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Pathway to Fractional Integrals, Fractional Differential Equations, and Role of the H-Function

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Abstract: In this paper, the pathway model for the real scalar variable case is re-explored and its connections to fractional integrals, solutions of fractional differential equations, Tsallis statistics and superstatistics in statistical mechanics, the reaction-rate probability integral, Krätzel transform, pathway transform, etc., are explored. It is shown that the common thread in these connections is their H-function representations. The pathway parameter is shown to be connected to the fractional order in fractional integrals and fractional differential equations.

Keywords: pathway model; fractional integral; fractional differential equations; reaction-rate probability integral; Krätzel integral; pathway transform

MSC: 6A33; 26B15; 33C99; 33E12; 49K05; 62E15; 94A15

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1. Introduction

The pathway model was first introduced for the real rectangular matrix-variate case [1], then extended to cover the complex rectangular matrix-variate case in [2]. If we consider a real scalar (1×1 matrix) variable, then the model in [1] reduces to the following model:

$$g_1(x) = C_1|x-\mu|^{\gamma}[1-b(1-\alpha)|x-\mu|^{\delta}]^{\frac{\rho}{1-\alpha}}, \alpha < 1 \quad (1)$$

for $b > 0, \alpha < 1, \gamma > -1, \delta > 0, \rho > 0, 1-b(1-\alpha)|x-\mu|^{\delta} > 0$, and zero elsewhere, where μ is a location parameter and C_1 is the normalizing constant, g_1 is a real-valued scalar function, and $g_1(x-\mu) = 0$ elsewhere. If $\alpha > 1$, then $(1-\alpha) = -(\alpha-1), \alpha > 1$, and the model in (1) changes to the following model:

$$g_2(x) = C_2|x-\mu|^{\gamma}[1+b(\alpha-1)|x-\mu|^{\delta}]^{-\frac{\rho}{\alpha-1}}, \alpha > 1 \quad (2)$$

for $-\infty < x-\mu < \infty, b > 0, \alpha > 1, \delta > 0, \gamma > -1, \rho > 0$. Note that the models in (1) and (2) belong to the beta family of functions. In (1), we have an extended and power-transformed real scalar type-1 beta model, whereas in (2) we have an extended power-transformed real scalar type-2 beta model. When $\alpha \to 1_-$ in (1) and $\alpha \to 1_+$ in (2), the models in (1) and (2) will tend to the following model:

$$g_3(x-\mu) = C_3|x-\mu|^{\gamma}\mathrm{e}^{-b\rho|x-\mu|^{\delta}} \quad (3)$$

for $b > 0, \rho > 0, \delta > 0, \gamma > -1, -\infty < x < \infty$, which is a generalized gamma density. The pathway idea is that through the pathway parameter α it is possible to reach g_1, g_2 and g_3, as well as the transitional stages from g_1 to g_3 and g_2 to g_3. This idea proves very useful in model building situations. Whatever be the nature of the data, as long as they are from the wider families of generalized type-1 or type-2 beta families of functions or generalized gamma family of functions, then one member of the pathway family will be a good fit to the data. It is also possible to provide the following interpretation in a physical situation: for example, a Gaussian model may describe the stable situation, but may not capture the data or fit well in a physical situation, as the physical situation may be somewhere in the unstable neighborhood. The pathway model can capture both the stable unstable neighborhoods through the pathway parameter. If g_1, g_2 and g_3 are treated as statistical densities, then the normalizing constants are as follows:

$$C_1 = \delta[b(1-\alpha)]^{\frac{\gamma+1}{\delta}} \frac{\Gamma(\frac{\rho}{1-\alpha}+1+\frac{\gamma+1}{\delta})}{2\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{\rho}{1-\alpha}+1)} \tag{4}$$

$$\text{for } \gamma > -1, b > 0, \alpha < 1, \rho > 0, \delta > 0$$

$$C_2 = \delta[b(\alpha-1)]^{\frac{\gamma+1}{\delta}} \frac{\Gamma(\frac{\rho}{\alpha-1})}{2\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{\rho}{\alpha-1}-\frac{\gamma+1}{\delta})} \tag{5}$$

$$\text{for } \rho > 0, \delta > 0, b > 0, \alpha > 1, \gamma > -1, \frac{\rho}{\alpha-1} - \frac{\gamma+1}{\delta} > 0$$

$$C_3 = \frac{\delta(b\rho)^{\frac{\gamma+1}{\delta}}}{2\Gamma(\frac{\gamma+1}{\delta})}, b > 0, \rho > 0, \delta > 0. \tag{6}$$

Note that Tsallis statistics in non-extensive statistical mechanics ([3]) are available from (1), (2), and (3) for $\gamma = 0, b = 1, \delta = 1, \rho = 1, \alpha < 1, \alpha > 1, \alpha \to 1$. Superstatistics in statistical mechanics ([4]) are available from (2) and (3) for $b = 1, \delta = 1, \rho = 1$ and for $\alpha > 1, \alpha \to 1$.

