

Article

# Extensions of Orders to a Power Set vs. Scores of Hesitant Fuzzy Elements: Points in Common of Two Parallel Theories

Esteban Induráin <sup>1,\*</sup>, Ana Munárriz <sup>2,†</sup> and M. Sergio Sara <sup>3,†</sup>

<sup>1</sup> InaMat2 (Institute for Advanced Materials and Mathematics), Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, 31006 Pamplona, Spain

<sup>2</sup> Inarbe (Institute for Advanced Research in Business and Economics), Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, 31006 Pamplona, Spain; ana.munarriz@unavarra.es

<sup>3</sup> Departamento de Estadística, Informática y Matemáticas, Universidad Pública de Navarra, 31006 Pamplona, Spain; sara.142587@e.unavarra.es

\* Correspondence: steiner@unavarra.es; Tel.: +34-948169551

† These authors contributed equally to this work.

**Abstract:** We deal with two apparently disparate theories. One of them studies extensions of orderings from a set to its power set. The other one defines suitable scores on hesitant fuzzy elements. We show that both theories have the same mathematical substrate. Thus, important possibility/impossibility results concerning criteria for extensions can be transferred to new results on scores. And conversely, conditions imposed a priori on scores can give rise to new extension criteria. This enhances and enriches both theories. We show examples of translations of classical results on extensions in the context of scores. Also, we state new results concerning the impossibility of finding a utility function representing some kind of extension order if some restrictions are imposed on the utility function considered as a score.

**Keywords:** extensions of orders from a set to its power set; criteria of extensions of orderings; possibility and impossibility results; hesitant fuzzy elements and sets; scores



**Citation:** Induráin, E.; Munárriz, A.; Sara, M.S. Extensions of Orders to a Power Set vs. Scores of Hesitant Fuzzy Elements: Points in Common of Two Parallel Theories. *Axioms* **2024**, *13*, 549. <https://doi.org/10.3390/axioms13080549>

Academic Editors: Ta-Chung Chu and Wei-Chang Yeh

Received: 5 June 2024

Revised: 5 August 2024

Accepted: 8 August 2024

Published: 13 August 2024



**Copyright:** © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

**MSC:** 03A72; 06A06; 91B02

## 1. Introduction

### 1.1. Motivation of the Manuscript

Sometimes it happens that two apparently disparate theories have a parallel development. In a sense, one could be considered a part of the other, even if they belong to totally different frameworks. When this occurs, the results of each theory can be applied to the other, thereby reinforcing both.

In the present manuscript, we compare two such parallel theories.

One theory involves extending total orders from a set to its power set, following some criteria established a priori. This had already been studied by people working in mathematical economics from 1980 on (see [1,2]) when dealing with the problem of ranking sets of objects. Depending on the chosen criteria, we may reach either possibility or impossibility results.

On the other hand, we have the theory of scores on hesitant fuzzy sets. In terms of extensions of orderings, we start with the usual order on the unit interval  $[0, 1]$  and, using scores (see [3]), we get a new ordering on the subsets of  $[0, 1]$  (also known as the hesitant fuzzy elements). Again, we seek to ensure that the scores accomplish some pre-defined criteria. Depending on these criteria, we may have a suitable score or not (possibility or impossibility, again).

At this stage, we notice that the second setting derives from the first.

In fact, some criteria used by mathematical economists may be unknown to fuzzy set theorists, and vice versa. This observation suggests the potential to broaden the scope of criteria and possibility/impossibility results.

### 1.2. Aim and Objectives

This work arises from our professional experiences as mathematicians working with two different collectives, namely economists and engineers, who sometimes, unfortunately, overlook each other's mathematical applications and achievements. These distinct achievements—economists have their own, and engineers have theirs—could actually be mutually beneficial and lead to future collaborations. This is especially true once we recognize and highlight that the mathematical foundation is the same, and the results obtained by economists could indeed be translated into the context used by engineers, and vice versa.

We aim to highlight the fact that problems in mathematical economics, such as extending orders from a set to its power set in the spirit of “ranking sets of objects”, and problems related to defining and analyzing scores of hesitant fuzzy elements encountered by mathematicians and engineers in fuzzy set theory and artificial intelligence approaches are, in a sense, equivalent. Furthermore, technical results obtained in one approach can often be translated into new results in the other approach, enriching both theories, which often ignore each other despite sharing the same mathematical substrate.

Specifically, we want to show the analogy between the study of extensions of orderings from a set to its power set and the definition of scores of hesitant fuzzy elements.

### 1.3. Contents of the Manuscript

Bearing these ideas in mind, the contents of the manuscript are as follows:

After the Introduction and the section on Preliminaries, we introduce, on the one hand, classical criteria to extend orderings from a set to its power set (Section 3) and analyze their compatibility (Section 4). On the other hand, we introduce different classes of scores for hesitant fuzzy sets (Section 5) and analyze compatibility among extra required features on scores. In Section 6, we explore the analogies between both approaches, showing that some (in)compatibility results in one setting could be translated in some way to the other setting. A final section of concluding remarks and lines for future research closes the paper.

## 2. Preliminaries

### 2.1. Extension of Orderings from a Set to Its Power Set

Let  $X$  denote a nonempty set.

**Definition 1.** A preorder  $\preceq$  on  $X$  is a binary relation on  $X$  that is reflexive and transitive. An antisymmetric preorder is said to be an order. A complete (or total) preorder  $\preceq$  on a set  $X$  is a preorder such that if  $a, b \in X$ , then  $(a \preceq b) \vee (b \preceq a)$  holds. If  $\preceq$  is a preorder on  $X$ , then as usual, we denote the associated asymmetric relation by  $\prec$  and the associated equivalence relation by  $\sim$ ; these are defined by  $a \prec b \Leftrightarrow (a \preceq b) \wedge \neg(b \preceq a)$  and  $a \sim b \Leftrightarrow (a \preceq b) \wedge (b \preceq a)$ . If this asymmetric relation is transitive, then  $\preceq$  is said to be a quasi-transitive preorder. For any preorder  $\preceq$  on  $X$ , the indifference part of  $\preceq$ , denoted by  $\text{ind}(\preceq)$  is the binary relation on  $X$  defined by  $(a, b) \in \text{ind}(\preceq)$  if and only if  $a \prec c \Leftrightarrow b \prec c$  as well as  $c \prec a \Leftrightarrow c \prec b$ , for any  $c \in X$ . A complete preorder  $\preceq$  defined on  $X$  is usually called a preference, and it is said to be representable if there exists a real-valued function  $u : X \rightarrow \mathbb{R}$  such that  $a \preceq b \Leftrightarrow u(a) \leq u(b)$  holds for every  $a, b \in X$ . Here  $u$  is said to be a utility function representing  $\preceq$ .

From now on, we will consider  $X$  endowed with a complete preorder  $\preceq$ . Sometimes we will work in the particular case in which  $\preceq$  is a total order.

**Definition 2.** An extension of the complete preorder  $\preceq$  from the set  $X$  to its power set  $\mathcal{P}(X)$  is another preorder  $\preceq_E$ , now defined in  $\mathcal{P}(X)$  such that the following property holds true for any elements  $a, b \in X$ :  $a \preceq b \Leftrightarrow \{a\} \preceq_E \{b\}$ .

**Remark 1.**

- (i) Among possible extensions, the literature pays special attention to the following situations: (a) Both  $\preceq$  and  $\preceq_E$  are total orders; (b) both  $\preceq$  and  $\preceq_E$  are complete preorders; (c)  $\preceq$  is a total order, but  $\preceq_E$  is a complete preorder.
- (ii) Some authors do not consider  $\emptyset$  in the extensions, so they define the extensions not on the whole power set  $\mathcal{P}(X)$  but instead on  $\pi(X) = \mathcal{P}(X) \setminus \{\emptyset\}$ .
- (iii) The most classical criteria appeared from 1950 on, mainly in several papers related to social choice and decision making (see, e.g., [1,4]). They usually consider a total order  $\preceq$  defined on a finite set  $X$ , whereas the extension  $\preceq_E$  can be either a total order or just a complete preorder depending on the context.

