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Exploring Fixed-Point Theorems in k-Fuzzy Metric Spaces: A Comprehensive Study

Muhammad Nazam [1], Seemab Attique [1,*], Aftab Hussain [2] and Hamed H. Alsulami [2]

[1] Department of Mathematics, Allama Iqbal Open University, H-8, Islamabad 44000, Pakistan; muhammad.nazam@aiou.edu.pk
[2] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; aniassuirathka@kau.edu.sa (A.H.); hhaalsalmi@kau.edu.sa (H.H.A.)
* Correspondence: seemabattique@gmail.com

Abstract: Recently, k -fuzzy metric spaces were introduced by connecting the degree of nearness of two points with k parameters $(t_1, t_2, t_3, \cdots, t_k)$ and the authors presented an analogue of Grabiec's fixed-point result in k-fuzzy metric spaces along with other necessary notions. The results presented only addressed continuous mappings. For discontinuous mappings, there is no result in k-fuzzy metric spaces. In this paper, we obtain some fixed-point results stating necessary conditions for the existence of fixed points of mappings eliminating the continuity requirement in k-fuzzy metric spaces. We illustrate the hypothesis of our findings with examples. We provide a common fixed-point theorem and fixed-point theorems for single-valued k-fuzzy Kannan type contractions. As an application, we use a fixed-point result to ensure the existence of solution of fractional differential equations.

Keywords: k-fuzzy metric space; fixed point; fuzzy contraction

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1. Introduction

The fuzzy set was introduced by Zadeh [1]. The distance function on a fuzzy set was introduced by Kramosil and Michalek [2] by applying the concept of t-norm. Later, George and Veeramani [3] amended this definition to show that a fuzzy metric space (fuzzy metric space) inherited a Hausdorff topology. These measurements are particularly appealing since they have the following two benefits over traditional metrics: First, regardless of the type of distance notion being measured, values provided by fuzzy metrics are in the range $[0, 1]$. This suggests that distinct distance requirements, which may initially be in fairly different ranges, are simple to integrate when using fuzzy metrics, which bring about a common range. Thus, it is possible to merge various distance requirements in a simple manner. Second, as the value provided by using a fuzzy metric can be used immediately or understood as a fuzzy certainty level, fuzzy metrics ideally complement the application of other fuzzy approaches. This makes it possible to directly include fuzzy measures into more intricate fuzzy systems.

Grabiec [4] proposed the fixed-point (fixed point) theory in an fuzzy metric space and extended the Banach and Edelstin fixed point theorems to an fuzzy metric space in the sense of Kramosil and Michalek [2]. We know that a Banach contraction is a continuous mapping. Thus, there arises a question about the existence of fixed points of discontinuous mappings. To answer this question, Reich [5] introduced the Kannan contraction principle and also addressed the existence of fixed points of discontinuous mappings. Furthermore, because they offer a flexible framework for proving the existence and uniqueness of fixed points in MSs, Kannan mappings are crucial to mathematics. They are valuable tools for studying numerical methods, nonlinear analysis, and other scientific and technical fields. Their ability to ascertain the convergence of iterative algorithms and resolve a broad variety of mathematical and real-world issues accounts for their prominence. Actually, one of mathematics' most notable discoveries is Kannan's fixed-point theorem, particularly with reference to entire MSs. The theorem specifies the conditions under which a unique fixed point in such spaces can be derived in the absence of continuous mapping. This is significant since many conventional fixed-point theorems require continuity as a prerequisite. After Banach's fixed point theorem in fuzzy metric spaces, we only had a knowledge of fixed points of continuous mapping in an fuzzy metric space. What about discontinuous ones? To answer this question and realizing the importance of Kannan's contraction, Romaguerra [6] introduced two new Kannan type fuzzy contractions in fuzzy metric spaces and proved some fixed point theorems in fuzzy metric spaces (fuzzy metric spaces). He also characterized the completeness of fuzzy metric spaces. Similarly, many authors introduced different forms of Kannan type contractions in fuzzy metric spaces [7].

Recently, Gopal et al. [8] extended the idea of fuzzy metric spaces and introduced a new space called k-fuzzy metric space. It is important to note that the study of a k-fuzzy metric space is more versatile than the fuzzy metric space created by George and Veeramani in 1994. We know that the fuzzy gap between two points in an fuzzy metric space is determined by how close together they are in relation to the parameter p between 0 and ∞. For illustration, we can consider p as the amount of time needed to go between two places in space, A and B. When we gauge the degree of proximity in relation to other (more than one) criteria, an interesting situation arises. Let us say we fly from Romania (labeled A) to the USA (labeled by B) using a plane, and we use several planes with different fuel efficiency to gauge how close A and B are in terms of time and fuel usage. Therefore, it follows that the degree of nearness will vary for various planes at the same time p, as well as for the same plane but for various time periods. The scenario described in the previous sentence served as the impetus for the introduction of the idea of k-fuzzy metric spaces, which is an extension and generalization of the concept of fuzzy metric spaces first proposed by George and Veeramani (1994) [3]. The fuzzy distance between two locations in a k-fuzzy metric space is determined by their proximity to one or more of the k parameters.

Recently, Alnaro ANTON-SANCHO provided fixed-point theorems over a compact algebraic curve [9] and compact Reimann surface [10,11]. Gopal et al. [8] highlighted the importance of this concept and proved Banach's fixed point theorems in the setting of k-fuzzy metric spaces. Motivated by the research work done in [6–8] and recognizing the significance of Kannan and Chatterjea's contractions, in this article, we introduce different forms of Kannan contraction (Sections 4 and 7) and Chatterjea contraction (Section 5) in a k-fuzzy metric space. Using Kannan type contraction, we provide fixed point theorems and common fixed point theorems in k-fuzzy metric spaces. We provide an application of an fixed point theorem that insures the existence of a solution of a fractional differential equation.

2. Preliminaries

In this section, we give some important definitions that are useful in proving our result.

Definition 1 ([12])**.** *A continuous triangular norm or continuous t-norm is defined as a binary operation* $\circ : [0,1] \times [0,1] \to [0,1]$ *satisfying the following properties* $\forall\ a,b,c,d \in [0,1]$ *:*

(1) $a \circ b = b \circ a$ *(commutative);*
(2) $(a \circ b) \circ c = a \circ (b \circ c)$ *(associativity);*
(3) $a \circ 1 = a, \forall\ a \in [0,1]$ *(identity law);*
(4) $a \circ b \leq c \circ d$ *whenever* $a \leq c$ *and* $b \leq d$ *(monotonicity);*
(5) $\circ$ *is continuous.*

Let X be any arbitrary set, $\circ$ a continuous t-norm, F a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all $a,b,c \in X$ and $s,p > 0$:

$$(FM0) : F(a,b,0) = 0;$$
$$(FM1) : F(a,b,p) > 0;$$
$$(FM2) : F(a,b,p) = 1 \iff a = b;$$
$$(FM3) : F(a,b,p) = F(b,a,p);$$
$$(FM4) : F(a,b,p) \circ F(b,c,s) \leq F(a,c,p+s);$$
$$(FM5) : F(a,b,.) : (0,\infty) \to [0,1] \text{ is left-continuous mapping.}$$
$$(FM6) : F(a,b,.) : (0,\infty) \to [0,1] \text{ is continuous mapping.}$$

Definition 2 ([2])**.** *The triplet $(X,F,\circ)$ is called an fuzzy metric space if it satisfies (FM0), (FM2), (FM3), (FM4), and (FM5).*

Later on, A. George and P. Veeramani modified the above definition to define a Hausdorff topology in an fuzzy metric space.

Definition 3 ([3])**.** *The triplet (X, F, ∘) is called an fuzzy metric space if it satisfies (FM1), (FM2), (FM3), (FM4), and (FM6).*

3. k-Fuzzy Metric Spaces

The idea of k-fuzzy metric spaces was given by Gopal et al. [8]. The main motivation behind this idea was to find the degree of nearness of two points $a, b \in X$ with respect to more than one parameters $p \in (0,\infty)$. In this section, we highlight some properties of such spaces.

Definition 4 ([8])**.** *Let X be a nonempty set, $\circ$ a continuous t-norm, k a positive integer, and F a fuzzy set on $X^2 \times (0,\infty)^k$. An ordered triple $(X,F,\circ)$ is called a k-fuzzy metric space if the following conditions are satisfied for all $a, b \in X$, $p, s > 0$, and $p_1, p_2, \ldots, p_k > 0$,*

$$
\begin{aligned}
& (KF1) : F(a,b,p_1,p_2,\ldots,p_k) > 0; \\
& (KF2) : F(a,b,p_1,p_2,\ldots,p_k) = 1 \iff a = b; \\
& (KF3) : F(a,b,p_1,p_2,\ldots,p_k) = F(b,a,p_1,p_2,\ldots,p_k); \\
& (KF4) : \text{ for any } l \in \{1,2,3,\ldots,k\}, \text{ we have} \\
& F(a,b,p_1,p_2,\ldots,p_{l-1},p,p_{l+1},\ldots,p_k) \circ F(b,c,p_1,p_2,\ldots,p_{l-1},s,p_{l+1},\ldots,p_k) \\
\leq\ & F(a,c,p_1,p_2,\ldots,p_{l-1},p+s,p_{l+1},\ldots,p_k); \\
& (KF5) : F(a,b,.) : (0,\infty)^k \to [0,1] \text{ is continuous mapping.}
\end{aligned}
$$

Definition 5 ([8])**.** *A k-fuzzy metric space $(X,F,\circ)$ is an l-natural k-fuzzy metric space if there exists $l \in \{1,2,\ldots,k\}$, such that*

$$\lim_{p_l \to \infty} F(a,b,p_1,p_2,\ldots,p_k) = 1, \ \forall\ a,b \in X.$$

For simplification, we denote $F(a,b,p_1,p_2,\ldots,p_k)$ by $F(a,b,p_i^k)$.