Note that (3) for $\gamma = 0, \delta = 2$ is the Gaussian density. For $x > 0, \mu = 0$, (3) produces the generalized gamma density, Weibull density, gamma density, chi-square density, exponential density, Rayleigh density, Maxwell–Boltzmann density, etc. An exponentiation in (2) produces the generalized logistic density of [5] by transforming $x \to y$ through $x = \mathrm{e}^{-cy}, c > 0$. The model in (2) can also produce the Cauchy density, Student's t-density, and F-density; as limiting forms after exponentiation, it is also possible to obtain the Fermi–Dirac density from (2) and the Bose–Einstein density from (1).

The notations used in this paper are as follows: real scalar variables, whether mathematical or random, are denoted by lowercase letters such as x, y. Vector ($1 \times p$ or $p \times 1$, $p > 1$ matrix)/matrix variables, whether mathematical or random, are denoted by capital letters such as X, Y. Let $Y = (y_{jk})$ be a $p \times q$ matrix where the y_{jk}s are distinct real scalar variables; then, the wedge product of differentials is denoted by $\mathrm{d}Y = \wedge_{j=1}^{p} \wedge_{k=1}^{q} \mathrm{d}y_{jk} = \mathrm{d}Y'$, where Y' means the transpose of Y. For two real scalar variables u and v, the wedge product of the differentials is defined as $\mathrm{d}u \wedge \mathrm{d}v = -\mathrm{d}v \wedge \mathrm{d}u$ such that $\mathrm{d}u \wedge \mathrm{d}u = 0, \mathrm{d}v \wedge \mathrm{d}v = 0$. For a real-valued scalar function $f(X)$, $\int_X f(X)\mathrm{d}X$ means the integral over X. For a $p \times p$ matrix Y, we denote the determinant of Y by $|Y|$ or $\det(Y)$. Finally, $\mathrm{tr}(Y)$ denotes the trace of Y when Y is a square matrix. Other notations are explained whenever they occur for the first time.

The rest of this paper is organized as follows: Section 2 deals with the derivation of densities through entropy optimization. Here, the materials dealing with vectors and matrices were made available in the lectures of the first author. In Section 3, a connection is established between the pathway model and fractional integrals. Section 4 covers Mellin convolution of products and ratios along with the connection to statistical distribution theory. In Section 5, a connection is established between the pathway parameter and fractional order index. This material is believed to be new. Section 6 establishes the connection to fractional order index and the coefficient of the complex variable in the Mellin–Barnes representation of the H-function. This material is believed to be new as well. Section 7 provides results on fractional differential equations; some of the materials on the H-function representation of these solutions are likely to be new. The aim of the various sections is to derive statistical distributions known as Mathai's pathway models through optimization of Mathai entropy, then to establish connections among the pathway parameters in statistical distribution theory, fractional order index in fractional calculus, and coefficient of the complex variable s in the Mellin–Barnes integral representation of the H-function, ultimately showing that the H-function can be taken as a common thread between these different topics.

2. Optimization of Entropy

In (1) to (3), we have the distributional pathway or pathway through densities. We can also provide an entropic pathway by deriving (1), (2), and (3) through optimization of an entropy measure. For a real scalar variable x with density function $f(x)$, the Shannon entropy, denoted by $S(f)$, is defined as follows:

$$S(f) = -K\int_x f(x)\ln f(x)\mathrm{d}x \tag{7}$$

where K is a constant and x is a real scalar variable. An α-generalized entropy, known in the literature as Havrda–Charvat entropy, is as follows, denoted by $H(f)$:

$$H_\alpha(f) = \frac{\int_x [f(x)]^\alpha \mathrm{d}x - 1}{2^{1-\alpha} - 1}, \alpha \neq 1. \tag{8}$$

A modified $H_\alpha(f)$ is Tsallis' entropy $T_q(f)$, defined as follows:

$$T_q(f) = \frac{\int_x [f(x)]^q \mathrm{d}x - 1}{1-q}, q \neq 1. \tag{9}$$

In the limit when $\alpha \to 1$, $H_\alpha(f)$ reduces to $S(f)$. In the limit when $q \to 1$, $T_q(f)$ also reduces to $S(f)$. Here, α is a parameter; hence, $H_\alpha(f)$ is called the α-generalized entropy. Evidently, $T_q(f)$ is a q-generalized Shannon entropy $S(f)$. In (8) and (9), $f(x)$ is the density of a real scalar variable. Mathai defined a general statistical density as a real-valued scalar function $g(X)$ such that $g(X) \geq 0$ in the domain of X and $\int_X g(X)\mathrm{d}X = 1$; the differential element $\mathrm{d}X$ is defined in Section 1, where X can be scalar, vector, matrix, or a sequence of matrices in the real or complex domain. With this general definition for a density $g(X)$, Mathai's entropy is defined as follows, denoted by $M_\alpha(g)$ (for an earlier version, see [6]):

$$M_\alpha(g) = \frac{\int_X [g(X)]^{\frac{1-\alpha+\rho}{\rho}} \mathrm{d}X - 1}{\alpha - 1}, \rho > 0, \alpha < 1 + \rho. \tag{10}$$