2.2. Hesitant Fuzzy Elements, Hesitant Fuzzy Sets and Scores

Now we present some nomenclature (see, e.g., [5]) to be used henceforward. By  $X$ , we will denote a nonempty set, also called the universe.

**Definition 3 ([6]).** A (type-1) fuzzy subset  $H$  of  $X$  is defined as a function  $\mu_H : X \rightarrow [0, 1]$ . Here  $\mu_H$  is called the membership function of  $H$ . If  $\mu_H$  takes values just in  $\{0, 1\}$ , the corresponding subset  $\mu_H$  is a classical crisp (i.e., non-fuzzy) subset of  $X$ .

Throughout the manuscript, the following notation will be used:

- $\mathcal{P}([0, 1])$  denotes the collection of all subsets of the unit interval  $[0, 1]$ ,
- $\Pi([0, 1])$  denotes the collection of all nonempty subsets of  $[0, 1]$ ,
- $\mathcal{I}([0, 1])$  constitutes the subset of all intervals in  $[0, 1]$ ,
- $\mathcal{I}_C([0, 1])$  consists of all closed intervals in  $[0, 1]$ ,
- $\mathcal{I}^\cup([0, 1])$  consist of all finite unions of intervals in  $[0, 1]$ ,
- $\mathcal{F}([0, 1])$  denotes the collection of all nonempty finite subsets of  $[0, 1]$ , and
- $\mathcal{F}_n([0, 1])$  denotes the collection of all nonempty subsets of  $[0, 1]$  with at most  $n$  elements.

If  $X, Y \subseteq \mathbb{R}$ , we write  $Y > X$  to mean  $y > x$  for all  $y \in Y, x \in X$ . Notice that  $Y > X$  implies  $X \cap Y = \emptyset$ .

The main objects to be handled in our analysis are called the hesitant fuzzy elements (HFEs for short):

**Definition 4 ([7,8]).** A subset  $E$  of  $[0, 1]$  is said to be a hesitant fuzzy element (HFE). Also, a function  $h : X \rightarrow \mathcal{P}([0, 1])$  is called a hesitant fuzzy set (HFS) over  $X$ .

**Remark 2.** Classical fuzzy sets extend crisp subsets with the assistance of a first level of uncertainty: the membership function  $\mu_H$  maps any element  $x$  of the universe  $X$  with its “uncertainty degree”. Thus, mapping  $x$  into a number from  $[0, 1]$  graduates the acceptability of the claim that this element belongs to the fuzzy subset  $H$ , which therefore generalizes the idea of classical subsets. HFSs are a particular case of “type-2 fuzzy sets” [9–11].

Basically, a type-1 fuzzy set on a universe  $X$  is a map  $h$  from  $X$  into  $[0, 1]$ . Concerning type-2 fuzzy subsets, they are defined as functions from  $X$  to the family of type-1 fuzzy subsets of  $[0, 1]$ ; see, e.g., [12–14]. In that sense, a type-1 fuzzy set has a grade of membership that is crisp, namely a number in the unit interval. And a type-2 has grades of membership that are fuzzy subsets of  $[0, 1]$ , that is, maps from  $[0, 1]$  into itself. Notice that HFSs correspond to the particular case of type-2 fuzzy subsets of  $X$  such that the grade maps  $[0, 1]$  into  $\{0, 1\}$ .

In many practical applications, the HFSs considered map each element of  $X$  to a finite subset of  $[0, 1]$ . This particular case is well known in the literature:

**Definition 5** ([15]). A function  $h: X \rightarrow \mathcal{F}([0, 1])$  is said to be a typical hesitant fuzzy set (THFS) of  $X$ . An element of  $\mathcal{F}([0, 1])$  is called a typical hesitant fuzzy element (THFE).

Scores are now formally defined:

**Definition 6.** Given a family  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ , a score on  $\mathcal{G}$  is defined as a function  $s: \mathcal{G} \rightarrow [0, 1]$  that satisfies the following properties:

1. The score assigned to the empty set is zero. That is,  $s(\emptyset) = 0$  provided that  $\emptyset \in \mathcal{G}$ ;
2. For all  $E \in \mathcal{G}$ , we have that the score assigned to  $E$  must lie between the infimum and the supremum of  $E$ . That is,  $\inf(E) \leq s(E) \leq \sup(E)$ . (In particular,  $s(\{t\}) = t$  holds true for each  $t$  in the unit interval).

A score on  $\mathcal{G} = \mathcal{P}([0, 1])$  (respectively, on  $\mathcal{G} = \mathcal{F}([0, 1])$ ) is called total (respectively, typical). Additionally, a score on  $\mathcal{G} = \mathcal{I}([0, 1])$  is said to be an interval score.

**Remark 3.** One significant application of scores defined on HFEs is that they enable the reduction of uncertainty by one level. If  $h: X \rightarrow \mathcal{P}([0, 1])$  defines an HFS, and  $s$  is a score defined on  $\mathcal{P}([0, 1])$ , then the composition  $s \circ h$  directly defines a (type-1) fuzzy set over  $X$ .

### 3. Classical Criteria to Extend Orderings from a Set to Its Power Set

The most classical criteria appeared from 1950 on, mainly in several papers related to social choice and decision making (see, e.g., [1,4,16]). They usually consider a total order  $\succsim$  defined on a finite set  $X$ , whereas the extension  $\succsim_E$  can be either a total order or just a complete preorder depending on the context.

In order to classify criteria, we may focus on several important aspects, usually related to some objective that we want to achieve from the restrictions imposed.

In the first classification of extension criteria, we may focus on the following facts:  
 (i) maxima and/or minima of sets; (ii) means; (iii) monotonicity.

Needless to say, there are some other possible classes, so that this is just a first approximation. In addition, these classes are not always pairwise disjoint.

In general, what we may expect a priori is that if we take criteria from different classes, these give rise to an incompatibility result. However, this is not always true.

Here are several criteria based on the maxima or minima of a finite set.

**Definition 7.** Let  $X$  stand for a finite set endowed with a total order  $\succsim$ . An extension  $\succsim_E$  satisfies the Pure Maximality Criterion [PMA] (respectively, the Pure Minimality Criterion [PMI]) if for every nonempty subsets  $A, B \subseteq X$  it holds true that  $\max A \prec \max B \Rightarrow A \prec_E B$  (respectively,  $\min A \prec \min B \Rightarrow A \prec_E B$ ). Of course, here maxima and minima are taken as regards the given total order  $\succsim$  on  $X$ . They exist because  $X$  is finite.

**Definition 8.** Let  $X$  stand for a finite set endowed with a total order  $\succsim$ . An extension  $\succsim_E$  satisfies the Gärdenfors Principle [G] (see [4]) if for every nonempty subset  $A \subseteq X$  and any element  $x \notin A$  it holds true that  $x \prec \min A \Rightarrow A \cup \{x\} \prec_E A$ , and also  $\max A \prec x \Rightarrow A \prec_E A \cup \{x\}$ . Again, maxima and minima are taken here with respect to the given total order  $\succsim$  on  $X$ .

**Definition 9.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the Barberà-Pattanaik Property [BP] if for every  $x, y \in X$  with  $x \prec y$  it holds true that  $\{x\} \prec_E \{x, y\} \prec_E \{y\}$ .

In addition,  $\succsim_E$  satisfies the generalized Barberà-Pattanaik Property [GBP] if for every nonempty subsets  $A, B \subseteq X$  such that  $a \prec b$  holds for any  $a \in A, b \in B$ , we also have that  $A \prec_E A \cup B \prec B$ .