Proposition 1 ([8])**.** *Let $(X,F,\circ)$ be a k-fuzzy metric space, $p, p_1, p_2, \ldots, p_k > 0$. If $p_l < p$ for some $l \in \{1,2,\ldots,k\}$, then*

$$F(a,b,p_i^k) \leq F(a,b,p_1,p_2,...p_{l-1},p,p_{l+1},\ldots,p_k)$$

$\forall\ a,b \in X$.

Remark 1 ([8])**.** *In a k-fuzzy metric space $(X,F,\circ)$, if*

$$F(a, b, p_i^k) > 1 - \epsilon,$$

$\forall\ a, b \in X, p_1, p_2, \ldots, p_k > 0$, *and* $\epsilon \in (0, 1)$, *then for each* $l \in \{1, 2, \ldots, k\}$, *we can find* $p \in (0, p_l)$ *s.t*

$$F(a, b, p_1, p_2, \ldots, p_{l-1}, p, p_{l+1}, \ldots, p_k) > 1 - \epsilon.$$

When defining the k-fuzzy metric from an application standpoint, one should tend to consider the physical nature of the quantities. For instance, due to the various dimensions of these values, one is unable to use the formulas for the degree of nearness as given in the examples above if one evaluates the degree of the nearness of two points A and B in a space with respect to time and fuel required in going from A to B. An illustration of one such situation is provided below.

Example 1 ([8])**.** *Let d on X be the customary distance in the Euclidean space represented by $X = \mathbb{R}^3$. Assume that t_1 is the amount of time and t_2 is the amount of fuel used to travel from point A to point B in X. The 2- fuzzy metric F on $X^2 \times (0, \infty)^2$ provided by (1) can then be used to determine how close A and B are, subject to t_1 and t_2.*

$$F(a, b, t_1, t_2) = e^{-d(a,b)\left(\frac{q}{t_1} + \frac{w}{t_2}\right)}, \tag{1}$$

for all $a, b \in X$, $t_1, t_2 > 0$, where q and w are constants selected to have appropriate physical dimensions.

Definition 6 ([8])**.** *Let $(X, F, \circ)$ be a k-fuzzy metric space. A sequence $\{a_n\}$ in X is said to be convergent and converges to a point a in X iff for every real $\epsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ s.t*

$$F(a_n, a, p_i^k) > 1 - \epsilon$$

$for\ all\ n \geq n_0$ *and* $p_1, p_2, \ldots, p_k > 0$.

Lemma 1 ([8])**.** *Let $(X, F, \circ)$ be a k-fuzzy metric space. A sequence $\{a_n\}$ in X converges to $a \in X$ if and only if*

$$\lim_{n \to \infty} F(a_n, a, p_i^k) = 1\ \forall\ p_1, p_2, \ldots, p_k > 0.$$

Definition 7 ([8])**.** *Let $(X, F, \circ)$ be a k-fuzzy metric space, and $\{a_n\}$ be a sequence in X.*

(1) *$\{a_n\}$ is called an F-Cauchy sequence if for every $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that*

$$F(a_n, a_m, p_i^k) > 1 - \epsilon$$

$\forall n, m \geq n_0$ *and* $p_1, p_2, \ldots, p_k > 0$,

(2) *$\{a_n\}$ is called a G-Cauchy sequence if*

$$\lim_{n \to \infty} F(a_n, a_{n+p}, p_i^k) = 1$$

$\forall\ p_1, p_2, \ldots, p_k > 0$ *and* $p > 0$,

Remark 2. *The F-Cauchy sequence and G-Cauchy sequence are two different Cauchy sequences; see [13,14].*

Example 2. *Let* $(\mathbb{R}, d)$ *be a metric space,* $\circ$ *the product (minimum)* $[a \circ b = a.b$ *or* $a \circ b = min(a, b)]$ *t-norm,* $\omega > 0$*, and* k *a positive integer. Define a fuzzy set* F *on* $\mathbb{R}^2 \times (0, \infty)^k$ *by*

$$F(a, b, p_r^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + d(a, b)}$$

for all $a, b \in \mathbb{R}$ *and* $p_r > 0$*. Then,* F *is a k-fuzzy metric on* $\mathbb{R}$ *[8]. Consider* $Q_s = 1 + 1/2 + 1/3 + \ldots + 1/s$*, for* $s \in \mathbb{N}$*; then,*

$$F(Q_{s+p}, S_s, p_r^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + |Q_{s+p} - Q_s|}$$

$$= \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + 1/(s+1) + \ldots + 1/(s+p)}.$$

Thus, $F(Q_{s+p}, Q_s, p_r^k) \to 1$ *as* $n \to \infty$ *for all* $p > 0$*. Hence,* Q_s *is a G-Cauchy sequence but obviously, it is not an F-Cauchy sequence. Suppose on the contrary that* Q_s *is an F-Cauchy sequence; then,*

$$F(Q_m, Q_s, p_r^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + |Q_m - Q_s|}.$$

This implies that $\{Q_s\}$ *is F-Cauchy iff it is Cauchy in the standard metric space* $\mathbb{R}$*. We know that* $|Q_m - Q_s| \approx \ln(m/s)$*, and it grows much when* m *is very large as compared to* s*. We infer that* $\{Q_s\}$ *is not Cauchy in the standard metric space* $\mathbb{R}$ *and consequently, it is not F-Cauchy in a k-fuzzy metric space on* $\mathbb{R}$*.*

Definition 8 ([8])**.** *Let* $(X, F, \circ)$ *be a k-fuzzy metric space.*

(1) $(X, F, \circ)$ *is said to be F-complete if every F-Cauchy sequence in* X *converges to some* $a \in X$
(2) $(X, F, \circ)$ *is said to be G-complete if every G-Cauchy sequence in* X *converges to some* $a \in X$

Lemma 2. *The fuzzy metric function* $F(a, b, p_i^k)$ *is non-decreasing for all* $a, b \in X$*.*

Proof. Suppose on the contrary that for $0 < p_l < p_s$,

$$F(a, b, p_1, \ldots, p_{l-1}, p_l, p_{l+1}, \ldots, p_k) > F(a, b, p_1, \ldots, p_{l-1}, p_s, p_{l+1}, \ldots, p_k).$$

By keeping in mind the fact that $F(b, b, p_1, \ldots, p_{l-1}, p_s - p_l, p_{l+1}, \ldots, p_k) = 1$ and using (KF4), we have

$$\begin{aligned} F(a, b, p_1, \ldots, p_{l-1}, p_l, p_{l+1}, \ldots, p_k) \circ\ & F(b, b, p_1, \ldots, p_{l-1}, p_s - p_l, p_{l+1}, \ldots, p_k) \\ \leq\ & F(a, b, p_1, \ldots, p_{l-1}, p_s, p_{l+1}, \ldots, p_k) \\ <\ & F(a, b, p_1, \ldots, p_{l-1}, p_l, p_{l+1}, \ldots, p_k). \end{aligned}$$

This shows a contradiction. Hence, $p_l < p_s$ implies

$$F(a, b, p_1, \ldots, p_{l-1}, p_l, p_{l+1}, \ldots, p_k) < F(a, b, p_1, \ldots, p_{l-1}, p_s, p_{l+1}, \ldots, p_k).$$

□

Remark 3. *It follows from Lemma 2 that if for all* $a, b \in X,$ *for some* $p_l > 0,$ $F_l(a, b, p_1^k) > 1 - p_l$*, then* $a = b$*.*

Kramosil and Michalek [2] initiated the concept of fuzzy metric spaces. This work provides the basis for the construction of an fixed point theory in fuzzy metric spaces. Grabiec [4] initiated the fixed point theory in fuzzy metric spaces. He defined completeness on an fuzzy metric space (known as a G-complete fuzzy metric space), extended the Banach contraction principle to a G-complete fuzzy metric space, and proved the fuzzy Banach contraction theorem.

Theorem 1. *(Fuzzy Banach contraction theorem) [4]. Let* $(X, F, \circ)$ *be a complete fuzzy metric space with an additional property that*

$$\lim_{p \to \infty} F(a, b, p) = 1 \text{ for all } a, b \in X.$$

If the mapping $\alpha : X \to X$ *satisfies the following inequality*

$$F(\alpha(a), \alpha(b), kp) \geq F(a, b, p)$$

$\forall a, b \in X, 0 < k < 1$, *then* α *admits a unique fixed point.*

Following Grabiec's work, many authors obtained fixed point theorems for contractive mappings in fuzzy metric spaces. In the same context, Salvador [6] addressed discontinuous contractive mappings and established a new contraction principle in an fuzzy metric space.

Theorem 2 ([6])**.** *(Salvador's contraction theorem). Let* $(X, F, \circ)$ *be a complete fuzzy metric space. If the mapping* $\alpha : X \to X$ *satisfies the following inequality for* $c \in (0, 1), a, b \in X$, *and* $p > 0$

$$\min\{F(a, \alpha(a), p), F(b, \alpha(b), p)\} > 1 - p \Rightarrow F(\alpha(a), \alpha(b), p) > 1 - cp,$$

then α *admits a unique fixed point.*

Gopal et al. [8] generalized and extended the concept of fuzzy metric spaces by adopting the concept of degree of nearness of two points subject to more than one parameters and introduced the notion of k-fuzzy metric spaces. Gopal et al. [8] proved that a k-fuzzy metric space was a countable and Hausdorff topological space. Finally, by extending the idea of Grabiec, they initiated an fixed point theory in k-fuzzy metric spaces and obtained two important fixed point theorems.