It can be observed that (10) can also be treated as an expected value of $[g(X)]^{\frac{1-\alpha}{\rho}}$ and that when $\rho = 1$ it is Kerridge's measure of inaccuracy; see [7]. In order to derive the scalar version of the pathway density, we can optimize (10) when X is a real scalar quantity x, under the following restrictions:

$$\int_x x^{\gamma(\frac{1-\alpha}{\rho})} g(x)\mathrm{d}x = \text{ given, and } \int_x x^{\gamma(\frac{1-\alpha}{\rho})+\delta} g(x)\mathrm{d}x = \text{ given} \tag{11}$$

for the case $\gamma > -1, \alpha < 1, \delta > 0$. When the integrals in (11) exist, if we are using calculus of variation for optimization, then the Euler equation

$$\frac{\partial}{\partial g}[g^{\frac{1-\alpha+\rho}{\rho}} - \lambda_1 x^{\gamma(\frac{1-\alpha}{\rho})} g + \lambda_2 x^{\gamma(\frac{1-\alpha}{\rho})+\delta} g] = 0$$

leads to

$$g_1(x) = \lambda_3 x^\gamma (1 - \lambda_4 x^\delta)^{\frac{\rho}{1-\alpha}} \tag{12},$$

where λ_1 and λ_2 are the Lagrangian multipliers and λ_3 and λ_4 are some constants; here, λ_3 can act as the normalizing constant if g_1 is a statistical density. For $\lambda_4 = b(1-\alpha), \alpha < 1, b > 0$, we have the pathway model g_1 in (1). The entropy in (10) is also a modified version of Havrda–Charvat α-generalized entropy; see [7] in the real scalar case.

Optimization of (9) under the restrictions in (11) for $\gamma = 0, \delta = 1$ in an associated escort density produces Tsallis statistics in (9). Direct optimization of (9) produces the q-exponential function. However, the pathway model is available directly from (10) under the restrictions in (11); this produces (12), from which Tsallis statistics and superstatistics are available as special cases. Superstatistics considerations can be explained in terms of statistical language as the construction of an unconditional density when the conditional density and marginal density belong to generalized gamma families of functions. Such a procedure can produce only the cases $\alpha > 1$ and $\alpha \to 1$, not $\alpha < 1$. However, optimization of (9) through an escort density can produce special cases of the pathway model for all of cases $\alpha < 1, \alpha > 1, \alpha \to 1$. This is the advantage of Tsallis statistics over superstatistics.

Now, let us consider the vector-variate case. In (10), let X be a $p \times 1$ vector and assume the following restrictions based on the moments of the ellipsoid of concentration, namely, $(X-\mu)'A(X-\mu) = c > 0$, where μ is the expected value of X and $A > O$ is $p \times p$ real positive definite, which is the inverse of the covariance matrix in X. Consider the following moments:

$$\int_X [(X-\mu)'A(X-\mu)]^{\gamma\frac{(\alpha-1)}{\rho}} f(X)\mathrm{d}X = \text{ given} \tag{13}$$

and

$$\int_X [(X-\mu)'A(X-\mu)]^{\gamma\frac{(\alpha-1)}{\rho}+\delta} f(X)\mathrm{d}X = \text{ given.} \tag{14}$$

Optimization of (10) under the restrictions in (13) and (14) leads to the following density:

$$h_1(X) = c_1[(X-\mu)]A(X-\mu)]^{\gamma}[1 + b(\alpha-1)\{(X-\mu)'A(X-\mu)\}^{\delta}]^{-\frac{\rho}{\alpha-1}}, \alpha > 1. \tag{15}$$

From here, we can move on to the densities for $\alpha < 1$ and $\alpha \to 1$. For $\alpha \to 1$, we have a multivariate version of the real scalar generalized gamma density, namely,

$$h_2(X) = c_2[(X-\mu)'A(X-\mu)]^{\gamma}\mathrm{e}^{-b\rho[(X-\mu)'A(X-\mu)]^{\delta}}. \tag{16}$$

We can make a connection from (15) and (16) to random points $p \leq n$ in Euclidean n-space. Evidently, for $\gamma = 0, \delta = 1$, (16) is the p-variate Gaussian density with mean value vector μ and covariance matrix A^{-1}.