**Remark 4.** Notice that, unlike the Gärdenfors principle, here we do not ask  $X$  to be finite. If  $X$  is finite, the Barberà-Pattanaik Property [BP] is much weaker than the Gärdenfors Principle [G], of which it is an immediate consequence in that case. In addition, we could also think that [BP]

reminds us of an idea of a mean, so that if an element  $y \in X$  is better than another one  $x$ , in the extension to the power set, the subset  $\{x, y\}$  should lie between the subsets  $\{x\}$  and  $\{y\}$ . This becomes much clearer when the extension is represented by a numerical function (utility) so that the value for  $\{x, y\}$  represents, in a sense, a mean of the values for  $\{x\}$  and  $\{y\}$ .

Let us now introduce some more criteria based on the intuitive idea of a mean.

**Definition 10.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the Kelly Criterion [K] (see [17]) if given two nonempty subsets  $A, B \in \mathcal{P}(X)$  with  $A \neq B$  and such that  $y \succ x$  holds for every  $x \in A; y \in B$ , then  $B \prec_E A$  also holds true.

**Remark 5.** Notice that this criterion [K] tells us that “the elements of  $A$  are at least as good as those in  $B$ ”, but there is at least one that is better since  $A \neq B$ . Thus, at least intuitively, an “average value” or a “mean” should be higher in  $A$ .

**Definition 11.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the Criterion of Useful Elements [CUE]. If for every subset  $A \subseteq X$  such that  $A$  has at least two elements, there is  $a \in A$  such that  $A \setminus \{a\} \prec_E A$ . Such an element  $a \in A$  is said to be a useful element for the subset  $A$ .

**Definition 12.** Let  $X$  be a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the Criterion of Singular Elements [CSE] if for every subset  $A \subseteq X$  such that  $A$  has at least two elements, there is  $a \in A$  such that  $A \prec_E A \setminus \{a\}$ . Such an element  $a \in A$  is said to be a singular element for the subset  $A$ . Also  $\succsim_E$  satisfies the Robustness Criterion [R] if for every nonempty subsets  $A, B, C \in \mathcal{P}(X)$  such that  $y \prec x$  holds for any  $x \in A \cup B; y \in C$ , it holds that  $B \prec_E A \Rightarrow B \cup C \prec_E A$ .

Let us pay attention now to criteria based on monotonicity properties.

**Definition 13.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the criterion of Monotonicity for Inferior Element [MIE] (see [1], where the terminology is different) if for every  $x, y, z \in X$  such that  $z \prec y \prec x$  it holds true that  $\{y, z\} \prec_E \{x, z\}$ . Similarly, the extension  $\succsim_E$  satisfies the criterion of Monotonicity for Superior Element [MSE] (see [1]) if for every  $x, y, z \in X$  such that  $z \prec y \prec x$  it holds true that  $\{z, x\} \prec_E \{y, x\}$ .

**Definition 14.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the criterion of Monotonicity for Sets [MS] if for every nonempty subsets  $A, B, C \subseteq X$  with  $C \cap (A \cup B) = \emptyset$  it holds that  $A \prec_E B \Leftrightarrow A \cup C \prec_E B \cup C$ . Moreover,  $\succsim_E$  satisfies the criterion of Simple Monotonicity for Elements [SME] (see [1]) if for every nonempty subsets  $A, B \subseteq X$  and  $x \in X$  such that  $x \notin (A \cup B)$  it holds true that  $A \prec_E B \Rightarrow A \cup \{x\} \prec_E B \cup \{x\}$ . Also,  $\succsim_E$  satisfies the criterion of Strong Monotonicity for Elements [STRME] (see [1]) if for every nonempty subsets  $A, B \subseteq X$  and  $x \in X$  with  $x \notin (A \cup B)$  it holds that  $A \prec_E B \Leftrightarrow A \cup \{x\} \prec_E B \cup \{x\}$ .

**Definition 15.** Let  $X$  stand for a set endowed with a complete preorder  $\succsim$ . An extension  $\succsim_E$  satisfies the General Criterion of Monotonicity for Superior Elements [CSUP] (see [1], with a different nomenclature) if for every nonempty subsets  $A, B \subseteq X$ , and any element  $x \in X$  such that  $y \prec x$  holds true for every  $y \in A \cup B$ , it holds then true that  $B \prec_E A \Rightarrow B \cup \{x\} \prec_E A \cup \{x\}$ . Also,  $\succsim_E$  satisfies the General Criterion of Monotonicity for Inferior Elements [CINF] (see [1]) if for every nonempty subsets  $A, B \subseteq X$ , and any element  $x \in X$  such that  $x \prec y$  holds true for every  $y \in A \cup B$ , it holds then true that  $B \prec_E A \Rightarrow B \cup \{x\} \prec_E A \cup \{x\}$ .

**Remark 6.** Notice that [CINF] is a consequence of [R] if we consider  $C = \{x\}$  in the corresponding definition of Robustness. Therefore [R] implies [CINF].



**Definition 16.** Let  $X$  stand for a set endowed with a complete preorder  $\preceq$ . An extension  $\preceq_E$  satisfies the Criterion of Monotonicity Relative to Disjoint Sets [MDS] (see once more [1], with a different terminology) if for every nonempty sets  $A, B \subseteq X$  such that  $A \cap B = \emptyset$ , and any  $x \in X$  such that  $x \notin (A \cup B)$ , it holds then true that  $B \prec_E A \Rightarrow B \cup \{x\} \prec_E A \cup \{x\}$ . Also,  $\preceq_E$  satisfies the Criterion of Monotonicity Relative to Nested Sets [MNS] (see [1]) if for every nonempty sets  $A, B \subseteq X$  such that  $B \subseteq A$ , and any element  $x \in X$  such that  $x \notin A$ , it holds then true that  $A \prec_E A \cup \{x\} \Rightarrow B \prec_E B \cup \{x\}$ .

**4. Compatibility of Criteria of Extension of Orderings**

**4.1. Complete Preorders vs. Total Orders**

In the classical theory, it is typical to analyze extensions in which we start with a total order  $\preceq$  on a set  $X$  and we want an extension  $\preceq_E$  that is also a total order on the power set of  $X$ . However, sometimes this will not be possible. As a matter of fact, some combination of criteria imposed on the extension may oblige it to be a complete preorder instead of a total order, as shown in Proposition 1.

**Proposition 1.** Let  $X$  be a finite set with at least three elements, endowed with a total order of  $\preceq$ . There is no extension  $\preceq_E$  to a total order satisfying [SME] and [B].

**Proof.** Let  $X = \{a, b, c\}$  with  $a \prec b \prec c$ . It follows that  $\{a\} \prec_E \{b\} \prec_E \{c\}$ . Using [B], we have  $\{a\} \prec_E \{a, b\} \prec_E \{b\} \prec_E \{b, c\} \prec_E \{c\}$ . Now, by [SME], we would arrive at  $\{a, c\} \prec_E \{a, b, c\}$ , since  $\{a\} \prec_E \{a, b\}$ . Again, by [SME], it follows that  $\{a, b, c\} \prec_E \{a, c\}$ , because  $\{b, c\} \prec_E \{c\}$ . We arrive at a contradiction. Therefore,  $\preceq_E$  cannot be a total order. □

**Remark 7.** Under Proposition 1, we wonder if  $\preceq_E$  could be a complete preorder. The answer is affirmative. An example is the extension given as follows:  $\{a\} \prec_E \{a, b\} \prec_E \{b\} \sim_E \{a, c\} \sim_E \{a, b, c\} \prec_E \{b, c\} \prec_E \{c\} \prec_E \emptyset$ .

If  $X$  has at most two elements, then there exist extensions  $\preceq_E$  that are total orders and satisfy [G] and [SME]. For instance, if  $X = \{a, b\}$  and  $a \prec b$ , we may take the extension  $\preceq_E$  given by  $\emptyset \prec_E \{a\} \prec_E \{a, b\} \prec_E \{b\}$ .