For simplification, we write

$$F_l(\alpha(a), \alpha(b), p_1, p_2, \ldots, p_{l-1}, \lambda p_l, p_{l+1}, \ldots, p_k) = F_l^{1/\lambda}(\alpha(a), \alpha(b), p_i^k)$$

Theorem 3 ([8])**.** *Let* $(X, F, \circ)$ *be a G-complete k-fuzzy metric space and* $\alpha : X \to X$ *be a mapping satisfying the following condition:*

$$F_l^{1/\lambda}(\alpha(a), \alpha(b), p_i^k) \geq F(a, b, p_i^k),$$

for all $a, b \in X$, $p_1, p_2, \cdots, p_k > 0$, $l \in \{1, 2, \ldots, k\}$, *and* $\lambda \in (0, 1)$ *is a constant. Suppose that* $(X, F, \circ)$ *is an l-natural k-fuzzy metric space. Then,* α *admits a unique fixed point.*

Definition 9 ([8])**.** *Let* $(X, F, \circ)$ *be a k-fuzzy metric space. A mapping* $\alpha : X \to X$ *is called a k-fuzzy contraction mapping if*

$$\frac{1}{F(\alpha(a), \alpha(b), p_i^k)} - 1 \leq \lambda \left[\frac{1}{F(a, b, p_i^k)} - 1 \right],$$

for all $a, b \in X$, $p_1, p_2, \ldots, p_k > 0$, *where* $\lambda \in [0, 1)$ *is a constant*

Theorem 4 ([8])**.** *(k-Fuzzy contraction theorem) Let* $(X, F, \circ)$ *be a G-complete k-fuzzy metric space and* $\alpha : X \to X$ *a k-fuzzy contraction mapping. Then,* α *admits a unique fixed point.*

4. Fixed Points of $(1k)$ and $(1/2k)$-Fuzzy Contractions

In this section, we extend Gopal et al.'s idea to define $(1k)$ and $(1/2k)$-fuzzy contractions and prove an fixed point result in k-fuzzy metric spaces by eliminating the continuity requirement that was a necessary assumption for existing fixed point theorems in k-fuzzy metric spaces [8]. We provide an example to show that every (1k)-fuzzy contraction need not be (1/2k)-fuzzy contraction. We begin by the definition of a k-fuzzy contraction in a k-fuzzy metric space.

Let

$$F_l^{\frac{1}{c}}(\alpha(a),\alpha(b),p_i^k) = F(\alpha(a),\alpha(b),p_1,p_2,\ldots,p_{l-1},cp_l,p_{l+1},\cdots,p_k).$$

Definition 10. *Let $(X,F,\circ)$ be a k-fuzzy metric space. We say that mapping $\alpha : X \to X$ is a (1k)-fuzzy contraction on X if for any $a,b \in X$ and $p > 0$, there exist a constant $c \in (0,1)$ and $l \in \{1,2,\ldots,k\}$ such that*

$$\left.\begin{array}{l} F_l(a,\alpha(a),p_i^k) > 1-p_l \\ \textit{and} \\ F_l(b,\alpha(b),p_i^k) > 1-p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c}}(\alpha(a),\alpha(b),p_i^k) > 1-cp_l. \tag{2}$$

We say that a mapping $\alpha : X \to X$ is a $(1/2k)$-fuzzy contraction on X, if for any $a,b \in X$ and $p > 0$, there exist a constant $c \in \left(0,\frac{1}{2}\right)$ and $l \in \{1,2,\ldots,k\}$ such that

$$\left.\begin{array}{l} F_l(a,\alpha(a),p_i^k) > 1-p_l \\ \textit{and} \\ F_l(b,\alpha(b),p_i^k) > 1-p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c}}(\alpha(a),\alpha(b),p_i^k) > 1-cp_l. \tag{3}$$

Remark 4. *Every $(1/2k)$-fuzzy contraction is a $(1k)$-fuzzy contraction but the converse needs not be true.*

Now, we prove the main result of this paper.

Theorem 5. *Every (1k)-fuzzy contraction on a complete k-fuzzy metric space admits a unique fixed point.*

Proof. We know that $(X,F,\circ)$ is a complete k-fuzzy metric space and $\alpha : X \to X$ is a $(1k)$-fuzzy contraction; then, there exist $c \in (0,1)$ and $l \in \{1,2,\ldots,k\}$ such that

$$\left.\begin{array}{l} F_l(a,\alpha(a),p_i^k) > 1-p_l \\ \text{and} \\ F_l(b,\alpha(b),p_i^k) > 1-p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(a),\alpha(b),p_i^k) > 1-cp_l. \tag{4}$$

for all $a,b \in X$ and $p > 0$. By (2), we can infer that

$$\left.\begin{array}{l} F_l(a,\alpha(a),p_i^k) > 1-p_l \\ \text{and} \\ F_l(\alpha(a),\alpha^2 a,p_i^k) > 1-p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(a),\alpha^2 a,p_i^k) > 1-cp_l. \tag{5}$$

Replacing a with b in (5), we obtain

$$\left.\begin{array}{l} F_l(b,\alpha(b),p_i^k) > 1-p_l \\ \text{and} \\ F_l(\alpha(b),\alpha^2(b),p_i^k) > 1-p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(b),\alpha^2(b),p_i^k) > 1-cp_l. \tag{6}$$

By (5) and (6), we can infer that

$$\left.\begin{array}{l} F_l^{\frac{1}{c}}(\alpha(a),\alpha^2(a),p_i^k) > 1-cp_l \\ \text{and} \\ F_l^{\frac{1}{c}}(\alpha(b),\alpha^2(b),p_i^k) > 1-cp_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c^2}}(\alpha^2(a),\alpha^2(b),p_i^k) > 1-c^2p_l.$$

Repeating these steps n times, we have

$$\left.\begin{array}{l} F_l^{\frac{1}{c^{n-1}}}(\alpha^{n-1}(a),\alpha^n(a),p_i^k) > 1-c^{n-1}p_l \\ \text{and} \\ F_l^{\frac{1}{c^{n-1}}}(\alpha^{n-1}(b),\alpha^n(b),p_i^k) > 1-c^{n-1}p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c^n}}(\alpha^n(a),\alpha^n(b),p_i^k) > 1-c^np_l. \quad (7)$$

Let $a_0 \in X$. We define the sequence $\{a_n\}_{n\in\mathbf{N}}$ by

$$a_n = \alpha^n(a_0).$$

We show that $\{a_n\}$ is a Cauchy sequence in $(X, F, \circ)$.

For this purpose, for a given $\epsilon \in (0,1)$ and $p_l > 0$, there exists $K_0 \in \mathbf{N}$ such that $c^n p_l = \min\{\epsilon, p_l\}$ for all $n \geq K_0$. Suppose that $m > n \geq K_0$; then, we can have $m = n + o$ for some $o \in \mathbf{N}$, and consequently, we have the following information:

$$\begin{aligned} F(a_n,a_m,p_i^k) &= F(\alpha^n(a_0),\alpha^n\alpha^o(a_0),p_i^k) \\ &\geq F_l^{\frac{1}{c^n}}(\alpha^n(a_0),\alpha^n\alpha^o a_0,p_i^k) \\ &> 1-c^np_l \\ &> 1-\epsilon. \end{aligned}$$

This implies that $\{a_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $(X, F, \circ)$. By the completeness of $(X, F, \circ)$, we infer that there is a $z \in X$ s.t the sequence $\{a_n\}_{n\in\mathbf{N}}$ converges to z and by definition of a convergent sequence, there exists $K_0 \in \mathbb{N}$ s.t

$$F(a_n, z, p_i^k) > 1-\epsilon, \ \forall n \geq K_0.$$

Now, we show that z is an fixed point of α. Let $r, s > 0$ s.t $c < s < r < 1$. By induction, we show that for each $o \in \mathbf{N}$,

$$F_l^{\frac{1}{r^o}}(z,\alpha(z),p_i^k) > 1-r^op_l. \quad (8)$$

With no loss of generality, suppose that $r^o p_l \leq 1$ and for each $o \in \mathbb{N}$, define

$$A_{ors} = \left\{\epsilon \in (0,1) : \epsilon + sr^{o-1}p_l < r^op_l\right\}.$$

Let $o = 1$. Then, we have

$$F_l(z,\alpha(z),p_i^k) > 1-p_l.$$

and

$$F_l(a_n,a_{n+1},p_i^k) > 1-p_l. \quad (9)$$

By Lemma 2 and (4), we obtain

$$\begin{aligned} F_l^{1/s}(\alpha(z),a_{n+1},p_i^k) &\geq F_l^{1/c}(\alpha(z),a_{n+1},p_i^k) \\ &> 1-cp_l > 1-sp_l. \end{aligned}$$

Since $\{a_n\}_{n\in\mathbf{N}}$ converges to z for any $\epsilon \in A_{1rs}$, there exists $K_0 \in \mathbb{N}$ such that

$$F(z,a_n,p_i^k) > 1-\epsilon, \ \forall \ n \geq K_0.$$

Thus,

$$\begin{aligned} F_l^{1/r}(z,\alpha(z),p_i^k) &> F(z,\alpha(z),p_1,\ldots,p_{l-1},\epsilon+sp_l,p_{l+1},\ldots,p_k) \\ &\geq F(z,a_{K_0},p_1,\ldots,p_{l-1},\epsilon,p_{l+1},\ldots,p_k)\circ F(a_{K_0},\alpha(z),p_1,\ldots,p_{l-1},sp_l,p_{l+1},\ldots,p_k) \\ &> (1-\epsilon)\circ(1-sp_l)\geq(1-\epsilon)\circ(1-rp_l)\geq 1-rp_l. \end{aligned}$$

