end{cases} \tag{8}$$

for $\forall (z,\zeta) \in \mathrm{GHE_{IV}}$.

Proof.

$$\begin{aligned} |f(z,\zeta)| &= |f(0,0) + \int_0^1 \langle \nabla f(tz,t\zeta), \overline{(z,\zeta)}\rangle \mathrm{d}t| \\ &\leq |f(0,0)| + \int_0^1 |\nabla f(tz,t\zeta)||\overline{(z,\zeta)}|\mathrm{d}t \\ &= |f(0,0)| + |\overline{(z,\zeta)}| \int_0^1 \frac{[(1+t^4|zz'|^2-2t^2|z|^2)^k - \|t\zeta\|_\varphi^2]^\alpha |\nabla f(tz,t\zeta)|}{[(1+t^4|zz'|^2-2t^2|z|^2)^k - \|t\zeta\|_\varphi^2]^\alpha}\mathrm{d}t \\ &\leq |f(0,0)| + |\overline{(z,\zeta)}| \int_0^1 \frac{\|f\|_{\mathcal{B}^{(\alpha,k)}}}{[(1+t^4|zz'|^2-2t^2|z|^2)^k - \|t\zeta\|_\varphi^2]^\alpha}\mathrm{d}t \\ &\leq \left[1 + \int_0^1 \frac{|\overline{(z,\zeta)}|}{[(1+t^4|zz'|^2-2t^2|z|^2)^k - \|t\zeta\|_\varphi^2]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= \left[1 + \int_0^1 \frac{|\overline{(z,\zeta)}|}{[1-(1-(1+t^4|zz'|^2-2t^2|z|^2)^k + \|t\zeta\|_\varphi^2)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}}. \end{aligned}$$

According to Lemma 8, we obtain

$$\begin{aligned} &\left[1 + \int_0^1 \frac{|\overline{(z,\zeta)}|}{[1-(1-(1+t^4|zz'|^2-2t^2|z|^2)^k + \|t\zeta\|_\varphi^2)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\leq \left[1 + M\int_0^1 \frac{\sqrt{1-(1+|zz'|^2-2|z|^2)^{\frac{k}{q}} + \|\zeta\|_\varphi^{\frac{2}{q}}}}{[1-t^2(1-(1+|zz'|^2-2|z|^2)^k + \|\zeta\|_\varphi^2)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}}. \end{aligned}$$

By the elementary inequality $a - b \leq q(a^{\frac{1}{q}} - b^{\frac{1}{q}})$, we obtain

$$\begin{aligned} &\left[1 + M\int_0^1 \frac{\sqrt{1-(1+|zz'|^2-2|z|^2)^{\frac{k}{q}} + \|\zeta\|_\varphi^{\frac{2}{q}}}}{[1-t^2(1-(1+|zz'|^2-2|z|^2)^k + \|\zeta\|_\varphi^2)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\leq \left[1 + M\int_0^1 \frac{\sqrt{1-\frac{1}{q}((1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2)}}{[1-t^2(1-(1+|zz'|^2-2|z|^2)^k + \|\zeta\|_\varphi^2)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\leq \left[1 + M\int_0^1 \frac{\sqrt{1-\frac{1}{q}((1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2)}}{[1-t^2(1-\frac{1}{q}((1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2))]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= \left[1 + M\int_0^1 \frac{K}{[1-t^2K^2]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= \left[1 + M\int_0^1 \frac{K}{[(1-tK)(1+tK)]^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\leq \left[1 + M\int_0^1 \frac{K}{(1-tK)^\alpha}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}}, \end{aligned}$$

where $K = \sqrt{1 - \frac{1}{q}((1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2)}$.
Below is a classification discussion of α:
Case $\mathcal{B}_1$: $0 < \alpha < 1$,

$$\begin{aligned}|f(z,\zeta)| &\le \left\{1 + \frac{M}{1-\alpha}[1 - (1-K)^{1-\alpha}]\right\}\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le (1 + \frac{M}{1-\alpha})\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le C\|f\|_{\mathcal{B}^{(\alpha,k)}},\end{aligned} \tag{9}$$

where $C = 1 + \frac{M}{1-\alpha}$.
Case $\mathcal{B}_2$: $\alpha = 1$,

$$\begin{aligned}|f(z,\zeta)| &\le \left[1 + M\int_0^1 \frac{K}{1-tK}\mathrm{d}t\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= \left[1 + M\ln\frac{1}{1-K}\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= \left[1 + M\ln\frac{1+K}{(1-K)(1+K)}\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le \left[1 + M\ln\frac{2}{1-K^2}\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le \left[\frac{1}{\ln 2}\ln\frac{2}{1-K^2} + M\ln\frac{2}{1-K^2}\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le \left[\frac{1}{\ln 2} + M\right]\ln\frac{2}{1-K^2}\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= C\|f\|_{\mathcal{B}^\alpha}\ln\frac{2q}{(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2},\end{aligned} \tag{10}$$

where $C = \frac{1}{\ln 2} + M$.
Case $\mathcal{B}_3$: $\alpha > 1$,

$$\begin{aligned}|f(z,\zeta)| &\le \left[1 + \frac{M}{\alpha-1}\left(\frac{1}{(1-K)^{\alpha-1}} - 1\right)\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le \left[C' + C'\left(\frac{1}{(1-K)^{\alpha-1}} - 1\right)\right]\|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &= C'\|f\|_{\mathcal{B}^{(\alpha,k)}}\frac{1}{(1-K)^{\alpha-1}} \\ &= C'\|f\|_{\mathcal{B}^{(\alpha,k)}}\frac{(1+K)^{\alpha-1}}{[(1-K)(1+K)]^{\alpha-1}} \\ &\le 2^{\alpha-1}C'\|f\|_{\mathcal{B}^{(\alpha,k)}}\frac{1}{(1-K^2)^{\alpha-1}} \\ &= C\|f\|_{\mathcal{B}^{(\alpha,k)}}\frac{1}{[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^{\alpha-1}},\end{aligned} \tag{11}$$

where $C = (2q)^{\alpha-1}C'$, $C' = \max\{1, \frac{M}{\alpha-1}\}$. By combining (9)–(11), the proof is completed. □