**Definition 17.** Let  $X$  be a nonempty set endowed with a complete preorder  $\preceq$ . Let  $\preceq_E$  be an extension of  $\preceq$  to the power set of  $X$ . Then we say that  $\preceq_E$  satisfies the Weak Monotonicity Criterion [WM] if for any nonempty subsets  $A, B \subseteq X$  and any element  $x \in X \setminus (A \cup B)$  it holds true that  $A \prec_E B \Rightarrow A \cup \{x\} \preceq_E B \cup \{x\}$ .

**Proposition 2.** The [SME] criterion implies [WM]. The converse is false.

**Proof.** The implication follows directly from definitions. Now let  $X = \{a, b, c\}$  and  $a \prec b \prec c$ . Consider the extension  $\preceq_E$  given by  $\emptyset \prec_E \{a\} \prec_E \{b\} \prec_E \{b\} \sim_E \{a, b\} \sim_E \{a, c\} \sim_E \{b, c\} \sim_E \{a, b, c\}$ . It satisfies [WM] but not [SME]. □

**Lemma 1.** Let  $X$  be a nonempty finite set endowed with a total order  $\preceq$ . Let  $\preceq_E$  be a quasi-transitive extension satisfying [G] and [WM]. Then, for every nonempty subset  $A$  of  $X$ , it holds true that  $A \sim_E \{\min(A), \max(A)\}$ .

**Proof.** The proof can be seen in [2], but we include it here for the sake of completeness, since it is decisive in what follows then. Assume that  $X$  has at least three elements, since otherwise the result becomes evident. So let  $n \geq 3$ , and  $X = \{a_1, a_2, \dots, a_n\}$  with  $a_1 \prec a_2 \prec \dots \prec a_n$ . Iterating [G] and using the transitivity of  $\prec_E$ , it follows that  $\{a_1\} \prec_E \{a_1, a_2\} \prec_E \dots \prec_E \{a_1, \dots, a_{n-1}\}$ . Using [WM], we get  $\{a_1, a_n\} \preceq_E X$ . Since, by [G] again,  $\{a_2, \dots, a_n\} \prec_E \{a_3, \dots, a_n\} \prec_E \dots \prec_E \{a_n\}$ , once more, by [WM], we arrive at  $X \preceq_E \{a_1, a_n\}$ . Therefore,  $X \sim_E \{a_1, a_n\}$ . □

### 4.2. On Kannai-Peleg Theorem

Kannai and Peleg, in a seminal paper published in 1984 (see [2]), proved an impossibility theorem where the Gärdenfors Principle [G], the Weak Monotonicity Criterion [WM], and the fact that  $X$  is finite but has at least six different elements were crucial. Now we pay attention to some variants of the original theorem.

**Proposition 3** (Kannai-Peleg Theorem –1984–, see [2]). *Let  $X$  be a finite set endowed with a total order  $\preceq$ . If  $X$  has at least 6 elements, then there is no extension  $\preceq_E$  that is also a complete preorder and satisfies both [G] and [WM].*

**Proof.** See the proof in the classical reference [2]. We omit it here. Instead, we will prove in Proposition 5 below a result whose proof follows, so-to-say, similar ideas.  $\square$

**Proposition 4.** *Let  $X$  be a finite set whose cardinality is at most five, endowed with a complete preorder  $\preceq$ . Then there exists an extension  $\preceq_E$  that is also a complete preorder and satisfies [G] and [WM].*

**Proof.** There is no loss of generality in assuming that  $X$  has five elements, that is,  $X = \{a_1, a_2, a_3, a_4, a_5\}$ , and it is endowed with a total order  $\preceq$  such that  $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5$ . Consider now the extension  $\preceq_E$  given by:  $\emptyset \prec_E \{a_1\} \prec_E \{a_1, a_2\} \prec_E \{a_1, a_3\} \sim_E \{a_1, a_2, a_3\} \prec_E \{a_1, a_4\} \sim_E \{a_1, a_2, a_4\} \sim_E \{a_1, a_2, a_3, a_4\} \sim_E \{a_1, a_3, a_4\} \prec_E \{a_2\} \prec_E \{a_2, a_3\} \prec_E \{a_3\} \sim_E \{a_2, a_4\} \sim_E \{a_2, a_3, a_4\} \sim_E \{a_1, a_5\} \sim_E \{a_1, a_2, a_5\} \sim_E \{a_1, a_3, a_5\} \sim_E \{a_1, a_4, a_5\} \sim_E \{a_1, a_2, a_3, a_5\} \sim_E \{a_1, a_2, a_4, a_5\} \sim_E \{a_1, a_3, a_4, a_5\} \sim_E \{a_1, a_2, a_3, a_4, a_5\} \prec_E \{a_3, a_4\} \prec_E \{a_4\} \prec_E \{a_2, a_5\} \sim_E \{a_2, a_3, a_5\} \sim_E \{a_2, a_4, a_5\} \sim_E \{a_2, a_3, a_4, a_5\} \prec_E \{a_3, a_5\} \sim_E \{a_3, a_4, a_5\} \prec_E \{a_4, a_5\} \prec_E \{a_5\}$ .  $\square$

**Definition 18.** *Let  $X$  be a nonempty set endowed with a complete preorder  $\preceq$ . Let  $\preceq_E$  be an extension of  $\preceq$  to the power set of  $X$ . Then we say that  $\preceq_E$  satisfies the Independence Criterion [IND] if for any nonempty subsets  $A, B \subseteq X$  and any element  $x \in X \setminus (A \cup B)$  it holds true that  $A \preceq_E B \Rightarrow A \cup \{x\} \preceq_E B \cup \{x\}$ .*

*Notice that the independence criterion [IND] implies [WM].*

**Proposition 5.** *Let  $X$  be a finite set endowed with a total order  $\preceq$ . If  $X$  has at least five elements, there is no extension  $\preceq_E$  being a complete preorder satisfying [G] and [IND].*

**Proof.** By Lemma 1, since [IND] implies [WM], given a nonempty subset  $A \subseteq X$ , we have that  $A \sim_E \{\min(A), \max(A)\}$ . There is no loss of generality in assuming now that  $X = \{a_1, a_2, a_3, a_4, a_5\}$  and  $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5$ . Assume that there is an extension  $\preceq_E$  that satisfies [G] and [IND]. First, we prove that  $\{a_2, a_4\} \preceq_E \{a_3\}$ . To do so, we will assume that this is false, and this will lead us to a contradiction: In fact, if  $\{a_3\} \prec_E \{a_2, a_4\}$ , using [IND] as regards  $a_5$ , we get  $\{a_3, a_5\} \preceq_E \{a_2, a_4, a_5\}$ . By Lemma 1, this implies  $\{a_3, a_5\} \sim_E \{a_3, a_4, a_5\}$ ,  $\{a_2, a_4, a_5\} \sim_E \{a_2, a_5\} \sim_E \{a_2, a_3, a_4, a_5\}$ , so that, in particular, we have  $\{a_3, a_4, a_5\} \preceq_E \{a_2, a_3, a_4, a_5\}$ . Let now  $B = \{a_3, a_4, a_5\}$ . Since  $a_2 \prec a_3 \prec a_4 \prec a_5$ , by [G] we get  $B \cup \{a_2\} \prec_E B$  or equivalently  $\{a_2, a_3, a_4, a_5\} \prec_E \{a_3, a_4, a_5\}$ . This is a contradiction. Therefore  $\{a_2, a_4\} \preceq_E \{a_3\}$ .