This implies that

$$F_l^{1/r}(z,\alpha(z),p_i^k)\geq 1-rp_l.$$

Therefore, the inequality in (8) holds for $o=1$. Now, suppose that the inequality in (8) holds for $o=j; j\in\mathbf{N}$, that is,

$$F_l^{1/r^j}(z,\alpha(z),p_i^k)\geq 1-r^jp_l,$$

and we show that the inequality in (8) is also true for $o=j+1$. Since $\{a_n\}$ is a Cauchy sequence, there exist $n_j\in\mathbf{N}$ s.t

$$F_l^{1/r^j}(a_n,a_{n+1},p_i^k)>1-r^jp_l,\ \forall\ n\geq n_j.$$

By (1), we obtain

$$F_l^{1/cr^j}(\alpha(z),a_{n+1},p_i^k)>1-cr^jp_l. \tag{10}$$

Since $s>c$, by (10), we have the following inequality:

$$F_l^{1/sr^j}(\alpha(z),a_{n+1},p_i^k)>1-sr^jp_l.$$

For any $\epsilon\in A_{(j+1)rs}$, we have $\epsilon+sr^jp_l<r^{j+1}p_l$, satisfying

$$F(z,a_{n(\epsilon)},p_i^k)>1-\epsilon,\ \forall\ n(\epsilon)\geq K_0.$$

$$\begin{aligned} F_l^{1/r^{j+1}}(z,\alpha(z),p_i^k) &> F(z,\alpha(z),p_1,\ldots,\epsilon+sr^jp_l,\ldots,p_k) \\ &\geq F(z,a_{n(\epsilon)},p_1,\cdots,\epsilon,\cdots,p_k)\circ F(a_{n(\epsilon)},\alpha(z),p_1,\ldots,sr^jp_l,\ldots,p_k) \\ &\geq (1-\epsilon)(1-sr^jp_l) \\ &\geq (1-\epsilon)(1-r^{j+1}p_l). \end{aligned}$$

This shows that

$$F_l^{1/r^{j+1}}(z,\alpha(z),p_i^k)\geq 1-r^{j+1}p_l.$$

Hence, the induction method assures that the inequality in (8) holds for all $o\in\mathbf{N}$. Now, for a given $p_l>0$, there exists $o\in\mathbf{N}$ s.t $r^op_l<p_l$, and we have

$$\begin{aligned} F_l(z,\alpha(z),p_i^k) &\geq F_l^{1/r^o}(z,\alpha(z),p_i^k) \\ &> 1-r^op_l>1-p_l. \end{aligned}$$

This implies that $\alpha(z)=z$; hence, z is an fixed point of α. Now, we show that z is unique. On the contrary, let u,z be two distinct fixed points. Then, by (2),

$$\left.\begin{array}{l} F(z,\alpha(z),p_i^k)=1 \\ \text{and} \\ F(u,\alpha(u),p_i^k)=1 \end{array}\right\}\Rightarrow F_l^{\frac{1}{c}}(\alpha(z),\alpha(u),p_i^k)>1-cp_l>1-p_l.$$

This implies $z=\alpha(z)=\alpha(u)=u$. This completes the proof. □

Corollary 1. *Every (1/2k)-fuzzy contraction on a complete fuzzy metric space admits a unique fixed point.*

We give an important example of a (1k)-fuzzy contraction for $c = 1/2$ that is not a (1/2k)-fuzzy contraction.

Example 3. *Let $A = [0, 3/2]$ and let d be a usual metric on A. Define a function $F^d : A^2 \times [0, \infty)^k \to [0, 1]$ by*

$$F^d(a, b, p_i^k) = \begin{cases} 1 & \text{if } d(a, b) < p_l, \\ 0 & \text{if } d(a, b) \geq p_l \text{ for some } \ l \in \{0, 1, \ldots, k\}. \end{cases}$$

Then, $(F^d, \wedge)$ is a k-fuzzy metric on A for a continuous t-norm $\wedge$, and hence $(A, F^d, \wedge)$ is a complete k-fuzzy metric space. Define $\alpha : A \to A$, as $\alpha(a) = 0$ for all $a \in [0, 1]$ and $\alpha(a) = a/3$ for all $a \in (1, 3/2]$. We first show that α is not a (1/2k)-fuzzy contraction. Choose $c \in (0, 1/2)$. Take $a = 0, b = 3/2$, and $p_l = 1/2c$, then $p_l > 1$, so

$$\min\left\{F^d(a, \alpha(a), p_i^k), F^d(b, \alpha(b), p_i^k)\right\} = 1 > 1 - p_l.$$

However,

$$F^{d1/c}(\alpha(a), \alpha(b), p_i^k) = F_{01l}^{d1/c}(0, 1/2, p_1, \ldots, 1/2, \ldots, p_k).$$

Since $d(a, b) = 1/2 < p_l$,

$$F^{d1/c}(\alpha(a), \alpha(b), p_i^k) = 0 < 1 - cp_l.$$

Thus, α is not a (1/2k)-fuzzy contraction on A. Now, we prove that α is a (1k)-fuzzy contraction for $c = 1/2$. We divide it into three cases:

Case 1. *$a, b \in [0, 1]l$; then, for any $p_l > 0$,*

$$\begin{aligned} F^{d1/c}(\alpha(a), \alpha(b), p_l^k) &= F^{d1/c}(\alpha(a), \alpha(b), p_1, \ldots p_1/2, \ldots, p_k) \\ &= 1 \\ &> 1 - cp_l. \end{aligned}$$

Case 2. *$a \in [0, 1], b \in (1, 3/2]$; if $p_l > 1$, we have $b/3 \leq 1/2 < p_l/2$, so*

$$\begin{aligned} F^{d1/c}(\alpha(a), \alpha(b), p_1, \ldots, cp_l, \ldots, p_k) &= F^{d1/c}(0, b/3, p_1, \ldots, p_l/2, \ldots, p_k) \\ &= 1 \\ &> 1 - cp_l. \end{aligned}$$

If $p_l \leq 1$ and assuming that

$$\min\left\{F^d(a, \alpha(a), p_i^k), F^d(b, \alpha(b), p_i^k)\right\} > 1 - p_l,$$

(otherwise (Definition 10) has no meaning)

we deduce that $F^d(a, 0, p_i^k) = F^d(b, b/3, p_i^k) = 1$ and $(a < p_l, 2b/3 < p_l)$, so we obtain

$$\begin{aligned} F^{d1/c}(\alpha(a), \alpha(b), p_1, \ldots, cp_l, \ldots p_k) &= F^{d1/c}(0, b/3, p_1, \ldots, p_l/2, \ldots, p_k) \\ &= 1 \\ &> 1 - cp_l. \end{aligned}$$

Case 3. *Let $a, b \in (1, 3/2]$; we assume that $a > b$; if $p > 1$, we have $(a - b)/3 < a/3 \leq 1/2 < p_l/2$. Thus,*

$$\begin{aligned} F^{d1/c}(\alpha(a), \alpha(b), p_1, \ldots, cp_l, \ldots, p_k) &= F^{d1/c}(a/3, b/3, p_1, \ldots, p_l/2, \ldots, p_k) \\ &= 1 \\ &> 1 - cp_l. \end{aligned}$$

Now, if $p \leq 1$ and assuming that

$$\min\Big\{F^d(a,\alpha(a),t_i^k), F^d(b,\alpha(b),t_i^k)\Big\} > 1 - p_l,$$

(otherwise (Definition 10) has no meaning) we deduce that $F^d(a,a/3,p_i^k) = F^d(b,b/3,p_i^k) = 1$ *and* $(2a/3 < p_l, 2b/3 < p_l)$*, so we obtain*

$$\begin{aligned} F^{d1/c}(\alpha(a),\alpha(b),p_1,\ldots,cp_l,\ldots,p_k) &= F^{d1/c}(a/3,b/3,p_1,\ldots,p_l/2,\ldots,p_k) \\ &= 1 \\ &> 1 - cp_l. \end{aligned}$$

Thus, (2) is satisfied for all cases, and hence α is a (1k)-fuzzy contraction on A.

5. Fixed Points of $(1c)$ and $(1/2c)$-Fuzzy Contractions

In this section, we extend Gopal et al.'s idea to define $(1c)$ and $(1/2c)$-fuzzy contractions and prove a related fixed point result in k-fuzzy metric spaces.

Definition 11. *Let $(X,F,\circ)$ be a k-fuzzy metric space. We say that mapping $\alpha : X \to X$ is a (1c)-fuzzy contraction on X if for any $a,b \in X$ and $p > 0$, there exists a constant $c \in (0,1)$ and $l \in \{1,2,\ldots,k\}$ such that*

$$\left.\begin{array}{l} F_l(a,\alpha(b),p_i^k) > 1 - p_l \\ \text{and} \\ F_l(b,\alpha(a),p_i^k) > 1 - p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c}}(\alpha(a),\alpha(b),p_i^k) > 1 - cp_l. \tag{11}$$

We say that a mapping $\alpha : X \to X$ is a $(1/2c)$-fuzzy contraction on X, if for any $a,b \in X$ and $p > 0$, there exists a constant $c \in \left(0,\frac{1}{2}\right)$ and $l \in \{1,2,\ldots,k\}$ such that

$$\left.\begin{array}{l} F_l(a,\alpha(b),p_i^k) > 1 - p_l \\ \text{and} \\ F_l(b,\alpha(a),p_i^k) > 1 - p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c}}(\alpha(a),\alpha(b),p_i^k) > 1 - cp_l. \tag{12}$$

Proposition 2. *If α is a (1c) contraction on k-fuzzy metric space $(X,F,\circ)$, then the sequence $\{a_n\}$ defined by $a_n = \alpha^n(a_0)$ for $a_0 \in X$ is a Cauchy sequence.*

Proof. Let $(X,F,\circ)$ be a k-fuzzy metric space and $\alpha : X \to X$ a $(1c)$-fuzzy contraction; then, there exist $c \in (0,1)$ and $l \in \{1,2,\ldots,k\}$ such that

$$\left.\begin{array}{l} F_l(a,\alpha(b),p_i^k) > 1 - p_l \\ \text{and} \\ F_l(b,\alpha(a),p_i^k) > 1 - p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(a),\alpha(b),p_i^k) > 1 - cp_l. \tag{13}$$

for all $a,b \in X$ and $p > 0$. By (12), we can infer that

$$\left.\begin{array}{l} F_l(b,\alpha(a),p_i^k) > 1 - p_l \\ \text{and} \\ F_l(\alpha(b),\alpha^2(a),p_i^k) > 1 - p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(a),\alpha^2(a),p_i^k) > 1 - cp_l. \tag{14}$$