We can extend this procedure to rectangular the matrix-variate case as well. In (10), let X be a $p \times q, p \leq q$ matrix of rank p in the real domain. Consider the following moment-type restrictions:

$$\int_X [\mathrm{tr}(AXBX')]^{\gamma\frac{(\alpha-1)}{\rho}} f(X)\mathrm{d}X = \text{ given} \tag{17}$$

and

$$\int_X [\mathrm{tr}(AXBX')]^{\gamma\frac{(\alpha-1)}{\rho}+\delta} f(X)\mathrm{d}X = \text{ given} \tag{18}$$

where $A > O$ is a $p \times p$ and $B > O$ a $q \times q$ constant real positive definite matrix, $q > 1$, $\rho > 0$, $\delta > 0$. Optimization of (10) under the restrictions in (17) and (18) leads to the density

$$h_3(X) = c_3[\mathrm{tr}(AXBX')]^{\gamma}[1 + b(\alpha-1)\{\mathrm{tr}(AXBX')\}^{\delta}]^{-\frac{\rho}{\alpha-1}}, \alpha > 1. \tag{19}$$

When $\alpha \to 1$, h_3 provides h_4, where

$$h_4(X) = c_4[\mathrm{tr}(AXBX')]^{\gamma}\mathrm{e}^{-b\rho[\mathrm{tr}(AXBX')]^{\delta}} \tag{20}$$

for $b > 0, \rho > 0, \delta > 0, \gamma > -1$. This h_4 is a rectangular matrix-variate generalized gamma density, from which we have the real matrix-variate Gaussian density for $\gamma = 0$ and $\delta = 1$ (see also [8]).

3. Connection of the Pathway Model to Fractional Integrals

From the geometrical interpretation of the fractional integral, it is evident that a type-1 beta form must be present in the definition of a fractional integral if we wish to encompass all the different definitions of fractional integrals in current use. Hence a definition, covering all fractional integrals in current use, introduced in [9], is provided as Mellin convolutions of products and ratios, where the functions involved in the Mellin convolutions are of the following form for the real scalar variable case $x_1 > 0, x_2 > 0$:

$$f_1(x_1) = \phi_1(x_1)[1 - a(1-q)x_1^{\delta}]^{\frac{1}{1-q}} \ \& \ f_2(x_2) = \phi_2(x_2)f(x_2) \tag{21}$$

where ϕ_1 and ϕ_2 are prefixed functions and f is an arbitrary function, $a > 0, \delta > 0, q < 1$. If statistical densities are needed, then f_1 and f_2 can be multiplied by appropriate normalizing constants. In this case, ϕ_1, ϕ_2, f are to be restricted to be positive functions and $1 - a(1-q)x^{\delta} > 0$. Mellin convolution of the products corresponds to fractional integrals of the second kind or the right-sided integrals, while Mellin convolution of ratios corresponds to fractional integrals of the first kind or left-sided integrals.

3.1. Fractional Integrals of the Second Kind

Let $u = x_1x_2, v = x_2$ or $x_2 = v, x_1 = \frac{u}{v}$ and let the Jacobian be $\frac{1}{v}$. Then, the Mellin convolution of the product, denoted by $g_2(u)$, is as follows, taking the pathway model of (21) as $f_1(x_1)$ with $\phi_1(x_1) = x_1^{\gamma}, \phi_2(x_2) = 1$ for $q < 1$:

$$\begin{aligned} g_2(u) &= \int_v \frac{1}{v}(\frac{u}{v})^{\gamma}[1 - a(1-q)(\frac{u}{v})^{\delta}]^{\frac{1}{1-q}} f(v)\mathrm{d}v, q < 1 \\ &= u^{\gamma}\int_v v^{-\gamma-1}[1 - a(1-q)(\frac{u^{\delta}}{v^{\delta}})]^{\frac{1}{1-q}} f(v)\mathrm{d}v. \end{aligned} \tag{22}$$

Let us consider the Mellin transform of $g_2(u)$ with Mellin parameter s. Then,

$$M_{g_2}(s) = \int_{u=0}^{\infty} u^{\gamma+s-1}\int_{v>[a(1-q)]^{\frac{1}{\delta}}u}[1 - a(1-q)\frac{u^{\delta}}{v^{\delta}}]^{\frac{1}{1-q}} f(v)\mathrm{d}v. \tag{23}$$