Lemma 11. *Let $\phi = (\phi_1, \phi_2, \ldots, \phi_{n+r})$ be a holomorphic self-map of $\mathrm{GHE}_{\mathrm{IV}}$ and $\psi \in H(\mathrm{GHE}_{\mathrm{IV}})$. The weighted composition operator ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE}_{\mathrm{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE}_{\mathrm{IV}})$ is compact if and only if ${}_\psi C_\phi$ is bounded and for any bounded sequence $\{f_n\}_{n\ge1}$ in $\mathcal{B}^{(\alpha,k)}(\mathrm{GHE}_{\mathrm{IV}})$ converging to 0 uniformly on compact subsets of $\mathrm{GHE}_{\mathrm{IV}}$, $\|{}_\psi C_\phi f_n\|_{\mathcal{A}_{(\beta,k)}} \to 0$ as $n \to \infty$.*

Proof. This is similar to the proof of Lemma 12 in reference [32]. □

3. Boundedness of ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)} \to \mathcal{A}_{(\beta,k)}$

Theorem 1. *Assume that* $\alpha = 1$, $\beta > 0$, $0 < k \le \frac{1}{2}$ *and that* $q_j (j = 1, 2, \cdots, r)$ *are positive integers,* $q = \max\{q_1, q_2, \cdots, q_r\}$. *Let* $\phi = (\phi_1, \phi_2, \cdots, \phi_{n+r})$ *be a holomorphic self-map of* $\mathrm{GHE_{IV}}$, *with* $\psi \in H(\mathrm{GHE_{IV}})$ *and* $(z_\phi, \zeta_\phi) = \phi(z, \zeta)$. *If*

$$M_1 := \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta \times \ln \frac{2q}{(1+|z_\phi z_\phi'|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} < \infty, \tag{12}$$

then the weighted composition operator ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ *is bounded.*

Conversely, if the weighted composition operator ${}_\psi C_\phi : \mathcal{B}^{(\alpha.k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ *is bounded, then*

$$M_2 := \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta (1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^{1-k} \times \ln \frac{2q}{(1+|z_\phi z_\phi'|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} < \infty. \tag{13}$$

Proof. Assume that (12) holds and for $f \in \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$, we know that

$$\begin{aligned}&[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |({}_\psi C_\phi f)(z,\zeta)| \\ &= [(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta)||f(\phi(z,\zeta))|.\end{aligned}$$

From Lemma 10, we obtain

$$\begin{aligned}&[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta)||f(\phi(z,\zeta))| \\ &\le C|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta \times \ln \frac{2q}{(1+|z_\phi z_\phi'|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} \|f\|_{\mathcal{B}^{(\alpha,k)}} \\ &\le CM_1 \|f\|_{\mathcal{B}^{(\alpha,k)}}.\end{aligned}$$

For all $(z,\zeta) \in \mathrm{GHE_{IV}}$, we have

$$\|{}_\psi C_\phi f\|_{\mathcal{A}_{(\beta,k)}} = \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |({}_\psi C_\phi f)(z,\zeta)| \le CM_1 \|f\|_{\mathcal{B}^{(\alpha,k)}},$$

which implies that ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded.

Conversely, assume that ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded. For any $(\omega, v) \in \mathrm{GHE_{IV}}$, let us introduce a test function $f_{(\omega,v)} \in H(\mathrm{GHE_{IV}})$, such that

$$f_{(\omega,v)}(z,\zeta) := (1+|\omega\omega'|^2 - 2|\omega|^2)^{1-k} \ln \frac{2q}{(1+zz'\overline{\omega\omega'} - 2z\overline{\omega}')^k - \langle \zeta, v\rangle_\varphi}.$$

This means that

$$\begin{aligned}\frac{\partial f_{(\omega,v)}}{\partial z_l} &= \frac{k\cdot(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\cdot(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^{k-1}(2\overline{\omega_l}'-2z_l\overline{\omega\omega'})}{(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k - \langle\zeta,v\rangle_\varphi}, \\ &\quad l = 1,\cdots,n. \\ \frac{\partial f_{(\omega,v)}}{\partial \zeta_j} &= \frac{(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\cdot q_j \zeta_j^{q_j-1}\overline{v_j}^{q_j}}{(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k - \langle\zeta,v\rangle_\varphi}, \quad j = 1,\cdots,r.\end{aligned}$$

There exists a constant $C_1 > 0$, such that

$$
\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]|\nabla f_{(\omega,v)}(z,\zeta)| \\
&= \frac{[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2](1+|\omega\omega'|^2-2|\omega|^2)^{1-k}}{|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k - \langle\zeta,v\rangle_\varphi|} \\
&\times \left\{ k^2|1+zz'\overline{\omega\omega'}-2z\overline{\omega}'|^{2k-2} \times \sum_{l=1}^{n} |2\overline{\omega_l}'-2z_l\overline{\omega\omega'}|^2 + \sum_{j=1}^{r} |q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2 \right\}^{\frac{1}{2}} \\
&\le \frac{[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k| - \|\zeta\|_\varphi\|v\|_\varphi|} \\
&\times \left\{ k(1+|\omega\omega'|^2-2|\omega\omega|^2)^{1-k}|1+zz'\overline{\omega\omega'}-2z\overline{\omega}'|^{k-1}|2\overline{\omega}'-2z\overline{\omega\omega'}| \right. \\
&\left. + \left[\sum_{j=1}^{r} |q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}} \times (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \right\}.
\end{aligned}
$$