At this stage, again proceeding by contradiction, we prove that [G] is incompatible with [IND]. In fact, using now [IND], we get  $\{a_1, a_2, a_4\} \preceq_E \{a_1, a_3\}$ . In a similar way to the argument above, we use again Lemma 1 and obtain  $\{a_1, a_3\} \sim_E \{a_1, a_2, a_3\}$ ,  $\{a_1, a_2, a_4\} \sim_E \{a_1, a_4\} \sim_E \{a_1, a_2, a_3, a_4\}$ . Hence  $\{a_1, a_2, a_3, a_4\} \preceq_E \{a_1, a_2, a_3\}$ . Let now  $C = \{a_1, a_2, a_3\}$ . Since  $a_3 \prec a_4$  and  $a_3 = \max C$ , by [G] we have that  $C \prec_E C \cup \{a_4\}$ , or equivalently  $\{a_1, a_2, a_3\} \prec_E \{a_1, a_2, a_3, a_4\}$ . This is a contradiction.  $\square$

**Example 1.** The extension  $\lesssim_E$  introduced in the example that appears in the proof of Proposition 4 satisfies [WM]. However, it fails to satisfy [IND]. To see this, notice that it is  $\{a_3\} \sim_E \{a_2, a_4\}$ , so that in particular,  $\{a_3\} \lesssim_E \{a_2, a_4\}$ . However,  $\{a_3, a_5\} \not\lesssim \{a_2, a_4, a_5\}$  does not hold.

If in Proposition 5 above we substitute the condition of  $X$  having at least five elements, putting instead that  $X$  has at most four elements, we get compatibility.

**Proposition 6.** Let  $X$  be a finite set whose cardinality is at most four, endowed with a total order of  $\lesssim$ . Then there is an extension  $\lesssim_E$  that satisfies both [G] and [IND].

**Proof.** Without loss of generality, let  $X = \{a_1, a_2, a_3, a_4\}$  and  $\lesssim$  such that  $a_1 \prec a_2 \prec a_3 \prec a_4$ . Take the extension  $\lesssim_E$  given by  $\emptyset \prec_E \{a_1\} \prec_E \{a_1, a_2\} \prec_E \{a_1, a_3\} \sim_E \{a_1, a_2, a_3\} \prec_E \{a_2\} \prec_E \{a_2, a_3\} \sim_E \{a_1, a_4\} \sim_E \{a_1, a_2, a_4\} \sim_E \{a_1, a_3, a_4\} \sim_E \{a_1, a_2, a_3, a_4\} \prec_E \{a_3\} \prec_E \{a_2, a_4\} \sim_E \{a_2, a_3, a_4\} \prec_E \{a_3, a_4\} \prec_E \{a_4\}$ .  $\square$

If we substitute [IND] by [SME] in the statement of Proposition 5 above, not only does the result become true, but now, just with three elements, we get incompatibility, as already stated in Proposition 1.

Concerning compatibility of criteria, the following classical result that involves, as well as weak monotonicity [WM], the Kelly criterion [K] instead of the more demanding Gärdenfors property [G] was proved in [18].

**Proposition 7.** Let  $X$  stand for a finite nonempty set. Let  $\lesssim$  be a total order defined on  $X$ . Then there exists an extension  $\lesssim_E$  of  $\lesssim$  to the power set of  $X$  that is a complete preorder and satisfies both [K] and [WM].

**Proof.** See Proposition 2 in [18].  $\square$

In the same spirit, we now prove the following result on the compatibility of the criteria.

**Proposition 8.** Let  $X$  stand for a finite nonempty set. Let  $\lesssim$  be a total order defined on  $X$ . Then there exists an extension  $\lesssim_E$  of  $\lesssim$  to the power set of  $X$  that is a complete preorder and satisfies both [R] and [SME].

**Proof.** Just notice that given  $X = \{a_1, a_2, \dots, a_n\}$  with  $a_1 \prec a_2 \prec \dots \prec a_n$ , the lexicographic order (that considers each element of  $X$  as a letter and each subset of  $X$  as a word where its case letters are also ordered by means of  $\prec$ , and the words are ordered lexicographically, as in a dictionary) satisfies both [R] and [SME].  $\square$

### 5. Scores: Definitions, Hierarchies, and Incompatibility Results

A substantial part of this Section 5 already appeared in [3]. We have decided to include some definitions and results here for the sake of completeness and the well-understanding of the ideas.

#### 5.1. Some Background on Scores

The most commonly used scores (see, for example, [8,19–27]) are typical (in the sense of Definition 6), as they are defined on the family of all finite subsets of the unit interval. Some examples may be seen in [3]. From the second condition in Definition 6, a score can be understood as a “mean value” (see [28] for more details).

As a sample, we furnish now another example, not included in [3]. This defines a function  $s: \mathcal{F}([0, 1]) \rightarrow [0, 1]$  that is a typical score: For each  $E = \{e_1, \dots, e_n\} \in \mathcal{F}([0, 1])$ , with  $e_1 < \dots < e_n$ , and  $p$  a natural number, we define  $s(E) = (e_1^p + \dots + e_n^p)^{\frac{1}{p}}$ .

To classify scores on HFEs, we may pay attention to the following aspects:

- (i) Properties based on certain coherence features of the score (e.g., the addition of better elements should never decrease the score).



- (ii) Properties based on the specific type of HFEs for which the score will be considered (e.g., finite HFEs, interval HFEs, etc.).

**Remark 8.** *The classification will not necessarily produce mutually exclusive sets.*

### 5.2. On Coherence Features of Scores

First, we introduce some definitions and results related to coherence.

**Definition 19.** *Suppose  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ . A score  $s$  on  $\mathcal{G}$  is said to be best-worst monotonic for elements [BWME] if for any  $x, y \in [0, 1]$  such that  $x < y$ , and additionally  $\{x\}$ ,  $\{x, y\}$  and  $\{y\}$  belong to  $\mathcal{G}$ , it holds that  $s(\{x\}) < s(\{x, y\}) < s(\{y\})$ .*

*Notice that this property is essentially the Barberà-Pattanaik property [BP] (see Definition 9) adapted for scores.*

**Definition 20.** *Suppose that  $\mathcal{G} \subseteq \Pi([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  is strongly monotonic with respect to unions [SMU] when for each  $A, B, A \cup B \in \mathcal{G}$  and such that  $a < b$  for every  $a \in A, b \in B$ , it holds true that:  $s(A) < s(A \cup B) < s(B)$ .*

[SMU] captures the following intuition: Adding better elements to a subset should increase its score, and removing worse elements should also increase its score.

**Remark 9.** *Notice that [SMU] implies [BWME]. However, the converse is not true in general. A counterexample appears in [3].*

Given a non-empty subset  $A$  of the unit interval  $[0, 1]$  and real numbers  $\alpha, \beta > 0$ , we define the set  $\alpha A$  as  $\alpha A = \{\alpha \cdot t : t \in A\}$ . Also, we define the set  $\beta + A$  as follows:  $\beta + A = \{\beta + t : t \in A\}$ . Depending on  $A$  and  $\alpha, \beta$ , the resulting sets  $\alpha A$  and/or  $\beta + A$  may or may not be subsets of  $[0, 1]$ .