Let us first integrate out u; then, $0 \le u \le \frac{v}{[a(1-q)]^{\frac{1}{\delta}}}$. Integration over u provides the following:

$$\begin{aligned} &\int_{u=0}^{\frac{v}{[a(1-q)]^{\frac{1}{\delta}}}} u^{\gamma+s-1}[1 - a(1-q)\frac{u^{\delta}}{v^{\delta}}]^{\frac{1}{1-q}}\mathrm{d}u \\ &\quad = \frac{1}{\delta}\frac{v^{\gamma+s}}{[a(1-q)]^{\frac{\gamma+s}{\delta}}}\int_0^1 t^{\frac{\gamma+s}{\delta}-1}[1-t]^{\frac{1}{1-q}}\mathrm{d}t \\ &\quad = \frac{1}{\delta}\frac{\Gamma(\frac{1}{1-q}+1)}{[a(1-q)]^{\frac{\gamma+s}{\delta}}}\frac{\Gamma(\frac{\gamma}{\delta}+\frac{s}{\delta})}{\Gamma(\frac{1}{1-q}+\frac{\gamma}{\delta}+\frac{s}{\delta})} \end{aligned}$$

for $\Re(\gamma + s) > 0, \delta > 0, a > 0, q < 1$, where $\Re(\cdot)$ means the real part of $(\cdot)$. Now, taking the integral over v, we have the Mellin transform of the arbitrary function $f(v)$, denoted by $f^*(s)$. Therefore,

$$M_{g_2}(s) = \frac{\Gamma(\frac{1}{1-q}+1)}{\delta[a(1-q)]^{\frac{\gamma+s}{\delta}}}\frac{\Gamma(\frac{\gamma+s}{\delta})}{\Gamma(\frac{1}{1-q}+1+\frac{\gamma+s}{\delta})}f^*(s) \tag{24}$$

for $\Re(\gamma + s) > 0, a > 0, \delta > 0, q < 1$;

$$M_{g_2}(s) = \frac{\Gamma(\frac{1}{1-q}+1)}{\delta(1-q)^{\frac{s}{\delta}}}\frac{\Gamma(\frac{s}{\delta})}{\Gamma(\frac{1}{1-q}+1+\frac{s}{\delta})}f^*(s) \text{ for } a = 1, \gamma = 0; \tag{25}$$

$$M_{g_2}(s) = \frac{\Gamma(\frac{\gamma+s}{\delta})}{\delta a^{\frac{\gamma+s}{\delta}}} f^*(s) \text{ for } \delta > 0, a > 0, q \to 1. \tag{26}$$

Now, we can compare (24) with Theorem 3.4.2 of the Mellin transform of an Erdélyi–Kober fractional integral of the second kind for $\delta = 1$ (see [9]). It has the same structurem and $\frac{1}{1-q} + 1$ corresponds to the order of the fractional integral α. Comparing (25) with the Mellin transform of the Weyl fractional integral of the second kind for $\delta = 1$ ([9]), the fractional order α again corresponds to $\frac{1}{1-q} + 1$.

3.2. *Fractional Integral of the First Kind or Left-Sided Integral*

Let $u = \frac{x_2}{x_1}, v = x_2$ or $x_2 = v, x_1 = \frac{v}{u}$, with the Jacobian as $-\frac{v}{u^2}$. Taking $\phi_1(x_1) = x_1^{\gamma-1}, \phi_2(x_2) = 1$ and denoting the Mellin convolution of the ratio as $g_1(u)$, we have the following:

$$\begin{aligned} g_1(u) &= \int_v \frac{v}{u^2} (\frac{v}{u})^{\gamma-1} [1 - a(1-q)(\frac{v}{u})^{\delta}]^{\frac{1}{1-q}} f(v) \mathrm{d}v \\ &= u^{-\gamma-1-\frac{\delta}{1-q}} \int_v v^{\gamma} [u^{\delta} - a(1-q)v^{\delta}]^{\frac{1}{1-q}} f(v) \mathrm{d}v. \end{aligned} \tag{27}$$

The Mellin transform of $g_1(u)$ with Mellin parameter s is as follows:

$$M_{g_1}(s) = \int_{u=0}^{\infty} u^{-\gamma-1+s-1-\frac{\delta}{1-q}} \int_v v^{\gamma} [u^{\delta} - a(1-q)v^{\delta}]^{\frac{1}{1-q}} f(v) \mathrm{d}v.$$

First integrating out u, we have the integral over u as follows:

$$\begin{aligned} &\int_{u=[a(1-q)]^{\frac{1}{\delta}} v}^{\infty} [u^{\delta} - a(1-q)v^{\delta}]^{\frac{1}{1-q}} u^{-\gamma-1+s-1-\frac{\delta}{1-q}} \mathrm{d}u \\ &\qquad = \frac{1}{\delta} \int_{z=0}^{\infty} z^{\frac{1}{1-q}} [z + a(1-q)v^{\delta}]^{-\frac{1}{\delta}(\gamma+1+\frac{\delta}{1-q}-s)-1} \mathrm{d}z. \end{aligned}$$

This can be evaluated with the help of a type-2 beta integral. Then, integration over v provides the following final result:

$$M_{g_1}(s) = \frac{\Gamma(\frac{1}{1-q}+1)}{[a(1-q)]^{\frac{\gamma+1-s}{\delta}}} \frac{\Gamma(\frac{\gamma+1-s}{\delta})}{\Gamma(\frac{1}{1-q}+1+\frac{\gamma+1-s}{\delta})} f^*(s), \Re(\gamma+1-s) > 0. \tag{28}$$

The fractional order α corresponds to $\frac{1}{1-q} + 1$, as seen before. The pathway parameter for $q < 1$ is such that

$$\frac{1}{1-q} + 1 = \alpha, q < 1,$$

where α is the fractional order in the fractional integrals of the first and second kinds.