According to Lemma 9, we obtain

$$
\begin{aligned}
&\frac{[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k| - \|\zeta\|_\varphi\|v\|_\varphi|} \\
&\times \left\{ k(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}|1+zz'\overline{\omega\omega'}-2z\overline{\omega}'|^{k-1}|2\overline{\omega}'-2z\overline{\omega\omega'}| \right. \\
&\left. + \left[\sum_{j=1}^{r} |q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}} \times (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \right\} \\
&\le \frac{2[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]}{[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2] + [(1+|\omega\omega'|^2-2|\omega|^2)^k - \|v\|_\varphi^2]} \\
&\quad \times \left\{ k4^{1-k}|2\overline{\omega}'-2z\overline{\omega\omega'}| + \left[\sum_{j=1}^{r} |q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}} \times (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \right\} \\
&\le 2 \times \left\{ k4^{1-k}(2|\overline{\omega}'|+2|z||\overline{\omega\omega'}|) + \left[\sum_{j=1}^{r} |q_j|^2\right]^{\frac{1}{2}} \times (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \right\} \\
&\le 2 \times \left\{ k4^{2-k} + \left[\sum_{j=1}^{r} |q_j|^2\right]^{\frac{1}{2}} \times (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \right\} \\
&\le C_1.
\end{aligned}
$$

Since $f_{(\omega,v)}(0,0) = (1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\ln 2q \le \ln 2q$, one has

$$
\begin{aligned}
\|f_{(\omega,v)}\|_{\mathcal{B}^{(\alpha,k)}} &= |f_{(\omega,v)}(0,0)| + \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\alpha |\nabla f_{(\omega,v)}(z,\zeta)| \\
&\le C_1 + \ln 2q.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\infty &> (C_1+\ln 2q)\|{}_\psi C_\phi\|_{\mathcal{B}^{(\alpha,k)}\to\mathcal{A}_{(\beta,k)}} \\
&\ge \|{}_\psi C_\phi f_{(\omega,v)}\|_{\mathcal{A}_{(\beta,k)}} \\
&= \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta)f_{(\omega,v)}(\phi(z,\zeta))| \\
&\ge |\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta (1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \\
&\times \left| \ln \frac{2q}{(1+z_\phi z_\phi'\overline{\omega\omega'}-2z_\phi\overline{\omega}')^k - \langle\zeta_\phi,v\rangle_\varphi} \right|.
\end{aligned}
$$

Let us consider

$$(\omega,v) = (z_\phi,\zeta_\phi) = \phi(z,\zeta),$$

so that

$$\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^{1-k} \times \ln\frac{2q}{(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2}<\infty.$$

□

Theorem 2. *Assume that $\alpha>1,\beta>0$, $0<k\le\frac{1}{2}$ and that q_j $(j=1,2,\cdots,r)$ are positive integers. Let $\phi=(\phi_1,\phi_2,\cdots,\phi_{n+r})$ be a holomorphic self-map of $\mathrm{GHE_{IV}}$, with $\psi\in H(\mathrm{GHE_{IV}})$ and $(z_\phi,\zeta_\phi)=\phi(z,\zeta)$. If*

$$M_3:=\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}}|\psi(z,\zeta)|\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}}<\infty, \tag{14}$$

then the weighted composition operator ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded.

Conversely, if the weighted composition operator ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded, then

$$M_4:=\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}}|\psi(z,\zeta)|\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^{1-k}}{[(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}}<\infty. \tag{15}$$

Proof. Assuming that (14) holds and $f\in\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$, we have

$$\begin{aligned}&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f)(z,\zeta)|\\ &=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)\cdot(C_\phi f)(z,\zeta)|\\ &=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f(\phi(z,\zeta))|.\end{aligned}$$

From Lemma 10, we obtain

$$\begin{aligned}&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f(\phi(z,\zeta))|\\ &\le C|\psi(z,\zeta)|\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}}\|f\|_{\mathcal{B}^{(\alpha,k)}}\\ &\le CM_3\|f\|_{\mathcal{B}^{(\alpha,k)}}.\end{aligned}$$

For all $(z,\zeta)\in\mathrm{GHE_{IV}}$, we obtain

$$\|{}_\psi C_\phi f\|_{\mathcal{A}_{(\beta,k)}}=\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}}[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f)(z,\zeta)|\le CM_3\|f\|_{\mathcal{B}^{(\alpha,k)}}.$$

This implies that ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded.

Conversely, assume that ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded. For $(\omega,v)\in\mathrm{GHE_{IV}}$, we define a test function $f_{(\omega,v)}\in H(\mathrm{GHE_{IV}})$, such that

$$f_{(\omega,v)}(z,\zeta):=\frac{(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}}{[(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k-\langle\zeta,v\rangle_\varphi]^{\alpha-1}}.$$

For the test function f, we have

$$\begin{aligned}\frac{\partial f_{(\omega,v)}}{\partial z_l}&=\frac{k(\alpha-1)\cdot(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\cdot(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^{k-1}}{[(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k-\langle\zeta,v\rangle_\varphi]^\alpha}\\ &\quad\times(2\overline{\omega_l}'-2z_l\overline{\omega\omega'}), \qquad l=1,\cdots,n.\\ \frac{\partial f_{(\omega,v)}}{\partial \zeta_j}&=\frac{(\alpha-1)q_j\zeta_j^{q_j-1}\overline{t_j}^{q_j}\cdot(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}}{[(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k-\langle\zeta,v\rangle_\varphi]^\alpha}, \qquad j=1,\cdots,r.\end{aligned}$$