**Definition 21.** *Suppose now that  $\mathcal{G} \subseteq \Pi([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  is algebraically coherent with respect to a dilatation [ACD] when for each  $A \in \mathcal{G}$  and  $\alpha > 0$  such that  $\alpha A$  also belongs to  $\mathcal{G}$  it holds true that  $s(\alpha A) = \alpha s(A)$ .*

*Similarly, we say that a score  $s$  on  $\mathcal{G}$  is algebraically coherent with respect to a translation [ACT] when for each  $A \in \mathcal{G}$  and  $\beta > 0$  such that  $A + \beta$  also belongs to  $\mathcal{G}$  it holds true that  $s(A + \beta) = s(A) + \beta$ .*

**Remark 10.** *Many typical scores were introduced in [3] satisfy both [ACD] and [ACT].*

**Definition 22.** *Suppose  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  satisfies translation invariance [TI] when for each  $A \in \mathcal{G}$  such that  $A + \varepsilon \in \mathcal{G}$  (with  $\varepsilon > 0$ ), it holds true that  $s(A) < s(A + \varepsilon)$ .*

**Definition 23.** *Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  satisfies the (adapted) (We say here “adapted” Gärdenfors property since this property was introduced by Gärdenfors [4] in 1976 just to deal with finite subsets, so that the original definition was stated making reference to maxima and minima instead of suprema and infima. (Remember Definition 8 above)) Gärdenfors property [G] if for every  $A \in \Pi([0, 1])$  and any element  $x \notin A$  such that  $A, A \cup \{x\} \in \mathcal{G}$  the following two conditions hold:*

$$[G1] \ x < \inf A \Rightarrow s(A \cup \{x\}) < s(A); \quad [G2] \ \sup A < x \Rightarrow s(A) < s(A \cup \{x\}).$$

**Remark 11.** *Among the classical scores (see [3]) defined for finite subsets of the unit interval (TFHE’s), the minimum satisfies [G1], but not [G2]. Similarly, the maximum satisfies [G2] but not [G1].*

**Definition 24.** Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  satisfies the weak monotonicity property [WM] if for every  $A, B \in \Pi([0, 1])$  and  $x \notin A \cup B$ , such that  $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{G}$ , it holds true that  $s(A) < s(B) \Rightarrow s(A \cup \{x\}) < s(B \cup \{x\})$ .

**Remark 12.** Notice that [WM] is a particular case of Definition 17 above for extensions of orderings. It is not easy to find scores satisfying [WM], except maybe in situations dealing with special classes of finite subsets of the unit interval (see, e.g., [29,30]).

A classical theorem by Kannai and Peleg [2] states this incompatibility result.

**Proposition 9.** Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$  such that there exists a subset  $A \subseteq [0, 1]$  whose cardinality is at least 6, such that  $A$  and all its subsets belong to  $\mathcal{G}$ . Then no score  $s$  on  $\mathcal{G}$  satisfies both the adapted Gärdenfors property [G] and the weak monotonicity property [WM].

**Proof.** See [2], p. 174. Observe also that this Proposition 9 is also a direct consequence of Proposition 3 for extensions of orderings.  $\square$

### 5.3. Scores Defined on Special Classes of Sets

All the results of this subsection appear in [3]. For this reason, we do not include their proofs here. Nevertheless, the ideas involved in these results may affect Section 6.2 of the next Section 6 when trying to induce impossibility results for extensions of orderings mimicking the results got here for scores on HFEs.

Bearing this in mind, we now pay attention to scores that sometimes are not defined on the whole set  $\mathcal{P}([0, 1])$  but instead act only on THFEs (namely,  $\mathcal{F}([0, 1])$ ) or on intervals  $\mathcal{I}([0, 1])$ . In particular, we pay attention to scores defined on families  $\mathcal{G}$  that include the set of intervals  $\mathcal{I}([0, 1])$ .

This gives rise to new definitions and results concerning the compatibility of the newly introduced properties when imposed on those scores. These have been studied in depth in [3]. We keep here just a sample of these new definitions and results to motivate their use in Section 6 of the present paper when adapted to utility functions related to extensions of orderings.

**Definition 25.** Suppose  $\mathcal{I}([0, 1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0, 1])$ . A score  $s : \mathcal{G} \rightarrow [0, 1]$  is extremes monotonic [EM] when it satisfies the following two conditions: [EM1]:  $0 \leq b < b' \leq 1$  implies  $s([a, b]) < s([a, b'])$  for each  $a \in [0, b]$ ; [EM2]:  $0 \leq a < a' \leq 1$  implies  $s([a, b]) < s([a', b])$  for each  $b \in [a', 1]$ .

Moreover,  $s$  is strongly extreme monotonic [SEM] if for each  $a, b \in [0, 1]$  with  $a < b$  it holds that  $s([0, a]) < s([0, a]) < s([0, b])$ .

**Proposition 10.** Consider a score  $s : \mathcal{G} \rightarrow [0, 1]$  with  $\mathcal{I}([0, 1]) \subseteq \mathcal{G} \subseteq \mathcal{P}([0, 1])$ . Then  $s$  cannot satisfy the property [SEM] of strong extremes monotonicity.

Moreover,  $s$  cannot satisfy both [SMU] and [EM1].

**Proof.** See Lemmas 1 and 2 in [3].

## 6. Comparisons and Analogies between the Theory of Extension of Orderings and the Theory of Scores on Hesitant Fuzzy Elements

### 6.1. From Extensions of Orderings to Scores

The theory of scores on HFEs can be considered a particular case of that of extensions of orderings from a set to its power set. In fact, if we consider the usual total order  $\leq$  on the unit interval  $[0, 1]$  and a score  $s$  defined on  $\Pi([0, 1])$ , the order  $\leq$  is immediately extended to a complete preorder  $\leq_E$  on  $\Pi([0, 1])$ , but just declaring that  $A \leq_E B \Leftrightarrow s(A) \leq s(B)$ , for any nonempty subsets  $A, B$  of  $[0, 1]$ .

Even if we only consider the theory of extensions of orderings on finite sets, we can also adapt its results to particular cases of scores, namely scores defined on the family  $\mathcal{F}([0, 1])$

THFEs. Actually, to compare two nonempty finite sets of  $[0, 1]$ , say  $A = \{x_1, \dots, x_n\}$  and  $B = \{y_1, \dots, y_k\}$ , first we endow the finite set  $A \cup B$  with the total order  $\leq$  inherited from the usual order of  $[0, 1]$ , and, through a score  $s$  defined on  $\mathcal{F}([0, 1])$ , we extend it to a complete preorder  $\leq$  on the nonempty parts of  $A \cup B$  by declaring that  $C \leq D \Leftrightarrow s(C) \leq s(D)$ .

What is clear now is that the impossibility results arising from extensions of orderings from a set to its power set have a parallel result on scores: Proposition 3 on extensions induces Proposition 9 on scores.

**Definition 26.** Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$ . We say that a score  $s$  on  $\mathcal{G}$  satisfies the weak monotonicity property of second type [WM2] if for every  $A, B \in \Pi([0, 1])$  and  $x \notin A \cup B$ , such that  $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{G}$ , it holds true that  $s(A) < s(B) \Rightarrow s(A \cup \{x\}) \leq s(B \cup \{x\})$ . Similarly, it satisfies the weak monotonicity property of the third type [WM3] if for every  $A, B \in \Pi([0, 1])$  and  $x \notin A \cup B$ , such that  $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{G}$ , it holds true that  $s(A) \leq s(B) \Rightarrow s(A \cup \{x\}) \leq s(B \cup \{x\})$ .

Proposition 3 for extensions of orderings also implies the following Proposition 11.

**Proposition 11.** Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$  such that there exists a subset  $A \subseteq [0, 1]$  whose cardinality is at least 6, such that  $A$  and all its subsets belong to  $\mathcal{G}$ . Then there is no score  $s$  on  $\mathcal{G}$  that satisfies both the adapted Gärdenfors property [G] and the weak monotonicity property of the second type [WM2].

And Proposition 5 immediately gives rise to the following result for scores.

**Proposition 12.** Let  $\mathcal{G} \subseteq \mathcal{P}([0, 1])$  such that there exists a subset  $A \subseteq [0, 1]$  whose cardinality is at least 5, such that  $A$  and all its subsets belong to  $\mathcal{G}$ . Then there is no score  $s$  on  $\mathcal{G}$  that satisfies both the adapted Gärdenfors property [G] and the weak monotonicity property of the third type [WM3].

Needless to say, these cases are just a sample. That is, the key fact here is that impossibility results encountered in the framework of the extensions of orderings from a set to its power set (a.k.a, “ranking sets of objects”, see [1]) immediately induce parallel results on the theory of scores on hesitant fuzzy elements.