4. Mellin Convolutions of Products and Ratios for Other Functions

When f_1 is connected to the pathway model for $q < 1$ and f_2 is an arbitrary function, then the Mellin convolutions of products and ratios are seen to be connected to fractional integrals of the second and first kinds, respectively. Let f_1 and f_2 be generalized gamma functions which are actually the pathway model for $q \to 1$. Take (1) for $q \to 1_-$ and (2) for $q \to 1_+$. Let us observe what happens to the Mellin convolutions of products and ratios. Let $u = x_1 x_2, v = x_1, x_1 = \frac{u}{v}$, with the Jacobian as $\frac{1}{v}$. Let

$$f_1(x_1) = C_1 x^{\gamma_1 - 1} \mathrm{e}^{-a_1 x_1^{\delta_1}} \;\; \& \;\; f_2(x_2) = C_2 x_2^{\gamma_2 - 1} \mathrm{e}^{-a_2 x_2^{\delta_2}} \tag{29}$$

for $\gamma_j > 0, a_j > 0, \delta_j > 0, j = 1, 2$, where C_1 and C_2 can be normalizing constants if $f_j(x_j), j = 1, 2$ are to be treated as statistical densities. Let the Mellin convolution of a product again be denoted by $g_2(u)$. Then,

$$g_2(u) = C_1C_2 \int_v \frac{1}{v}(\frac{u}{v})^{\gamma_1-1} v^{\gamma_2-1} e^{-a_1(\frac{u}{v})^{\delta_1} - a_2 v^{\delta_2}}$$
$$= C_1C_2 u^{\gamma_1-1} \int_{v=0}^{\infty} v^{\gamma_2-\gamma_1-1} e^{-\frac{a_1 u^{\delta_1}}{v^{\delta_1}} - a_2 v^{\delta_2}} dv. \quad (30)$$

For $\delta_1 = 1, \delta_2 = 1$, (30) provides the basic Krätzel integral, which is associated with the Krätzel transform [10]; see [9] for some properties of the generalized Krätzel integral and its connection to statistical distribution theory and the reaction rate probability integral in nuclear reaction rate theory [11]. For $\delta_2 = 1, \delta_1 = \frac{1}{2}$, (30) provides the reaction rate probability integral. For $\delta_1 = 1, \delta_2 = 1$ and $\gamma_2 - \gamma_1 = -\frac{1}{2}$, the integrand in (30) provides the inverse Gaussian density in stochastic processes. The structure in (30) is also that of the unconditional density in a Bayesian setup, with the conditional density belonging to the generalized gamma family of type $b_1 x_1^{\gamma_1} e^{-a x_1^{\delta_1}/x_2^{\delta_1}}$ and the marginal density also belonging to generalized gamma family of form $x_2^{\gamma_2} e^{-b x_2^{\delta_2}}$. Then, the unconditional density has the form of the integral in (30). Observe that the generalized gamma functions we considered in (30) are nothing but the limiting forms of the pathway models in (1) and (2) for $q \to 1$ or the model in (3).

5. Connection of the Pathway Parameter to Fractional Indices in Fractional Differential Equations

For illustrative purposes, let us consider the fractional space–time diffusion equation (see [8], where ${}_0D_t^{\beta}$ is a Riemann–Liouville fractional derivative of the first kind and is of left-sided order β; see also [12–20]:

$${}_0D_t^{\beta} N(x,t) = \eta \, {}_xD_{\theta}^{\alpha} N(x,t) \quad (31)$$

with the initial condition ${}_0D_x{}^{\beta-1} N(x,0) = \sigma(x), 0 \le \beta \le 1, \lim_{x\to\pm\infty} N(x,t) = 0$, and where η is the diffusion constant. Here, $\eta, t > 0, x \in R, \alpha, \theta, \beta$ are real parameters, $0 < \alpha \le 2, |\theta| \le \min\{\alpha, 2-\alpha\}$. The solution of (5.1) for the case $\alpha = \beta$ is available from [9, 21] in the following form for $\rho = \frac{\alpha-\theta}{2\alpha}$:

$$N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t\eta^{\frac{1}{\alpha}}}\Big|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(\alpha,1),(1,\rho)}\right] \quad (32)$$
$$= \frac{t^{\alpha-1}}{\alpha|x|}\frac{1}{2\pi i}\int_L \frac{\Gamma(1+\frac{s}{\alpha})\Gamma(1+s)\Gamma(-\frac{s}{\alpha})}{\Gamma(1-\rho s)\Gamma(\alpha+s)\Gamma(1+\rho s)}\left[\frac{|x|}{t\eta^{\frac{1}{\alpha}}}\right]^{-s} ds,$$

where L is an appropriate contour and $i = \sqrt{-1}$. Let

$$g(x,t) = \frac{\alpha|x|}{t^{\beta-1}} N(x,t)$$
$$= \frac{1}{2\pi i}\int_L \frac{\Gamma(1+\frac{s}{\alpha})\Gamma(-\frac{s}{\alpha})}{\Gamma(-\rho s)\Gamma(1-\rho+\rho s)}\frac{\Gamma(1+s)}{\Gamma(\alpha+s)}\left[\frac{|x|}{t\eta^{\frac{1}{\alpha}}}\right]^{-s} ds. \quad (33)$$

Therefore, the Mellin transform of $g(x,t)$ with Mellin parameter s and argument $\frac{|x|}{t\eta^{\frac{1}{\alpha}}}$ is provided by the following:

$$M_g(s) = \frac{\Gamma(1+s)}{\Gamma(\alpha+s)} f^*(s), \quad f^*(s) = \frac{\Gamma(1+\frac{s}{\alpha})\Gamma(-\frac{s}{\alpha})}{\Gamma(-\rho s)\Gamma(1+\rho s)}. \quad (34)$$

Now, compare (34) with (25) for $\delta = 1$. The pathway integrals of the second kind for $q < 1$ are of the form $c\frac{\Gamma(1+s)}{\Gamma(\alpha+s)} f^*(s)$, where c is a constant, $\alpha = \frac{1}{1-q} + 1$ is the order of the fractional integral of the second kind, and α is also the order of the fractional differential equation. The exception is that here (34) is the Mellin transform of the solution of the fractional space–time diffusion equation for the case $\alpha = \beta$, where α is the fractional index in the fractional differential equation. This is the same α appearing in (34), which is also equal to $\frac{1}{1-q} + 1 = \alpha$. Moreover, this is the same α appearing as the coefficient of s in the H-function representation, the coefficients being $\frac{1}{\alpha}$ and ρ, which is a function of α. Note that in (34) $f^*(s)$ has an interesting structure; both the numerator and denominator are of the form $\Gamma(z)\Gamma(1-z)$ for different z, hence, the gamma product can be written in terms of $\sin \pi z$. For further reading on fractional diffusion and the corresponding fractional differential equations, see [22–26].

6. The H-Function Thread

We have seen already that the pathway model, fractional integrals of the first and second kinds, and fractional differential equation are all connected through the pathway model and that the basic representation takes place in terms of the H-function. Solutions of simple fractional differential equations are available in terms of Mittag–Leffler functions. Mittag–Leffler functions are special cases of the H-function. For an overview of Mittag–Leffler functions and their properties, see [9]. Mittag–Leffler functions as solutions of fractional differential equations may be seen in [27,28]. For example, a three-parameter Mittag–Leffler function has the H-function representation

$$\begin{aligned} E^{\gamma}_{\alpha,\beta}(z) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-\alpha s)} z^{-s} \mathrm{d}s \\ &= \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}\left[z \Big|^{(1,1)}_{(0,1),(1-\beta,\alpha)} \right] \end{aligned} \quad (35)$$

for $i = \sqrt{-1}, 0 < c < \gamma, \alpha > 0, \beta > 0$. Note that the basic Mittag–Leffler parameter α enters into the H-function as the coefficient of the complex variable s. In all the H-functional representations of fractional integrals and solutions of fractional differential equations, the fractional index or fractional order α (or a function of α) enters the picture as the coefficient of the complex variable s in the H-function representation, as illustrated in Section 5 and detailed in the discussion at the end of Section 5.