There exists a constant $C_2 > 0$, such that

$$
\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha|\nabla f_{(\omega,v)}(z,\zeta)| \\
&= \frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k-\langle\zeta,v\rangle_\varphi|^\alpha}\times(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\times(\alpha-1) \\
&\times\left\{k^2|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^{k-1}|^2\times\sum_{l=1}^{n}|2\overline{\omega_l}'-2z_l\overline{\omega\omega'}|^2+\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right\}^{\frac{1}{2}} \\
&\leq \frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^\alpha}\times(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\times(\alpha-1) \\
&\times\left\{k^2|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^{k-1}|^2\times\sum_{l=1}^{n}|2\overline{\omega_l}'-2z_l\overline{\omega\omega'}|^2+\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right\}^{\frac{1}{2}} \\
&\leq \frac{(\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^\alpha}\times(1+|\omega\omega'|^2-2|\omega|^2)^{1-k} \\
&\times\left\{k|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')|^{k-1}\times|2\overline{\omega}'-2z\overline{\omega\omega'}|+\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\} \\
&= \frac{(\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^\alpha} \\
&\times\left\{k|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')|^{k-1}(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}|2\overline{\omega}'-2z\overline{\omega\omega'}|\right. \\
&\left.+(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\}.
\end{aligned}
$$

From Lemma 9, we obtain

$$
\begin{aligned}
&\frac{(\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^\alpha} \\
&\times\left\{k|(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')|^{k-1}(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}|2\overline{\omega}'-2z\overline{\omega\omega'}|\right. \\
&\left.+(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\} \\
&\leq \frac{(\alpha-1)2^\alpha[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]+[(1+|\omega\omega'|^2-2|\omega|^2)^k-\|v\|_\varphi^2]|^\alpha} \\
&\times\left\{4^{1-k}k|2\overline{\omega}'-2z\overline{\omega\omega'}|+(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\} \\
&\leq \frac{(\alpha-1)2^\alpha[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha}{|(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2|^\alpha} \\
&\times\left\{4^{1-k}k(|2\overline{\omega}'|+|2z\overline{\omega\omega'}|)+(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\} \\
&\leq (\alpha-1)2^\alpha\left\{4^{2-k}k+(1+|\omega\omega'|^2-2|\omega|^2)^{1-k}\left[\sum_{j=1}^{r}|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\} \\
&\leq C_2.
\end{aligned}
$$

Since $f_{(\omega,v)}(0,0) = (1 + |\omega\omega'|^2 - 2|\omega|^2)^{1-k} \le 1$, so that

$$\|f_{(\omega,v)}\|_{\mathcal{B}^{(\alpha,k)}} = |f_{(\omega,v)}(0,0)| + \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\alpha |\nabla f_{(\omega,v)}(z,\zeta)| \\ \le C_2 + 1.$$

It follows that

$$\begin{aligned}\infty &> (C_2+1)\|_\psi C_\phi\|_{\mathcal{B}^{(\alpha,k)}\to \mathcal{A}_{(\beta,k)}} \\ &\ge \|_\psi C_\phi f_{(\omega,v)}\|_{\mathcal{A}_{(\beta,k)}} \\ &= \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} (1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta) f_{(\omega,v)}(\phi(z,\zeta))| \\ &\ge |\psi(z,\zeta)| \frac{[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta (1 + |\omega\omega'|^2 - 2|\omega|^2)^{1-k}}{[(1 + z_\phi z'_\phi \overline{\omega\omega'} - 2z_\phi\overline{\omega}')^k - \langle \zeta_\phi, v\rangle_\varphi]^{\alpha-1}}.\end{aligned}$$

For $(\omega, v) = (z_\phi, \zeta_\phi) = \phi(z,\zeta)$, we obtain

$$\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} |\psi(z,\zeta)| \frac{[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta (1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^{1-k}}{[(1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2]^{\alpha-1}} < \infty.$$

□

4. Compactness of $_\psi C_\phi : \mathcal{B}^{(\alpha,k)} \to \mathcal{A}_{(\beta,k)}$

Theorem 3. *Assume $\alpha = 1$, $\beta > 0$, $0 < k \le \frac{1}{2}$ and that q_j $(j = 1, 2, \cdots, r)$ are positive integers, $q = \max\{q_1, q_2, \cdots, q_r\}$. Let $\phi = (\phi_1, \phi_2, \cdots, \phi_{n+r})$ be a holomorphic self-map of $\mathrm{GHE_{IV}}$, with $\psi \in H(\mathrm{GHE_{IV}})$ and $(z_\phi, \zeta_\phi) = \phi(z,\zeta)$. If $\psi \in \mathcal{A}_{(\beta,k)}$ and*

$$\lim_{\phi(z,\zeta)\to\partial \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta \ln \frac{2q}{(1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} = 0, \quad (16)$$

then the weighted composition operator $_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact.

Conversely, if the weighted composition operator $_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact, then $\psi \in \mathcal{A}_{(\beta,k)}$ and

$$\lim_{\phi(z,\zeta)\to\partial \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta (1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^{1-k} \\ \times \ln \frac{2q}{(1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} = 0. \quad (17)$$

Proof. Assume that (16) holds. We have

$$\sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} |\psi(z,\zeta)|[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta \ln \frac{2q}{(1 + |z_\phi z'_\phi|^2 - 2|z_\phi|^2)^k - \|\zeta_\phi\|_\varphi^2} < \infty,$$

then, $_\psi C_\phi$ is bounded. Consider the bounded sequence $\{f_k\}_{k\ge 1}$ in $\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$, which converges to 0 uniformly on compact subsets of $\mathrm{GHE_{IV}}$. Hence, there exists $Q_1 > 0$, such that $\|f_k\|_{\mathcal{B}^{(\alpha,k)}} \le Q_1, k = 1, 2, \cdots$. From (16), $\forall\ \varepsilon > 0$, $\exists\ \delta \in (0,1)$, such that for $\mathrm{dist}(\phi(z,\zeta), \partial \mathrm{GHE_{IV}}) < \delta$, we have

$$|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta \ln\frac{2q}{(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2}<\varepsilon. \tag{18}$$

According to Lemma 10, we obtain

$$\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f_k)(z,\zeta)|\\
&=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)\cdot(C_\phi f_k)(z,\zeta)|\\
&=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f_k(\phi(z,\zeta))|\\
&\le C|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta\|f_k\|_{\mathcal{B}^{(\alpha,k)}}\\
&\times\ln\frac{2q}{(1+|z_\phi z'_\phi|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2}\\
&\le CQ_1\varepsilon.
\end{aligned}\tag{19}$$