Replacing a with b in (13), we obtain

$$\left.\begin{array}{l} F_l(a,\alpha(b),p_i^k) > 1 - p_l \\ \text{and} \\ F_l(\alpha(a),\alpha^2(b),p_i^k) > 1 - p_l \end{array}\right\} \Rightarrow F_l^{1/c}(\alpha(b),\alpha^2(b),p_i^k) > 1 - cp_l. \tag{15}$$

By (14) and (15), we can infer that

$$\left.\begin{array}{l} F_l^{\frac{1}{c}}(\alpha(a),\alpha^2(a),p_i^k) > 1-cp_l \\ \text{and} \\ F_l^{\frac{1}{c}}(\alpha(b),\alpha^2(b),p_i^k) > 1-cp_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c^2}}(\alpha^2(a),\alpha^2(b),p_i^k) > 1-c^2p_l.$$

Repeating these steps n times, we have

$$\left.\begin{array}{l} F_l^{\frac{1}{c^{n-1}}}(\alpha^{n-1}(a),\alpha^n(a),p_i^k) > 1-c^{n-1}p_l \\ \text{and} \\ F_l^{\frac{1}{c^{n-1}}}(\alpha^{n-1}(b),\alpha^n(b),p_i^k) > 1-c^{n-1}p_l \end{array}\right\} \Rightarrow F_l^{\frac{1}{c^n}}(\alpha^n(a),\alpha^n(b),p_i^k) > 1-c^np_l. \quad (16)$$

Let $a_0 \in X$ be an arbitrary initial guess. We define the sequence $\{a_n\}_{n\in\mathbf{N}}$ by

$$a_n = \alpha^n(a_0).$$

We show that $\{a_n\}$ is a Cauchy sequence in $(X, F, \circ)$.

For this purpose, for a given $\epsilon \in (0,1)$, and $p_l > 0$, there exists $K_0 \in \mathbf{N}$ such that $c^n p_l = \min\{\epsilon, p_l\}$ for all $n \geq K_0$. Suppose that $m > n \geq K_0$; then, we can have $m = n + o$ for some $o \in \mathbf{N}$, and consequently, we have the following information:

$$\begin{array}{rcl} F(a_n, a_m, p_i^k) & = & F(\alpha^n(a)_0, \alpha^n\alpha^o a_0, p_i^k) \\ & \geq & F_l^{\frac{1}{c^n}}(\alpha^n a_0, \alpha^n\alpha^o a_0, p_i^k) \\ & > & 1 - c^n p_l \\ & > & 1 - \epsilon. \end{array}$$

This shows that $\{a_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $(X, F, \circ)$. □

Remark 5. *Let (X, d_F) be a metric space and $(X, F, \circ)$ a k-fuzzy metric space. For each $a, b \in X, l \in \{1, 2, \ldots, k\}$. Put*

$$d_F(a,b) = Sup\Big\{p_l \geq 0 : F(a,b,p_i^k) \leq 1 - p_l\Big\}.$$

Then, d_M satisfies the following condition

$$d_F(a,b) < p_l \Rightarrow F(a,b,p_i^k) > 1 - p_l \text{ for all } p > 0.$$

Proposition 3. *Every (1/2c) contraction on a complete k-fuzzy metric space has a unique fixed point but its converse may or may not be true.*

Proof. Let α be a $(1/2c)$ contraction (with $c \in (0, 1/2)$) on a complete k-fuzzy metric space. We show that α is a classical Chatterjea contraction on the complete metric space as constructed in Remark 5. Indeed, let $a, b \in X$ and $p_l \geq 0, l \in \{1, 2, \ldots, k\}$ satisfy $F(\alpha(a), \alpha(b), p_i^k) \leq 1 - p_l$. Then,

$$\min\Big\{F_l^c(a,\alpha(b),p_i^k), F_l^c(b,\alpha(a),p_i^k)\Big\} \leq 1 - p_l/c.$$

Thus, by Remark 5

$$\max\{d_F(a,\alpha(b)), d_F(b,\alpha(a))\} \geq p_l/c.$$

Hence, $p \leq c \max\{d_F(a,\alpha(b)), d_F(b,\alpha(a))\}$. Thus,

$$\begin{aligned} & d_F(\alpha(a),\alpha(b)) \\ = \;& \sup\Big\{p \geq 0 : F(\alpha(a),\alpha(b),p_i^k) \leq 1 - p_l\Big\} \\ \leq \;& c \max\{d_F(a,\alpha(b)), d_F(b,\alpha(a))\} \\ \leq \;& c[d_F(a,\alpha(b)) + d_F(b,\alpha(a))]. \end{aligned}$$

We have proved that α is a classical Chatterjea contraction on (X, d_F), so it has a unique fixed point. Now, conversely, since α is a classical Chatterjea contraction, the metric space associated with this is complete, but we cannot tell anything about the k-fuzzy metric space. □

Remark 6. *The results discussed in (Theorem 5) and (Proposition 3) weaken the conditions assumed in [8] by considering discontinuous mappings. Moreover, the results presented in [6] are a special case of (Theorem 5) and (Proposition 3) $(k = 1)$.*

Example 4. *Let (A, d) be a metric space where $A = [0, 1]$ and $d(a, b) = |a - b|$ for all $a \in A$. Define $F^d : A^2 \times [0, \infty)^k \to [0, 1]$ by*

$$F^d(a,b,p_i^k) = \begin{cases} 1 & \text{if } \; d(a,b) < p_l \\ 0 & \text{if } \; d(a,b) \geq p_l. \end{cases}$$

and fix $a_0 \in (0, 1/2)$. Define $\alpha : A \to A$ by

$$\alpha(a) = \begin{cases} a_0 & \text{if } \; a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We check that α is a (1/2c) contraction on $(A, F, \circ)$ with $c = a_0$. Indeed, let $a, b \in A, p > 0$, and $l = \{1, 2, \ldots, k\}$ such that

$$F^d(a,\alpha(b),p_i^k) > 1 - p_l \text{ and } F^d(b,\alpha(a),p_i^k) > 1 - p_l.$$

Suppose that $F^{d/c}(\alpha(a),\alpha(b),p_i^k) = 0 < 1 - cp_l$. Then, $d(\alpha(a),\alpha(b)) \geq cp_l = a_0 p_l$. There are two cases:

Case 1. $a = 1, b = [0, 1)$.

Since $d(\alpha(a),\alpha(b)) = d(\alpha(1),\alpha(b)) = d(a_0, 0) = a_0$, by $d(\alpha(a),\alpha(b)) \geq a_0 p_l$, we have $a_0 \geq a_0 p_l$. Thus, $p_l \leq 1$. Since, by hypothesis, $F^d(a,\alpha(b),p_i^k) > 1 - p_l$, we deduce that $F^d(a,\alpha(b),p_i^k) = 1$, thus $d(a,\alpha(b)) = d(1,0) = 1 < p_l$ is a contradiction.

Case 2. $b = 1, a = [0, 1)$.

By $d(\alpha(a),\alpha(b)) \geq a_0 p_1$, we have $p_l \leq 1$. Since, by hypothesis, $F_{d,01}(b,\alpha(a),p_i^k) > 1 - p_l$, we deduce that $F^d(b,\alpha(a),p_i^k) = 1$, thus $d(b,\alpha(a)) = d(1,0) = 1 < p_l$ is a contradiction. We conclude that $F^{d/c}(\alpha(a),\alpha(b),p_i^k) = 1 > 1 - cp_l$, so α is a $(1/2c)$ contraction on $(A, F, \circ)$.

6. Fixed Points of Generalized *k*-Fuzzy Contractions

In this section, we extend Gopal et al.'s idea to define a generalized (k)-fuzzy contraction and prove a related fixed point result in k-fuzzy metric spaces.

Definition 12. Let X be a nonempty set, F a fuzzy set, and T_{min} a minimum t-norm; then, (X, F, T_{min}) is a k-fuzzy metric space. We say that the mapping $h : X \to X$ is a generalized (k)-fuzzy contraction if there exist $q \in (0, 1/2)$ and $l \in \{1, 2, \ldots, k\}$ such that

$$F(ha,hb,p_i^k) \geq \min\left\{ \begin{array}{l} F_l^q(a,b,p_i^k), F_l^q(ha,a,p_i^k), \\ F_l^q(hb,b,p_i^k), F_l^q(a,hb,p_i^k), \\ F_l^q(ha,b,p_i^k) \end{array} \right\} \tag{17}$$

for all $a, b \in X,$ and $p_l > 0$.

Before presenting the proof of the main result, we need the following lemma.

Lemma 3. *Let $\{a_n\}$ be a sequence in k-fuzzy metric space (X, M, T_{min}). If there exist $q \in (0,1), p_l > 0, n \in \mathbb{N}$, and $l \in \{1,2,\ldots,k\}$ such that*

$$F(a_n, a_{n+1}, p_i^k) \geq F_l^q(a_{n-1}, a_n, p_i^k) \tag{18}$$

and

$$\begin{aligned} &\lim_{n\to\infty} T_{i=n}^{\infty} F(a_0, a_1, p_1, p_2, \ldots, p_{l-1}, 1/\mu^i, p_{l+1}, \ldots, p_k) \\ = \;& 1, \end{aligned} \tag{19}$$

then $\{a_n\}$ is a Cauchy sequence provided $\mu \in (0,1)$.