A remarkable fact here is that, to the extent of what we know and perceive, a large part of these results of the impossibility of extensions of orderings turn out to be little known, if not ignored, by a substantial part of the researchers in fuzzy set theory, dealing with scores on hesitant fuzzy elements.

### 6.2. From Scores to Extensions of Orderings

The converse situation, that is, using some results on scores on hesitant fuzzy elements to get new (im)possibility theorems on extensions of total orders from a nonempty set to its power set, is not so direct. Since the extension of orderings is a more general setting, it may happen that some results on scores could not be translated into one for extensions of orderings, in the most general case (extensions that start from a total order defined on a nonempty set  $X$ , not necessarily finite).

In addition, the results on scores have  $[0, 1]$  as a starting point, that is, as a totally ordered set through the usual order  $\leq$ , on whose power set we want to define some score  $s$  accomplishing some properties or criteria. To translate this to some abstract set  $X$  endowed with a total order  $\preceq$ , we will need that  $X$  has some additional structure (e.g., topological) that could, in some way, remind us of  $[0, 1]$ .

Let us now discuss how to adapt Definition 25, Proposition 10, and some other definitions and results introduced in [1] to the framework of extensions of orderings from a set to its power set.

In what follows,  $X$  will stand for a nonempty set, and we will also assume that it is endowed with a total order  $\preceq$ . We want to extend  $\preceq$  to a complete preorder (i.e., now we will admit ties), say  $\preceq_E$ , defined on the power set of  $X$ .

Given  $a, b \in X$  with  $a \prec b$  we denote by  $[a, b] = \{x \in X : a \preceq x \preceq b\}$ ;  $(a, b) = \{x \in X : a \prec x \prec b\}$ ;  $[a, b) = \{x \in X : a \preceq x \prec b\}$ ;  $(a, b] = \{x \in X : a \prec x \preceq b\}$ . Also, given  $a \in X$ , we denote  $(a, \rightarrow) = \{x \in X : a \prec x\}$ ;  $[a, \rightarrow) = \{x \in X : a \preceq x\}$ ;  $(\leftarrow, a) = \{x \in X : x \prec a\}$ ;  $(\leftarrow, a] = \{x \in X : x \preceq a\}$ .

**Definition 27.** An extension  $\preceq_E$  of  $\preceq$  to the power set of  $X$  satisfies the property of interval compatibility [IC] if given  $a, b \in X$  with  $a \prec b$ , it holds true that  $(a, b) \sim_E [a, b] \sim_E [a, b) \sim_E (a, b]$ .

**Remark 13.** This property [IC] is very demanding. It forces the total order  $\preceq$  on  $X$  to be dense-in-itself. That is, if  $a \prec b$ , there is some other element  $c \in X$  such that  $a \prec c \prec b$ . The reason is that the extension  $\preceq_E$  should accomplish, on the one hand, that  $\{a\} \prec_E \{b\}$ . But, on the other hand, by [IC] we have that  $[a, b) = \{a\}$ , whereas  $(a, b] = \{b\}$ , and [IC] then would imply that  $\{a\} \sim_E \{b\}$ , a contradiction. As a clear consequence  $X$  should be infinite.

**Definition 28.** An extension  $\preceq_E$  of  $\preceq$  to the power set of  $X$  is extreme monotonic [EM] when it satisfies the following two conditions:

[EM1] If  $b \prec c$ , then  $[a, b] \prec_E [a, c]$  holds for every  $a \preceq b$  ( $a, b, c \in X$ );

[EM2] If  $b \prec c$ , then  $[b, a] \prec_E [c, a]$  holds for every  $c \preceq a$  ( $a, b, c \in X$ ).

Moreover, it is strongly extreme monotonic [SEM] if for each  $a, b \in X$  with  $a \prec b \prec c$ , it holds that  $[a, b] \prec_E [a, b] \prec_E [a, c] \prec_E [a, c]$ .

**Remark 14.** This new property, [SEM], is also very demanding. If  $X$  has at least three elements, the total order  $\preceq$  is almost “dense-in-itself” in the following sense: If  $a \prec b \prec c$  there exists  $d \in X$  with  $b \prec d \prec c$ . Otherwise,  $[a, b] = [a, c]$ , so that  $[a, b] \prec_E [a, c]$  becomes impossible. In particular,  $X$  should be infinite, too.

The results we can obtain now for extensions of orderings, after adapting some impossibility results for scores, have a strong relationship with the existence of a utility function. The reason is the following: If a complete preorder  $\preceq$  on a nonempty set  $Z$  has a utility representation  $u$ , there is no loss of generality in considering that  $u$  takes values in  $[0, 1]$ . Consequently, impossibility results for scores, adapted now to criteria of extensions of orderings from a set to its power set (see, e.g., Proposition 13 below), tell us the following key fact Either the extension  $\preceq_E$  cannot be a complete preorder, or if it is, then it does not admit a utility function. The first situation is typical for impossibility results on finite sets. The reason is that any complete preorder on a finite set always admits a utility representation. Moreover, if a set  $X$  is finite, its power set is also finite (see e.g., the first three chapters in [31]).

**Proposition 13.** Let  $X$  be a set whose cardinality is at least that of the continuum. Suppose that  $X$  is endowed with a total order  $\preceq$ . Let  $\preceq_E$  be a complete preorder on the power set of  $X$  that is an extension of  $\preceq$ . Then, if  $\preceq_E$  satisfies [SEM], it does not admit a utility representation.

**Proof.** This is analogous to the first part of Proposition 10 for scores. Suppose by contradiction that  $u$  is a utility function for  $\preceq_E$ . Take  $x \in X$  with either  $[x, \rightarrow)$  or  $(\leftarrow, x]$  having the cardinality of the continuum. (We will assume the first possibility, without loss of generality). Now, observe that for any  $x, y \in X$  with  $x \prec y, z$ , we have that  $a = u([x, y)) < b = u([x, y]) < c = u([x, z)) < d = u([x, z])$ . So there exists a rational number  $q_y \in (a, b)$  as well as another rational number  $q_z \in (c, d)$ , and by construction  $q_z > q_y$ . But this leads to a contradiction since  $\mathbb{Q}$  is countable.  $\square$

**Proposition 14.** Let  $X$  be a set whose cardinality is at least that of the continuum. Suppose that  $X$  is endowed with a total order  $\preceq$ . Let  $\preceq_E$  be a complete preorder on the power set of  $X$  that is an extension of  $\preceq$ . Then, if  $\preceq_E$  satisfies [GBP], it does not admit a utility representation.

**Proof.** This is analogous to the second part of Proposition 10 for scores. Suppose by contradiction that  $u$  is a utility function for  $\succsim_E$ . Take  $x \in X$  with either  $[x, \rightarrow)$  or  $(\leftarrow, x]$  having the cardinality of the continuum. (We will assume the first possibility, without loss of generality). Now, observe that for any  $x, y, z \in X$  with  $x \prec y, z$ , we have that  $[x, y) \prec_E [x, y] \prec_E \{y\}$  by [GBP]. Moreover, for the same reason,  $[x, y] \prec_E [x, z] \prec_E (y, z)$ . As in Proposition 14,  $a = u([x, y)) < b = u([x, y]) < c = u([x, z]) < d = u((y, z))$ , and we finally arrive at a contradiction.  $\square$

## 7. Concluding Remarks and Lines for Future Research

The main objective of this paper has been to show analogies between two apparently disparate theories. We have established some parallelism between the (more general) theory of extensions of ordering from a set to its power set and the theory of defining scores on hesitant fuzzy elements. Thus, each possibility or impossibility result encountered in the theory of ranking sets of objects (i.e., extending total orders from a set to its power set) immediately generates a parallel result of the same kind (possibility or impossibility) concerning scores on HFEs. Moreover, some possibility or impossibility results on scores on HFEs can be adapted somehow to obtain parallel results on extensions of orderings.