On the other hand, let us introduce the set

$$E_\delta:=\{(z,\zeta)\in \mathrm{GHE}_{\mathrm{IV}}:\mathrm{dist}(\phi(z,\zeta),\partial\mathrm{GHE}_{\mathrm{IV}})\ge\delta\},$$

which is a compact subset of $\mathrm{GHE}_{\mathrm{IV}}$. Assuming that $\{f_k\}$ converges to 0 uniformly on any compact subset of $\mathrm{GHE}_{\mathrm{IV}}$ and since $\psi\in\mathcal{A}_{(\beta,k)}$, for such ε, we know

$$\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f_k)(z,\zeta)|\\
&=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)\cdot(C_\phi f_k)(z,\zeta)|\\
&=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f_k(\phi(z,\zeta))|\\
&\le\|\psi\|_{\mathcal{A}_{(\beta,k)}}\varepsilon.
\end{aligned}\tag{20}$$

Combining (19) and (20), we have

$$\|{}_\psi C_\phi f_k\|_{\mathcal{A}_{(\beta,k)}}=\sup_{(z,\zeta)\in\mathrm{GHE}_{\mathrm{IV}}}[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f_k)(z,\zeta)|\to 0,\quad k\to\infty.$$

Hence, from Lemma 11, we finally have that ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE}_{\mathrm{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE}_{\mathrm{IV}})$ is compact.

Consequently, suppose ${}_\psi C_\phi:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE}_{\mathrm{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE}_{\mathrm{IV}})$ is compact. Letting $f\equiv 1$, we have

$$[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)|=[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f)(z,\zeta)|<\infty,$$

which shows that $\psi\in\mathcal{A}_{(\beta,k)}$. Consider now a sequence $(\omega^i,v^i)=\phi(z^i,\zeta^i)$ in $\mathrm{GHE}_{\mathrm{IV}}$, such that $\phi(z^i,\zeta^i)\to\partial\mathrm{GHE}_{\mathrm{IV}}$ as $i\to\infty$. If such a sequence does not exist, then condition (17) obviously holds. Moreover, let us introduce the following sequence of test functions $\{f_i\}_{i\ge1}$:

$$\begin{aligned}
f_i(z,\zeta)&=\left\{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}\right\}^{-1}\\
&\times\left\{\ln\frac{2q}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}\right\}^2\\
&\times(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k},\ \ i=1,2,\cdots.
\end{aligned}$$

Differentiating the above formula provides

$$\frac{\partial f_i}{\partial z_l} = \frac{2k(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^{i'}})^{k-1}(2\overline{\omega_l^i}'-2z_l\overline{\omega^i\omega^{i'}})(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}$$
$$\times\frac{\ln\frac{2q}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}, \qquad l=1,\cdots,n.$$

$$\frac{\partial f_i}{\partial \zeta_j} = \frac{2q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}\times(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}\times\frac{\ln\frac{2q}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}.$$
$$j=1,\cdots,r,\ \ i=1,2,\cdots.$$

There exists two constants $C_3>0$ and $C_4>0$, such that

$$[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]|\nabla f_i(z,\zeta)|$$
$$=\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2](1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\times\left|\frac{\ln\frac{2q}{(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi}}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\right|$$
$$\times\left\{4k^2|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^{i'}})^{k-1}|^2\times\sum_{l=1}^n|2\overline{\omega_l^i}'-2z_l\overline{\omega^i\omega^{i'}}|^2+4\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right\}^{\frac{1}{2}}$$
$$\le\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2](1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}}{||(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-|\langle\zeta,v^i\rangle_\varphi||}$$
$$\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\times\left\{2k|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^{i'}})|^{k-1}|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|\right.$$
$$\left.+2\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}\right\}$$
$$\le\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]}{||(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-|\langle\zeta,v^i\rangle_\varphi||}\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}$$
$$\times\left\{2k(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^{i'}})|^{k-1}\right.$$
$$\left.\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|+2\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}\right\},$$

from Lemma 9, we obtain

$$\begin{aligned}
&\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]}{||(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-|\langle\zeta,v^i\rangle_\varphi||}\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\times\left\{2k(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^{i'}})|^{k-1}\right.\\
&\left.\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|+2\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}\right\}\\
&\le\frac{2[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]}{(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2+(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}\\
&\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\times\left\{2k\times4^{1-k}|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|+2\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}\right\}\\
&\le\frac{2[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]}{(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2}\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|t^i\|_\varphi^2}}\\
&\times\left\{2k\times4^{1-k}(|2\overline{\omega^i}'|+|2z\overline{\omega^i\omega^{i'}}|)+2\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right]^{\frac{1}{2}}(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}\right\}\\
&\le2\times\left\{4^{2-k}+C_3\right\}\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le C_4\times\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}.
\end{aligned}$$

We now have two cases:

Case $\mathscr{C}$: If $|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|\le2q$, then

$$\begin{aligned}
\frac{\left|\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|}\right|+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}
&\le\frac{\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-|\langle\zeta,v^i\rangle_\varphi|}+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le\frac{\ln\frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-\|\zeta\|_\varphi^2\|v^i\|_\varphi^2}+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le\frac{\ln\frac{4q}{(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2+(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le\frac{\ln\frac{4q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}+\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le2+\frac{\pi}{\ln\frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}}\\
&\le C_5,
\end{aligned}$$

where $C_5 = 2 + \frac{\pi}{\ln 2}$.