Proof. Let $\nu \in (q,1)$ and $p_l > 0$, then $\sum_{i=1}^{\infty} \nu^i < \infty$. Therefore, there exists $n_0 = n_0(p_l)$, such that $\sum_{i=n_0}^{\infty} \nu^i \leq p_l$. Clearly, (17) implies that

$$F(a_n, a_{n+1}, p_i^k) \geq F_l^{q^n}(a_0, a_1, p_i^k).$$

For $n \geq n_0, n \in \mathbb{N}$, we have

$$\begin{aligned} & F(a_n, a_{n+m}, p_i^k) \\ \geq\; & F\begin{pmatrix} a_n, a_{n+m}, p_1, p_2, \ldots, p_{l-1}, \\ \sum_{i=n}^{\infty} \nu^i, p_{l+1}, \ldots, p_k \end{pmatrix} \\ \geq\; & F\begin{pmatrix} a_n, a_{n+m}, p_1, p_2, \ldots, p_{l-1}, \\ \sum_{i=n}^{n+m-1} \nu^i, p_{l+1}, \ldots, p_k \end{pmatrix} \\ \geq\; & \underbrace{T(T(\ldots T}_{(m-1)-times}\begin{pmatrix} F\begin{pmatrix} a_n, a_{n+1}, p_1, \ldots, \\ p_{l-1}, \nu^n, p_{l+1}, \ldots, p_k \end{pmatrix} \\ , \ldots, \\ F\begin{pmatrix} a_{n+m-1}, a_{n+m}, p_1, \ldots, \\ p_{l-1}, \nu^{n+m-1}, p_{l+1}, \ldots, p_k \end{pmatrix} \end{pmatrix})) \\ \geq\; & \underbrace{T(T(\ldots T}_{(m-1)-times}\begin{pmatrix} F\begin{pmatrix} a_0, a_1, p_1, \ldots, \\ p_{l-1}, \nu^n/q^n, p_{l+1}, \ldots, p_k \end{pmatrix} \\ , \ldots, \\ F\begin{pmatrix} a_0, a_1, p_1, \ldots, p_{l-1}, \\ \nu^{n+m-1}/q^{n+m-1}, \\ p_{l+1}, \ldots, p_k \end{pmatrix} \end{pmatrix})). \end{aligned}$$

Let $\mu = q/\nu \in (0,1)$. Then,

$$\begin{aligned} & F(a_n, a_{n+m}, p_i^k) \\ \geq\; & T_{i=n}^{n+m-1} F(a_0, a_1, p_1 \ldots p_{l-1}, 1/\mu^i, p_{l+1}, \ldots, p_k) \\ \geq\; & T_{i=n}^{\infty} F(a_0, a_1, p_1 \ldots p_{l-1}, 1/\mu^i, p_{l+1}, \ldots, p_k). \end{aligned}$$

By using (18)

$$\lim_{n\to\infty} F(a_n, a_{n+m}, p_i^k) = 1.$$

Hence, $\{a_n\}$ is a Cauchy sequence. □

Theorem 6. *Let $X \neq \emptyset$ and $h : X \to X$ be a generalized (k)-fuzzy contraction defined on a complete k-fuzzy metric space (X, F, T_{min}). Suppose that there exists $a_0 \in X$ such that*

$$\begin{aligned} & \lim_{n\to\infty} T_{i=n}^{\infty} F(a_0, ha_0, p_1, \ldots, p_{l-1}, 1/\mu^i, p_{l+1}, p_k) \\ = \; & 1, \mu \in (0,1). \end{aligned} \tag{20}$$

Then, h has a unique fixed point.

Proof. Let $a_0 \in X$ and $\{a_n\}$ be a Picard iterative sequence given by

$$a_n = h(a_{n-1}).$$

Substituting $a = a_{n-1}, b = a_n$ in (17), we have

$$\begin{aligned} F(a_n, a_{n+1}, p_i^k) &\geq \min \left\{ \begin{array}{l} F_l^q(a_{n-1}, a_n, p_i^k), \\ F_l^q(a_n, a_{n-1}, p_i^k), \\ F_l^q(a_{n+1}, a_n, p_i^k), \\ F_l^q(a_{n-1}, a_{n+1}, p_i^k), \\ F_l^q(a_n, a_n, p_i^k) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} F_l^q(a_{n-1}, a_n, p_i^k), \\ F_l^q(a_n, a_{n-1}, p_i^k), \\ \min \left\{ \begin{array}{l} F_l^{2q}(a_{n-1}, a_n, p_i^k), \\ F_l^{2q}(a_n, a_{n+1}, p_i^k) \end{array} \right\} \end{array} \right\} \end{aligned}$$

$p_l > 0, n \in \mathbb{N}$ and $l \in \{1, 2, \ldots, k\}$, if we choose

$$\begin{aligned} & \min\left\{ F_l^{2q}(a_{n-1}, a_n, p_i^k), F_l^{2q}(a_n, a_{n+1}, p_i^k) \right\} \\ = \; & F_l^{2q}(a_n, a_{n+1}, p_i^k), \end{aligned}$$

we obtain

$$F(a_n, a_{n+1}, p_i^k) \geq F_l^{2q}(a_n, a_{n+1}, p_i^k),$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$, $p_l > 0$ and $2q \in (0, 1/2)$

$$F(a_n, a_{n+1}, p_i^k) \geq F_l^{2q}(a_{n-1}, a_n, p_i^k).$$

By (Lemma 3), it follows that a_n is a Cauchy sequence. Since (X, F, T_{min}) is complete, there exists $a^\circ \in X$ such that $\lim_{n\to\infty} a_n = a^\circ$. We show that a° is a fixed point. Suppose on the contrary that $ha^\circ \neq a^\circ$. Put $a = a_n, b = a^\circ$ in (17)

$$F(a_{n+1}, ha^\circ, p_i^k) \geq \min \left\{ \begin{array}{l} F_l^q(a_n, a^\circ, p_i^k), \\ F_l^q(a_{n+1}, a_n, p_i^k), \\ F_l^q(ha^\circ, a^\circ, p_i^k), \\ F_l^q(a_n, ha^\circ, p_i^k), \\ F_l^q(ha_n, a^\circ, p_i^k) \end{array} \right\}.$$

By applying limit $n \to \infty$, we obtain for $p_l > 0$,

$$F(a^\circ, ha^\circ, p_i^k) \geq F_l^q(a^\circ, ha^\circ, p_i^k).$$

This is a contradiction. Hence, a° is an fixed point of h.

Uniqueness: Suppose that a° and b° are fixed points for h; then, by (19), for $p_l > 0$

$$F(ha^\circ, hb^\circ, p_i^k) \geq \min \left\{ \begin{array}{l} F_l^q(a^\circ, b^\circ, p_i^k), \\ F_l^q(ha^\circ, a^\circ, p_i^k), \\ F_l^q(hb^\circ, b^\circ, p_i^k), \\ F_l^q(a^\circ, hb^\circ, p_i^k), \\ F_l^q(ha^\circ, b^\circ, p_i^k) \end{array} \right\}.$$

Then, $F(a^\circ, b^\circ, p_i^k) \geq F_l^q(a^\circ, b^\circ, p_i^k)$. Thus, $a^\circ = b^\circ$. □

7. Common Fixed Point for k-Fuzzy Kannan Contraction

In this section, we generated a fixed point and a common fixed point solution for a k-fuzzy kannan contraction in the context of k-fuzzy metric spaces.

Definition 13. *Let $(X, F, \circ)$ be a complete k-fuzzy metric space. A self-mapping $Q : X \to X$ is called a k-fuzzy Kannan contraction of type-I if*

$$\frac{1}{F(Qa, Qb, p_i^k)} - 1 \leq \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(a, Qa, p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(b, Qb, p_i^k)} - 1\right) \end{array} \right\}$$

with $\alpha + \beta < 1/2$, for all $a, b \in X$, $p_l > 0$, $l \in \{1, 2, 3, \dots, k\}$

The primary outcome about the common fixed point for two single-valued mappings Q, R within the context of k-fuzzy metric spaces is as follows.

Theorem 7. *Suppose $(X, F, \circ)$ is a complete k-fuzzy metric space. Let $Q, R : X \to X$ be self-mappings s.t*

$$\frac{1}{F(Qa, Rb, p_i^k)} - 1 \leq \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(a, Qa, p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(b, Rb, p_i^k)} - 1\right) \end{array} \right\} \tag{21}$$

for $\alpha, \beta \in [0, 1/2)$, for all $(a, b) \in X \times X$. Then, Q and R have at most a common fixed point in X.

Proof. Suppose $a_0 \in X$, and define the sequence a_n by

$$Qa_{2j} = a_{2j+1} \text{ and } Ra_{2j+1} = a_{2j+2} \tag{22}$$

for $j = 0, 1, 2, \dots$ Using (21) and (22), we can write

$$\begin{aligned} & \left(\frac{1}{F(a_{2j+1}, a_{2j+2}, p_i^k)} - 1\right) \\ = & \left(\frac{1}{F(Qa_{2j}, Ra_{2j+1}, p_i^k)} - 1\right) \\ \leq & \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(a_{2j}, Qa_{2j}, p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(a_{2j+1}, Ra_{2j+1}, p_i^k)} - 1\right) \end{array} \right\} \\ = & \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(a_{2j}, a_{2j+1}, p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(a_{2j+1}, a_{2j+2}, p_i^k)} - 1\right) \end{array} \right\} \end{aligned}$$

This implies

$$
\begin{aligned}
&(1-\beta)\left\{\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right\} \\
\leq\ & \alpha\left\{\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right\} \\
\Rightarrow\ & \left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right) \\
\leq\ & \left(\frac{\alpha}{1-\beta}\right)\left(\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right) \\
=\ & \lambda\left(\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right)
\end{aligned}
$$

where $\left(\frac{\alpha}{1-\beta}\right)=\lambda<1$, since $\alpha,\beta\in(0,1/2)$. Similarly,

$$
\begin{aligned}
&\left(\frac{1}{F(a_{2j+2},a_{2j+3},p_i^k)}-1\right) \\
<\ & \left(\frac{\alpha}{1-\beta}\right)\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right) \\
=\ & \lambda\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right)
\end{aligned}
$$