7. Diffusion Equation

We consider the following diffusion model with fractional-order spatial and temporal derivatives:

$$ {}_0D_t^{\beta} N(x,t) = \eta \; {}_xD_\theta^{\alpha} N(x,t) \quad (36)$$

with the initial conditions ${}_0D_t^{\beta-1} N(x,0) = \sigma(x), 0 \le \beta \le 1, \lim_{x\to\pm\infty} N(x,t) = 0$, where η is a diffusion constant, $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints

$$0 < \alpha \le 2, |\theta| \min(\alpha, 2-\alpha),$$

and $\delta(x)$ is the Dirac delta function. Then, for the fundamental solution of (7.1) with initial conditions, the following formula holds [9]:

$$N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H^{2,1}_{3,3}\left[\frac{|x|}{(\eta t^{\beta})^{1/\alpha}} \Big|^{(1,1/\alpha),(\beta,\beta/\alpha),(1,\rho)}_{(1,1/\alpha),(1,1),(1,\rho)} \right], \alpha > 0 \quad (37)$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$. The following special cases of (1) are of special interest for fractional diffusion models:

(i) For $\alpha = \beta$, the corresponding solution of (36), denoted by N^{θ}_{α}, can be expressed in terms of the H-function as provided below, and can be defined for $x > 0$ as follows:

Non-diffusion: $0 < \alpha = \beta < 2; \theta \le \min\{\alpha, 2-\alpha\}$,

$$N_\alpha^\theta(x) = \frac{t^{\alpha-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{t\eta^{1/\alpha}}\left|\begin{matrix}(1,1/\alpha),(\alpha,1),(1,\rho)\\(1,1/\alpha),(1,1),(1,\rho)\end{matrix}\right.\right], \ \rho = \frac{\alpha-\theta}{2\alpha}. \quad (38)$$

(ii) When $\beta = 1, 0 < \alpha \le 2; \theta \le \min\{\alpha, 2-\alpha\}$, then (36) reduces to the space-fractional diffusion equation, which is the fundamental solution of the following space–time fractional diffusion model:

$$\frac{\partial N(x,t)}{\partial t} = \eta\ {}_xD_\theta^\alpha N(x,t), \eta > 0, x \in R \quad (39)$$

with the initial conditions $N(x, t = 0) = \sigma(x)$, $\lim_{x\to\pm\infty} N(x,t) = 0$, where η is a diffusion constant and $\sigma(x)$ is the Dirac delta function. Hence, for the solution of (1), the following formula holds:

$$N_\alpha^\theta(x) = \frac{1}{\alpha(\eta t)^{1/\alpha}}\ H_{2,2}^{1,1}\left[\frac{(\eta t)^{1/\alpha}}{|x|}\left|\begin{matrix}(1,1),(\rho,\rho)\\(\frac{1}{\alpha},\frac{1}{\alpha}),(\rho,\rho)\end{matrix}\right.\right], \ 0 < \alpha < 1, |\theta| \le \alpha \quad (40)$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$. The density represented by the above expression is known as the α-stable Lévy density; see [29,30]. Another form of this density is provided by

$$N_\alpha^\theta(x) = \frac{1}{\alpha(\eta t)^{1/\alpha}}\ H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{1/\alpha}}\left|\begin{matrix}(1-\frac{1}{\alpha},\frac{1}{\alpha}),(1-\rho,\rho)\\(0,1),(1-\rho,\rho)\end{matrix}\right.\right], 1 < \alpha < 2, |\theta| \le 2-\alpha. \quad (41)$$

(iii) Next, if we take $\alpha = 2, 0 < \beta < 2; \theta = 0$, then we obtain the time-fractional diffusion, which is governed by the following time-fractional diffusion model:

$$\frac{\partial^\beta N(x,t)}{\partial t^\beta} = \eta\frac{\partial^2}{\partial x^2}N(x,t), \eta > 0, x \in R,\ 0 < \beta \le 2 \quad (42)$$

with the initial conditions ${}_0D_t^{\beta-1}N(x,0) = \sigma(x), {}_0D_t^{\beta-2}N(x,0) = 0$, for $x \in r$, $\lim_{x\to\pm\infty} N(x,t) = 0$, where η is a diffusion constant and $\sigma(x)$ is the Dirac delta function, the fundamental solution of which is provided by the equation

$$N(x,t) = \frac{t^{\beta-1}}{2|x|}\ H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t^\beta)^{1/2}}\left|\begin{matrix}(\beta,\beta/2)\\(1,1)\end{matrix}\right.\right]. \quad (43)$$

(iv) If we set $\alpha = 2, \beta = 1$ and $\theta \to 0$, then for the fundamental solution of the standard diffusion equation

$$\frac{\partial}{\partial t}N(x,t) = \eta\frac{\partial^2}{\partial x^2}N(x,t) \quad (44)$$

with initial condition

$$N(x, t = 0) = \sigma(x), \lim_{x\to\pm\infty} N(x,t) = 0, \quad (45)$$

the following formula holds:

$$N(x,t) = \frac{1}{2|x|}H_{1,1}^{1,0}\left[\frac{|x|}{\eta^{1/2}t^{1/2}}\left|\begin{matrix}(1,1/2)\\(1,1)\end{matrix}\right.\right] = (4\pi\eta t)^{-1/2}\exp[-\frac{|x|^2}{4\eta t}] \quad (46)$$

which is the classical Gaussian density.

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