Case $\mathscr{D}$: If $|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i \rangle_\varphi| > 2q$, then

$$\begin{aligned}
\frac{\left|\ln \frac{2q}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi|}\right| + \pi}{\ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k - \|v^i\|_\varphi^2}} &= \frac{|\ln 2q - \ln|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi|| + \pi}{\ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k - \|v^i\|_\varphi^2}} \\
&\le \frac{\ln|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi| + \pi}{\ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k - \|v^i\|_\varphi^2}} \\
&\le \frac{\ln(|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k| + |\langle \zeta, v^i\rangle_\varphi|) + \pi}{\ln 2} \\
&\le \frac{\ln(G_0^k + \|\zeta\|_\varphi \|v^i\|_\varphi) + \pi}{\ln 2} \\
&\le \frac{\ln(G_0^k + 1) + \pi}{\ln 2} \\
&\le C_6,
\end{aligned}$$

where

$$G_0 = n^2 + 2n + 1 \ge |1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}'|.$$

By using both cases $\mathscr{C}$ and $\mathscr{D}$, we obtain that $[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]|\nabla f_i(z,\zeta)| \le QC_4$, then $\|f_i\|_{\mathcal{B}^{(\alpha,k)}} \le QC_4$, which means that $\{f_i\}$ is bounded, where $Q = \max\{C_5, C_6\}$. It follows that $f_i \in \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$ and

$$\begin{aligned}
|f_i(z,\zeta)| &= \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \\
&\quad \times \left|\ln \frac{2q}{(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi}\right|^2 (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\le \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \times (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\quad \times \left\{\left|\ln \frac{2q}{|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi|}\right| + \pi\right\}^2.
\end{aligned}$$

If $|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i\rangle_\varphi| \le 2q$, then

$$\begin{aligned}
|f_i(z,\zeta)| &\le \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \times (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\quad \times \left\{\ln \frac{2q}{|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k| - |\langle \zeta, v^i\rangle_\varphi|} + \pi\right\}^2 \\
&\le \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \times (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\quad \times \left\{\ln \frac{2q}{|(1 + zz'\overline{\omega^i\omega^{i'}} - 2z\overline{\omega^i}')^k| - \|\zeta\|_\varphi\|v^i\|_\varphi} + \pi\right\}^2 \\
&\le \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \times (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\quad \times \left\{\ln \frac{4q}{[(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2] + [(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2]} + \pi\right\}^2 \\
&\le \left\{\ln \frac{2q}{(1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2}\right\}^{-1} \times (1 + |\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
&\quad \times \left\{\ln \frac{4q}{(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2} + \pi\right\}^2.
\end{aligned}$$

Since $0 < (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \le 1$, we take $i \to \infty$ and obtain $(\omega^i, v^i) \to \partial\mathrm{GHE}_{\mathrm{IV}}$. This implies $(1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2 \to 0$, then $\left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \to 0$. Consider a compact subset E of $\mathrm{GHE}_{\mathrm{IV}}$. For $(z,\zeta) \in E$, it is easy to see that $(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2$ has a positive lower bound. Thus, we have $f_i(z,\zeta) \rightrightarrows 0,\ i \to \infty$ on all compact subsets of $\mathrm{GHE}_{\mathrm{IV}}$.

If $|(1 + zz'\overline{\omega^i \omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i \rangle_\varphi| > 2q$, then

$$\begin{aligned}
|f_i(z,\zeta)| \le & \left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \times (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \left\{ |\ln 2q - \ln(|(1 + zz'\overline{\omega^i \omega^{i'}} - 2z\overline{\omega^i}')^k - \langle \zeta, v^i \rangle_\varphi|)| + \pi \right\}^2 \\
\le & \left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \times (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \left\{ \ln(|(1 + zz'\overline{\omega^i \omega^{i'}} - 2z\overline{\omega^i}')^k| + |\langle \zeta, v^i \rangle_\varphi|) + \pi \right\}^2 \\
\le & \left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \times (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \{\ln(G_0^k + 1) + \pi\}^2.
\end{aligned}$$

Since $0 < (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \le 1$ and $\left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \to 0$ as $i \to \infty$, we have $f_i(z,\zeta) \to 0$.

The above proof shows that $f_i(z,\zeta) \rightrightarrows 0,\ i \to \infty$ on all compact subsets of $\mathrm{GHE}_{\mathrm{IV}}$. From Lemma 11, this implies that $\|_\psi C_\phi f_i\|_{\mathcal{A}_{(\beta,k)}} \to 0$. Hence, we conclude that

$$\begin{aligned}
0 \leftarrow & \|_\psi C_\phi f_i\|_{\mathcal{A}_{(\beta,k)}} \\
= & \sup_{(z,\zeta)\in \mathrm{GHE}_{\mathrm{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta)| (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \\
& \times \left| \ln \frac{2q}{(1 + z_\phi z_\phi \overline{\omega^i \omega^{i'}} - 2z_\phi \overline{\omega^i}')^k - \langle \zeta_\phi, v^i \rangle_\varphi} \right|^2 \\
\ge & [(1 + |z^i z^{i'}|^2 - 2|z^i|^2)^k - \|\zeta^i\|_\varphi^2]^\beta |\psi(z^i,\zeta^i)| (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \left\{ \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right\}^{-1} \times \left| \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2} \right|^2 \\
= & |\psi(z^i,\zeta^i)| [(1 + |z^i z^{i'}|^2 - 2|z^i|^2)^k - \|\zeta^i\|_\varphi^2]^\beta (1 + |\omega^i \omega^{i'}|^2 - 2|\omega^i|^2)^{1-k} \\
& \times \ln \frac{2q}{(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2}.
\end{aligned}$$

□

Theorem 4. *Assume $\alpha > 1$, $\beta > 0$, $0 < k \le \frac{1}{2}$ and that $q_j (j = 1, 2, \cdots, r)$ are some positive integers. Let $\phi = (\phi_1, \phi_2, \cdots, \phi_{n+r})$ be a holomorphic self-map of $\mathrm{GHE_{IV}}$, with $\psi \in H(\mathrm{GHE_{IV}})$, $(z_\phi, \zeta_\phi) = \phi(z, \zeta)$. If $\psi \in \mathcal{A}_{(\beta,k)}$ and*

$$\lim_{\phi(z,\zeta)\to\partial\mathrm{GHE_{IV}}} \frac{|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}} = 0, \tag{21}$$

then the weighted composition operator ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact.