As a line for future research, we suggest exploring in depth both theories, searching for (im)-possibility results that, being well known in one of the frameworks, have not been used or even commented on in the other. Therefore, we could complete (or, at least, enlarge) the panorama of possibility/impossibility theorems arising in both theories: extension of orderings from a set to its power set vs. scores on HFEs.

**Author Contributions:** Conceptualization, E.I., A.M. and M.S.S.; Methodology, E.I., A.M. and M.S.S.; Formal analysis, E.I., A.M. and M.S.S.; Investigation, E.I., A.M. and M.S.S.; Writing—original draft, E.I., A.M. and M.S.S. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported by the project of reference PID2022-136627NB-I00 from MCIN/AEI/10.13039/501100011033/FEDER, UE, and by ERDF A way of making Europe.

**Data Availability Statement:** The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

**Acknowledgments:** Thanks are given to the editors and three anonymous referees for their valuable suggestions and comments.

**Conflicts of Interest:** The authors have no conflicts of interest to declare. All co-authors have seen and agreed with the contents of the paper, and there is no financial interest to report. We certify that this is original work, and it is not under review at any other publication.

## References

1. Barberà, S.; Bossert, W.; Pattanaik, P. Ranking sets of Objects. In *Chapter 17 of Handbook of Utility Theory: Volume 2 Extensions*; Barberà, S., Hammond, P., Seidl, C., Eds.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2004; pp. 893–979.
2. Kannai, Y.; Peleg, B. A note on the extension of an order on a set to the power set. *J. Econ. Theory* **1984**, *32*, 172–175. [[CrossRef](#)]
3. Alcantud, J.C.R.; Campión, M.J.; Induráin, E.; Munárriz, A. Scores of hesitant fuzzy elements revisited: “Was sind und was sollen”. *Inf. Sci.* **2023**, *648*, 119500. [[CrossRef](#)]
4. Gärdenfors, P. Manipulation of social choice functions. *J. Econ. Theory* **1976**, *13*, 217–228. [[CrossRef](#)]
5. Alcantud, J.C.R.; Giarlotta, A. Necessary and possible hesitant fuzzy sets: A novel model for group decision making. *Inf. Fusion* **2019**, *46*, 63–76. [[CrossRef](#)]
6. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [[CrossRef](#)]
7. Torra, V. Hesitant fuzzy sets. *Int. J. Intell. Syst.* **2010**, *25*, 529–539. [[CrossRef](#)]
8. Xia, M.; Xu, Z. Hesitant fuzzy information aggregation in decision making. *Int. J. Approx. Reason.* **2011**, *52*, 395–407. [[CrossRef](#)]
9. Bustince, H.; Barrenechea, E.; Pagola, M.; Fernández, J.; Xu, Z.; Bedregal, B.; Montero, J.; Hagrás, H.; Herrera, F.; De Baets, B. A historical account of types of fuzzy sets and their relationships. *IEEE Trans. Fuzzy Syst.* **2016**, *24*, 179–194. [[CrossRef](#)]
10. Narukawa, Y.; Torra, V. An score index for hesitant fuzzy sets based on the Choquet integral. In Proceedings of the 2021 IEEE International Conference on Fuzzy Systems, Luxembourg, 11–14 July 2021; pp. 1–5.
11. Torra, V.; Narukawa, Y. On hesitant fuzzy sets and decision. In Proceedings of the 2009 IEEE International Conference on Fuzzy Systems, Jeju Island, Republic of Korea, 20–24 August 2009; pp. 1378–1382.
12. Mendel, J.M. Type-2 fuzzy sets: Some questions and answers. *IEEE Connect. Newsl. IEEE Neural Netw. Soc.* **2003**, *1*, 10–13.



13. Mendel, J.M.; John, R.I. Type-2 fuzzy sets made simple. *IEEE Trans. Fuzzy Syst.* **2002**, *10*, 117–127. [[CrossRef](#)]
14. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning. *Inf. Sci.* **1975**, *8*, 199–249. [[CrossRef](#)]
15. Bedregal, B.; Reiser, R.; Bustince, H.; López-Molina, C.; Torra, V. Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms. *Inf. Sci.* **2014**, *255*, 82–99. [[CrossRef](#)]
16. De Miguel, J.R.; Goicoechea, M.I.; Induráin, E.; Olóriz, E. Criterios de extensión al conjunto potencia de ordenaciones sobre un conjunto finito. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Madr.* **2000**, *94*, 83–92.
17. Kelly, J. Strategy proofness and social choice functions without singlevaluedness. *Econometrica* **1977**, *45*, 439–446. [[CrossRef](#)]
18. Barberà, S.; Pattanaik, P. Extending an order on a set to the power set: Some remarks on Kannai and Peleg’s approach. *J. Econ. Theory* **1984**, *32*, 185–191. [[CrossRef](#)]
19. Chen, N.; Xu, Z.; Xia, M. Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. *Appl. Math. Model.* **2013**, *37*, 2197–2211. [[CrossRef](#)]
20. Farhadinia, B. A novel method of ranking hesitant fuzzy values for multiple attribute decision-making problems. *Int. J. Intell. Syst.* **2013**, *28*, 752–767. [[CrossRef](#)]
21. Farhadinia, B. A series of score functions for hesitant fuzzy sets. *Inf. Sci.* **2014**, *277*, 102–110. [[CrossRef](#)]
22. Rodríguez, R.; Bedregal, B.; Bustince, H.; Dong, Y.; Farhadinia, B.; Kahraman, C.; Martínez, L.; Torra, V.; Xu, Y.; Xu, Z.; et al. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. *Inf. Fusion* **2016**, *29*, 89–97. [[CrossRef](#)]
23. Rodríguez, R.; Martínez, L.; Herrera, F. Hesitant fuzzy linguistic term sets for decision making. *IEEE Trans. Fuzzy Syst.* **2012**, *20*, 109–119. [[CrossRef](#)]
24. Rodríguez, R.; Martínez, L.; Torra, V.; Xu, Z.; Herrera, F. Hesitant fuzzy sets: State of the art and future directions. *Int. J. Intell. Syst.* **2014**, *29*, 495–524. [[CrossRef](#)]
25. Wang, B.; Liang, J.; Pang, J. Deviation degree: A perspective on score functions in hesitant fuzzy Sets. *Int. J. Fuzzy Syst.* **2019**, *21*, 2299–2317. [[CrossRef](#)]
26. Xu, Z. *Hesitant Fuzzy Sets Theory*; Volume 314 of Studies in Fuzziness and Soft Computing; Springer International Publishing: Berlin/Heidelberg, Germany, 2014.
27. Xu, Z.; Xia, M. Distance and similarity measures for hesitant fuzzy sets. *Inf. Sci.* **2011**, *181*, 2128–2138. [[CrossRef](#)]
28. Campión, M.J.; Candeal, J.C.; Catalán, R.G.; Giarlotta, A.; Greco, S.; Induráin, E.; Montero, J. An axiomatic approach to finite means. *Inf. Sci.* **2018**, *457*, 12–28. [[CrossRef](#)]
29. Bossert, W. Preference extension rules for ranking sets of alternatives with a fixed cardinality. *Theory Decis.* **1995**, *39*, 301–317. [[CrossRef](#)]
30. Maly, J.; Woltran, S. Ranking Specific Sets of Objects. In *Lecture Notes in Informatics (LNI)*, Gesellschaft für Informatik; Springer: Berlin/Heidelberg, Germany, 2017; pp. 193–201.
31. Bridges, D.S.; Mehta, G.B. *Representation of Preference Orderings*; Springer: Berlin/Heidelberg, Germany, 1995.

**Disclaimer/Publisher’s Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.