Continuing in this manner, we obtain

$$\left(\frac{1}{F(a_n,a_{n+1},p_i^k)}-1\right)<\lambda\left(\frac{1}{F(a_{n-1},a_n,p_i^k)}-1\right)$$

for all $n\in\mathbf{N}$

Repeating the above steps, we deduce

$$
\begin{aligned}
\left(\frac{1}{F(a_n,a_{n+1},p_i^k)}-1\right) &< \lambda\left(\frac{1}{F(a_{n-1},a_n,p_i^k)}-1\right) \\
&< \lambda^2\left(\frac{1}{F(a_{n-2},a_{n-1},p_i^k)}-1\right) \\
&\ \ \vdots \\
&< \lambda^n\left(\frac{1}{F(a_0,a_1,p_i^k)}-1\right)
\end{aligned}
$$

Hence,

$$\left(\frac{1}{F(a_n,a_{n+1},p_i^k)}-1\right)<\lambda^n\left(\frac{1}{F(a_0,a_1,p_i^k)}-1\right) \tag{23}$$

$n \in \mathbf{N}$. Taking (23) into account, we deduce

$$\begin{aligned}
&\sum_{k=n}^{m-1}\left\{\frac{1}{F(a_k, a_{k+1}, p_i^k)} - 1\right\} \\
= &\left\{\begin{array}{l}\left(\frac{1}{F(a_n,a_{n+1},p_i^k)} - 1\right) \\ +\left(\frac{1}{F(a_{n+1},a_{n+2},p_i^k)} - 1\right) \\ +\ldots+\left(\frac{1}{F(a_{m-1},a_m,p_i^k)} - 1\right)\end{array}\right\} \\
< &\left\{\begin{array}{l}\lambda^n\left(\frac{1}{F(a_0,a_1,p_i^k)} - 1\right) \\ +\lambda^{n+1}\left(\frac{1}{F(a_0,a_1,p_i^k)} - 1\right) \\ +\ldots+\lambda^{m-1}\left(\frac{1}{F(a_0,a_1,p_i^k)} - 1\right)\end{array}\right\} \\
< &\ \lambda^n(1+\lambda+\ldots+\lambda^{m-n-1})\left(\frac{1}{F(a_0, a_1, p_i^k)} - 1\right) \\
\leq &\ \frac{\lambda^n}{1-\lambda}\left(\frac{1}{F(a_0, a_1, p_i^k)} - 1\right), \quad m > n
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{\lambda^n}{1-\lambda}\right) = 0$, we must have

$$\lim_{n\to\infty}\sum_{k=n}^{m-1}\left\{\frac{1}{F(a_k, a_{k+1}, p_i^k)} - 1\right\} = 0$$

Hence,

$$\lim_{n\to\infty} F(a_k, a_{k+1}, p_i^k) = 1$$

for all $p_l > 0$. This means that a_n is a Cauchy sequence in X. Since $(X, F, \circ)$ is complete, there exists $c \in X$ such that

$$\lim_{n\to\infty} F(a_n, v, p_i^k) = 1$$

To prove c is an fixed point of Q, assume $F(Qv, v, p_i^k) > 0$.

$$\begin{aligned}
\left(\frac{1}{F(Qv, a_{2j+2}, p_i^k)} - 1\right) &= \left(\frac{1}{F(Qv, Ra_{2j+1}, p_i^k)} - 1\right) \\
&\leq \left\{\begin{array}{l}\alpha\left(\frac{1}{F(v,Qv,p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(a_{2j+1},Ra_{2j+1},p_i^k)} - 1\right)\end{array}\right\} \\
&= \left\{\begin{array}{l}\alpha\left(\frac{1}{F(v,Qv,p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)} - 1\right)\end{array}\right\} \\
&\to \alpha\left(\frac{1}{F(v, Qv, p_i^k)} - 1\right),
\end{aligned}$$

as $n \to \infty$. We arrive at

$$\begin{aligned} &\lim_{n\to\infty}\sup\left(\frac{1}{F(Qv, x_{2j+2}, p_i^k)} - 1\right) \\ \leq \quad &\alpha\left(\frac{1}{F(v, Qv, p_i^k)} - 1\right) \end{aligned}$$

$p_l > 0$

$$\Rightarrow (1-\alpha)\left(\frac{1}{F(v, Qv, p_i^k)} - 1\right) < 0$$

Since $\alpha + \beta < 1, F(v, Qv, p_i^k) = 1$, i.e., $Sv = v$. Similarly, we suppose that $\frac{1}{F(v,Rv,p_i^k)} - 1 > 0$, and we have

$$\begin{aligned} \left(\frac{1}{F(Rv, a_{2j+1}, p_i^k)} - 1\right) &= \left(\frac{1}{F(Rv, Qa_{2j}, p_i^k)} - 1\right) \\ &\leq \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(x_2j, Q_2j, p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(v,Rv,p_i^k)} - 1\right) \end{array} \right\} \\ &\to \beta\left(\frac{1}{F(v, Rv, p_i^k)} - 1\right), \end{aligned}$$

as $n \to \infty$. We arrive at

$$\begin{aligned} &\lim_{n\to\infty}\sup\left(\frac{1}{F(Rv, x_{2j+1}, p_i^k)} - 1\right) \\ \leq \quad &\beta\left(\frac{1}{F(v, Rv, p_i^k)} - 1\right) \end{aligned}$$

$p_l > 0$

$$\Rightarrow (1-\beta)\left(\frac{1}{F(v, Rv, p_i^k)} - 1\right) < 0$$

Since $\beta < 1, F(v, Rv, p_i^k) = 1$, i.e., $Rv = v$.

Now, let $u \in X$ be any fixed point of Q and R, i.e., $u \neq v$. Then,

$$\begin{aligned} \left(\frac{1}{F(u, v, p_i^k)} - 1\right) &= \left(\frac{1}{F(Qu, Rv, p_i^k)} - 1\right) \\ &\leq \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(u,Qu,p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(v,Rv,p_i^k)} - 1\right) \end{array} \right\} \\ &= \left\{ \begin{array}{l} \alpha\left(\frac{1}{F(u,u,p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(v,v,p_i^k)} - 1\right) \end{array} \right\} \\ &= 0. \end{aligned}$$

$$\Rightarrow F(u, v, p_i^k) = 1 \text{ and } u = v.$$

□

The convergence result for a type-I k-fuzzy Kannan contraction for a single-valued mapping Q is given below.

Corollary 2. *Suppose* $(X, F, \circ)$ *is a complete k-fuzzy metric space. Let* $Q : X \to X$ *be a self-mapping such that*

$$\frac{1}{F(Qa, Qb, p_i^k)} - 1 \leq \left\{ \begin{array}{c} \alpha\left(\frac{1}{F(a,Qa,p_i^k)} - 1\right) \\ +\beta\left(\frac{1}{F(b,Qb,p_i^k)} - 1\right) \end{array} \right\}$$

with $\alpha + \beta < 1/2$*, for all* $\alpha, \beta > 0, (a, b) \in X \times X, p_l > 0, l \in \{1, 2, 3, \ldots, k\}$*. Then, there is only one fixed point in* X *for map* T*.*

The common fixed point solution that is given below is based on the k-fuzzy Kannan contraction of type II within a k-fuzzy metric setup for two mappings Q and R.

Corollary 3. *Suppose* $(X, F, \circ)$ *is a complete k-fuzzy metric space. Let* $Q, R : X \to X$ *be self-mappings s.t*

$$\frac{1}{F(Qa, Rb, p_i^k)} - 1 \leq \gamma \left(\begin{array}{c} \frac{1}{F(a,Qa,p_i^k)} - 1 \\ +\frac{1}{F(b,Rb,p_i^k)} - 1 \end{array} \right) \tag{24}$$

for all $(a, b) \in X \times X$*. Then, Q and R have at most one common fixed point in* X*.*

Proof. Suppose a_0 is any point in X, and define the sequence a_n by

$$Qa_{2j} = a_{2j+1} \text{ and } Ra_{2j+1} = a_{2j+2} \tag{25}$$

for $j = 0, 1, 2, \ldots$ Using (24) and (25), we can write

$$\begin{aligned} & \left(\frac{1}{F(a_{2j+1}, a_{2j+2}, p_i^k)} - 1\right) \\ = & \left(\frac{1}{F(Qa_{2j}, Ra_{2j+1}, p_i^k)} - 1\right) \\ \leq & \gamma \left\{ \begin{array}{c} \left(\frac{1}{F(a_{2j},Qa_{2j},p_i^k)} - 1\right) \\ +\left(\frac{1}{F(a_{2j+1},Ra_{2j+1},p_i^k)} - 1\right) \end{array} \right\} \\ = & \gamma \left\{ \begin{array}{c} \left(\frac{1}{F(a_{2j},a_{2j+1},p_i^k)} - 1\right) \\ +\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)} - 1\right) \end{array} \right\} \end{aligned}$$

This implies

$$
\begin{aligned}
&(1-\gamma)\left\{\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right\} \\
\leq\ &\gamma\left\{\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right\} \\
\Rightarrow\ &\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right) \\
<\ &\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right) \\
=\ &\gamma'\left(\frac{1}{F(a_{2j},a_{2j+1},p_i^k)}-1\right)
\end{aligned}
$$

where $\left(\frac{\gamma}{1-\gamma}\right)=\gamma'<1/2$. Similarly,

$$
\begin{aligned}
&\left(\frac{1}{F(a_{2j+2},a_{2j+3},p_i^k)}-1\right) \\
<\ &\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right) \\
=\ &\gamma'\left(\frac{1}{F(a_{2j+1},a_{2j+2},p_i^k)}-1\right)
\end{aligned}
$$

The solution is the same as in the theorem to derive the common fixed point of the maps Q and R. □

The convergence result of a single mapping T for a k-fuzzy Kannan contraction of type II is given below.