Conversely, if the weighted composition operator ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact, then $\psi \in \mathcal{A}_{(\beta,k)}$ and

$$\lim_{\phi(z,\zeta)\to\partial\mathrm{GHE_{IV}}} \frac{|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-\frac{1}{k}}} = 0. \tag{22}$$

Proof. Assume that (21) holds. We have

$$\sup_{(z,\zeta)\in\mathrm{GHE_{IV}}} \frac{|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}} < \infty.$$

From Theorem 2, we have that ${}_\psi C_\phi : \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}) \to \mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded. Let $\{f_k\}_{k\ge1}$ be a bounded sequence in $\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})$ with $\{f_k\}$ that converges to 0 uniformly on compact subsets of $\mathrm{GHE_{IV}}$. There exists $\theta_2 > 0$, such that $\|f_k\|_{\mathcal{B}^{(\alpha,k)}} \le \theta_2$, $k = 1, 2, \cdots$. From (21) and for any $\varepsilon > 0$, there is a constant $\delta \in (0,1)$ for $\mathrm{dist}(\phi(z,\zeta), \partial\mathrm{GHE_{IV}}) < \delta$, such that

$$\frac{|\psi(z,\zeta)|[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}} < \varepsilon. \tag{23}$$

From Lemma 10, we have

$$\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f_k)(z,\zeta)| \\
&= [(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)\cdot(C_\phi f_k)(z,\zeta)| \\
&= [(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f_k(\phi(z,\zeta))| \\
&\le C|\psi(z,\zeta)|\|f_k\|_{\mathcal{B}^{(\alpha,k)}} \frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta}{[(1+|z_\phi z_\phi'|^2-2|z_\phi|^2)^k-\|\zeta_\phi\|_\varphi^2]^{\alpha-1}} \\
&\le C\theta_2\varepsilon.
\end{aligned} \tag{24}$$

On the other hand, if we set

$$E_\delta := \{(z,\zeta) \in \mathrm{GHE_{IV}} : \mathrm{dist}(\phi(z,\zeta), \partial\mathrm{GHE_{IV}}) \ge \delta\},$$

we have that E_δ is a compact subset of $\mathrm{GHE_{IV}}$. For ε defined in (23), $\{f_k\}$ converges to 0 uniformly on any compact subset of $\mathrm{GHE_{IV}}$. For $\psi \in \mathcal{A}_{(\beta,k)}$, we have

$$\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|({}_\psi C_\phi f_k)(z,\zeta)| \\
&= [(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)\cdot(C_\phi f_k)(z,\zeta)| \\
&= [(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\beta|\psi(z,\zeta)||f_k(\phi(z,\zeta))| \\
&\le \|\psi\|_{\mathcal{A}_{(\beta,k)}}\varepsilon.
\end{aligned} \tag{25}$$

According to inequalities (24) and (25), we see that

$$\|_{\psi}C_{\phi}f_k\|_{\mathcal{A}_{(\beta,k)}} = \sup_{(z,\zeta)\in \mathrm{GHE_{IV}}} [(1+|zz'|^2-2|z|^2)^k - \|\zeta\|_{\varphi}^2]^{\beta} |({}_{\psi}C_{\phi}f_k)(z,\zeta)| \to 0, \quad k\to\infty.$$

Consequently, from Lemma 11, ${}_{\psi}C_{\phi}:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact.

Conversely, suppose that ${}_{\psi}C_{\phi}:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is compact. Then, ${}_{\psi}C_{\phi}:\mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}})\to\mathcal{A}_{(\beta,k)}(\mathrm{GHE_{IV}})$ is bounded. Letting $f\equiv 1$, we obtain

$$[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\beta}|\psi(z,\zeta)| = [(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\beta}|({}_{\psi}C_{\phi}f)(z,\zeta)|<\infty.$$

This shows that $\psi\in\mathcal{A}_{(\beta,k)}$. Consider now a sequence $(\omega^i,v^i)=\phi(z^i,\zeta^i)$ in $\mathrm{GHE_{IV}}$, such that $\phi(z^i,\zeta^i)\to\partial\mathrm{GHE_{IV}}$ as $i\to\infty$. If such a sequence does not exist, then condition (22) obviously holds.

Moreover, let us introduce a sequence of test functions $\{f_i\}_{i\geq 1}$:

$$f_i(z,\zeta) := \frac{[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1+\alpha}}{[(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}]^{2\alpha-1}}, \quad i=1,2,\cdots.$$

Differentiation gives

$$\begin{aligned}
\frac{\partial f_i}{\partial z_l} &= \frac{(2\alpha-1)k[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1+\alpha}}{[(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}]^{2\alpha}} \\
&\quad \times(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^{k-1}(2\overline{\omega^i_l}'-2z_l\overline{\omega^i\omega^{i'}}), \quad l=1,\cdots,n. \\
\frac{\partial f_i}{\partial \zeta_j} &= \frac{(2\alpha-1)q_j\zeta_j^{q_j-1}\overline{t_j}^{q_j}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1+\alpha}}{[(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}]^{2\alpha}}, \\
&\quad j=1,\cdots,r, \quad i=1,2,\cdots.
\end{aligned}$$

It follows that there exists a constant $C_7>0$, such that

$$\begin{aligned}
&[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\alpha}|\nabla f_i(z,\zeta)| \\
&= \frac{(2\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\alpha}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1+\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}|^{2\alpha}} \\
&\times\left\{k^2|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^{k-1}|^2\times\sum_{l=1}^n|2\overline{\omega^i_l}'-2z_l\overline{\omega^i\omega^{i'}}|^2+\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}^{q_j}|^2\right\}^{\frac{1}{2}} \\
&\leq \frac{(2\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\alpha}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}|^{2\alpha}} \\
&\times\left\{k|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')|^{k-1}\times[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1}\right. \\
&\left.\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|+\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}'^{q_j}|^2\right]^{\frac{1}{2}}\times[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1}\right\} \\
&\leq \frac{(2\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_{\varphi}^2]^{\alpha}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_{\varphi}|^{2\alpha}} \\
&\times\left\{k|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')|^{k-1}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k]^{\frac{1}{k}-1}\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|\right. \\
&\left.+\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}'^{q_j}|^2\right]^{\frac{1}{2}}\times[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_{\varphi}^2]^{\frac{1}{k}-1}\right\}.
\end{aligned}$$

By the elementary inequality $\frac{a+b}{2} \geq \sqrt{ab}$ and Lemma 9, we have

$$
\begin{aligned}
&\frac{(2\alpha-1)[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^\alpha[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^\alpha}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|^{2\alpha}} \\
&\times\left\{k|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')|^{k-1}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k]^{\frac{1}{k}-1}\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|\right.\\
&\left.+\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}'^{q_j}|^2\right]^{\frac{1}{2}}\times[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1}\right\}\\
&\leq\frac{(2\alpha-1)\left\{\frac{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]+[(1+|\omega\omega'|^2-2|\omega|^2)^k-\|v\|_\varphi^2]}{2}\right\}^{2\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|^{2\alpha}}\\
&\times\left\{k|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')|^{k-1}(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^{1-k}\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|\right.\\
&\left.+\left[\sum_{j=1}^r|q_j\zeta_j^{q_j-1}\overline{v_j}'^{q_j}|^2\right]^{\frac{1}{2}}\times[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1}\right\}\\
&\leq(2\alpha-1)\frac{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^{2\alpha}}{||(1+zz'\overline{\omega\omega'}-2z\overline{\omega}')^k|-\|\zeta\|_\varphi\|v\|_\varphi|^{2\alpha}}\\
&\times\left\{k\cdot4^{(1-k)}\times|2\overline{\omega^i}'-2z\overline{\omega^i\omega^{i'}}|+C_7\right\}\\
&\leq(2\alpha-1)\times\left\{k\cdot4^{(1-k)}\times(|2\overline{\omega^i}'|+|2z\overline{\omega^i\omega^{i'}}|)+C_7\right\}\\
&\leq C''.
\end{aligned}
$$

This shows that $f_i \in \mathcal{B}^{(\alpha,k)}(\mathrm{GHE_{IV}}), i=1,2,\cdots$ and

$$
\begin{aligned}
|f_i(z,\zeta)|&=\frac{[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k-\langle\zeta,v^i\rangle_\varphi|^{2\alpha-1}}\\
&\leq\frac{[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{||(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-|\langle\zeta,v^i\rangle_\varphi||^{2\alpha-1}}\\
&\leq\frac{[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{|(1+zz'\overline{\omega^i\omega^{i'}}-2z\overline{\omega^i}')^k|-\|\zeta\|_\varphi\|v^i\|_\varphi|^{2\alpha-1}}\\
&\leq\frac{2^{2\alpha-1}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2+(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{2\alpha-1}}\\
&\leq\frac{2^{2\alpha-1}[(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{[(1+|zz'|^2-2|z|^2)^k-\|\zeta\|_\varphi^2]^{2\alpha-1}}.
\end{aligned}
$$

Taking $i\to\infty$, we have $(\omega^i,v^i)\to\partial\mathrm{GHE_{IV}}$. This implies that $(1+|\omega^i\omega^{i'}|^2-2|\omega^i|^2)^k-\|v^i\|_\varphi^2\to0$. If E is a compact subset of $\mathrm{GHE_{IV}}$, for $(z,\zeta)\in E$, we have that $(1+|zz'|^2-$

$2|z|^2)^k - \|\zeta\|_\varphi^2$ has a positive lower bound. Thus, we have $f_i(z,\zeta) \rightrightarrows 0,\ i \to \infty$ on all compact subsets of GHE_{IV}. According to Lemma 11, we have that $\|_\psi C_\phi f_i\|_{\mathcal{A}_{(\beta,k)}} \to 0$. Hence,

$$
\begin{aligned}
0 \leftarrow & \|_\psi C_\phi f_i\|_{\mathcal{A}_{(\beta,k)}} \\
& = \sup_{\phi(z,\zeta)\in GHE_{IV}} [(1+|zz'|^2 - 2|z|^2)^k - \|\zeta\|_\varphi^2]^\beta |\psi(z,\zeta)| \\
& \times \frac{[(1+|\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{|(1+z_\phi z'_\phi \overline{\omega^i\omega^{i'}} - 2z_\phi \overline{\omega^i}')^k - \langle \zeta_\phi, v^i\rangle_\varphi|^{2\alpha-1}} \\
& \geq \psi(z^i,\zeta^i)[(1+|z^i z^{i'}|^2 - 2|z^i|^2)^k - \|\zeta^i\|_\varphi^2]^\beta \\
& \times \frac{[(1+|\omega^i\omega^{i'}|^2 - 2|\omega^i|^2)^k - \|v^i\|_\varphi^2]^{\frac{1}{k}-1+\alpha}}{|(1+z^i_\phi z^{i'}_\phi \overline{\omega^i\omega^{i'}} - 2z^i_\phi \overline{\omega^i}')^k - \langle \zeta_\phi, v^i\rangle_\varphi|^{2\alpha-1}} \\
& = [(1+|z^i z^{i'}|^2 - 2|z^i|^2)^k - \|\zeta^i\|_\varphi^2]^\beta \frac{\psi(z^i,\zeta^i)}{[(1+|z^i_\phi z^{i'}_\phi|^2 - 2|z^i_\phi|^2)^k - \|\zeta^i_\phi\|_\varphi^2]^{\alpha-\frac{1}{k}}}.
\end{aligned}
$$

□

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