Theorem 8. *Suppose $(X,F,\circ)$ is a complete k-fuzzy metric space. Let $R:X\to X$ be self-mappings. Suppose that $\gamma\in[0,1/2)$ such that*

$$
\frac{1}{F(Ra,Rb,p_i^k)}-1\leq\gamma\left(\begin{array}{l}\frac{1}{F(a,Ra,p_i^k)}-1\\+\frac{1}{F(b,Rb,p_i^k)}-1\end{array}\right) \tag{26}
$$

for all $(a,b)\in X\times X$. Then, R has at most one common fixed point in X.

8. Existence of a Solution of Fractional Differential Equations

Physical systems having continuous distributions or interactions can be modeled and analyzed with the help of fractional differential equations or FDEs. They are often used to describe phenomena in more depth than differential equations can or to determine connections between numbers in engineering research. They provide a structure for understanding complex behaviors and interactions found in a range of engineering systems. There are several uses for implicit differential equations in engineering research, especially FDEs. This section establishes the existence of distinct FDE solutions in a k-fuzzy environment. There are several uses for these kinds of differential equations across numerous fields. Let us begin by going over the fundamental vocabulary used in fractional calculus. The Riemann–Liouville fractional derivative of order $n>0$ for a function $\rho\in C[0,1]$ is written as follows:

$$
(\Gamma(k-n))^{-1}\frac{d^k}{dg^k}\int_0^g\frac{\rho(\sigma)d\sigma}{(g-\sigma)^{n-k+1}}=D^n\rho(\sigma),
$$

Let us now consider the following FDE

$$^{\sigma}D^{\mu}\rho(g) + f(g, \rho(g)) = 0, \rho(0) = 0 = \rho(1), \quad (27)$$

$g \in [0,1]$ and $\mu \in (1,2]$; where f is a continuous function on $[0,1] \times \mathbb{R}$, and $^{\sigma}D^{\mu}$ is the Caputo fractional derivative having order μ, defined by

$$^{\sigma}D^{\mu} = \Gamma(k-\mu))^{-1} \int_0^g \frac{\rho^{(k)}(\sigma)d\sigma}{(g-\sigma)^{1+\mu-k}}$$

Denote Y the space $C([0,1], \mathbb{R})$ of all continuous functions taken on the interval $[0,1]$. Define a metric d on Y by

$$d(\rho, \rho^*) = \sup_{g \in [0,1]} |\rho(g) - \rho^*(g)|, \ \ \rho, \rho^* \in Y$$

Then, (Y, d) is a complete MS. Then, binary operation $\circ$ is defined by the product $\bullet - norm(P) : P(a, b) = a \cdot b$.

A standard k-fuzzy metric F is given by

$$\frac{1}{F(\rho, \rho^*, p_i^k)} - 1 = \frac{d(\rho, \rho^*)}{\omega \prod_i^k p},$$

for $p_l > 0, l \in \{1, 2, 3, \ldots, k\}$ and $\rho, \rho^* \in C$. Then, it can be easily verified that F is triangular and $(C, F, \circ)$ is a complete fuzzy metric space.

Theorem 9. *Consider the nonlinear FDE (27). If the following conditions are met,*

(i) *For $g \in [0,1]$ and $\rho, \rho^8 \in Y$, the following is true*

$$|f(g, \rho) - f(g, \rho^*)| \le |\rho(g) - T\rho(g)| + |\rho^*(g) - T\rho^*(g)|;$$

(ii) *There exits $r, 3 < r$, with*

$$\frac{1}{r} \ge \sup_{g \in [0,1]} \int_0^1 \delta(g, \sigma) d\sigma.$$

Then, FDE (27) has necessarily at most one solution in Y.

Proof. The equivalent IE for FDE (27) is the following

$$\rho(g) = \int_0^1 \delta(g, \sigma) f(\sigma, \rho(\sigma)) d\sigma,$$

for all $\rho \in Y$ and $g \in [0,1]$, where

$$\delta(g, \sigma) = \begin{cases} \frac{[g(1-\sigma)]^{\mu-1} - (g-\sigma)^{\mu-1}}{\Gamma(\mu)} & \text{if } 0 \le \sigma \le g \le 1 \\ \frac{g(1-\sigma)]^{\mu-1}}{\Gamma(\mu)} & \text{if } 0 \le g \le \sigma \le 1 \end{cases}$$

If the map $T : Y \to Y$ defined by

$$T\rho(g) = \int_0^1 \delta(g, \sigma) f(\sigma, \rho(\sigma)) d\sigma,$$

where $\rho^* \in Y$ is an fixed point, then ρ^* is a solution of Equation (27). Taking into account the given conditions, for $\rho, \rho^* \in Y$, we infer

$$\begin{aligned}
d(T\rho, T\rho^*) &= \sup_{0\leq g\leq 1} |T\rho(g) - T\rho^*(g)| \\
&= \sup_{0\leq g\leq 1} \left| \begin{array}{l} \int_0^1 \delta(g,\sigma) f(\sigma, \rho(\sigma)) d\sigma \\ - \int_0^1 \delta(g,\sigma) f(\sigma, \rho^*(\sigma)) d\sigma \end{array} \right| \\
&\leq \sup_{0\leq g\leq 1} \left\{ \int_0^1 |\delta(g,\sigma)| d\sigma . |f(\sigma,\rho(\sigma)) - f(\sigma,\rho^*(\sigma))| \right\} \\
&\leq \frac{1}{r} \sup_{0\leq g\leq 1} |f(g,\rho) - f(g,\rho^*)| \\
&\leq \frac{1}{r} \sup_{0\leq g\leq 1} [|\rho(g) - T\rho(g)| + |\rho^*(g) - T\rho^*(g)|].
\end{aligned}$$

This shows that

$$d(T\rho, T\rho^*) \leq \frac{1}{r}[d(\rho, T\rho) + d(\rho^*, T\rho^*)]$$

Using $1/r = \gamma < 1/2$, we can write

$$d(T\rho, T\rho^*) \leq \gamma[d(\rho, T\rho) + d(\rho^*, T\rho^*)]$$

The above expression can be written as

$$\frac{1}{F(T\rho, T\rho^*, p_i^k)} - 1 \leq \gamma \left\{ \begin{array}{l} \left(\frac{1}{F(\rho, T\rho, p_i^k)} - 1 \right) \\ + \left(\frac{1}{F(\rho^*, T\rho^*, p_i^k)} - 1 \right) \end{array} \right\} \tag{28}$$

for all $\rho, \rho^* \in Y$. This shows that T satisfies the k-fuzzy contraction of Theorem (8). Hence, T admits a unique fixed point in Y, implying that FDE (27) has a unique solution. □

9. Conclusions

We developed a new iterative method to show that the constructed sequence subject to a new fuzzy contraction was a Cauchy sequence. We adopted the new iterative method to obtain the fixed point's existence results for different k-fuzzy contractions. This methodology is useful to write new results for advanced k-fuzzy contractions. The main fixed point results proved in this research article state the necessary conditions for the existence of fixed points of mappings eliminating the continuity requirements in k-fuzzy metric spaces. The significance of these results is that they do not require a continuity condition unlike many conventional fixed points that require continuity as prerequisite. It is important to note that the study in k-fuzzy metric spaces is more versatile than that in fuzzy metric spaces. We know that the fuzzy distance between two points is determined by how close they are in relation to one parameter p, but the distance in a k-fuzzy metric space is determined by k parameters. Thus, the results proved in this article can provide a base for new research in k-fuzzy metric spaces and their application provides valuable tools for studying numerical methods, nonlinear analysis, and other scientific and technical fields. As an application, we applied the result to obtain the solution of an FDE.

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References

1. Zadeh, L.A. Fuzzy Sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Kramosil, I.; Michalek, J. Fuzzy metric and statistical metric spaces. *Kybernetica* **1975**, *15*, 326–334.
3. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
4. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389. [CrossRef]
5. Reich, S. Kannan's fixed point theorem. *Boll. Un. Mat. Ital.* **1971**, *4*, 1–11.
6. Romaguerra, S. A Fixed Point Theorem of Kannan Type That Characterizes Fuzzy Metric Completeness. *Filomat* **2020**, *34*, 4811–4819. [CrossRef]
7. Youunis, M.; Arfah, A.N.A. Noval Fuzzy contractions and Applications to Engineering Science. *Fractal Fractions* **2024**, *8*, 28. [CrossRef]
8. Gopal, D.; Wutiphol, S.; Abhay, S.; Satish, S. The investigation of k-fuzzy metric spaces with the contraction principle in such spaces. *Soft Comput.* **2023**, *27*, 11081–11089. [CrossRef]
9. Anton-Sancho, A. Fixed points of Principal E_6-Bundels over a Compact Algerbric Curve. *Quaest. Math.* **2024**, *47*, 501–513. [CrossRef]
10. Anton-Sancho, A. Fixed Points of Involutions of G-Higgs Bundle Moduli Spaces over a Compact Reimann Surface with Classical Complex Structure Group. *Front. Math.* **2024**. [CrossRef]
11. Anton-Sancho, A. Fixed Points of Automorphism of Vector Bundle Moduli Space over a compact Reimann Surface. *Mediterr. J. Math.* **2023**, *21*, 20. [CrossRef]
12. Schweizer, B.; Sklar, A. Statistical Metric Spaces. *Pac. J. Math.* **1960**, *10*, 313–334. [CrossRef]
13. Gregori, V.; Sapena, A. On fixed point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252. [CrossRef]
14. Vasuki, R.; Veeramani, P. Fixed point theorems and Cauchy sequences in fuzzy metric spaces. *Fuzzy Sets Syst.* **2003**, *135*, 415–417. [CrossRef]