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Theoretical Investigation of Fractional Estimations in Liouville–Caputo Operators of Mixed Order with Applications

Pshtiwan Othman Mohammed [1,2,*], Alina Alb Lupas [3,*], Ravi P. Agarwal [4], Majeed A. Yousif [5], Eman Al-Sarairah [6,7] and Mohamed Abdelwahed [8]

1 Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq
2 Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq
3 Department of Mathematics and Computer Science, University of Oradea, 410087 Oradea, Romania
4 Department of Mathematics and Systems Engineering, Florida Institute of Technology, Melbourne, FL 32901, USA
5 Department of Mathematics, College of Education, University of Zakho, Duhok 42001, Iraq; majeed.yousif@uoz.edu.krd
6 Department of Mathematics, Khalifa University of Science and Technology, Abu Dhabi P.O. Box 127788, United Arab Emirates
7 Department of Mathematics, Al-Hussein Bin Talal University, P.O. Box 20, Ma'an 71111, Jordan
8 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
* Correspondence: pshtiwansangawi@gmail.com (P.O.M.); dalb@uoradea.ro (A.A.L.)

Abstract: In this study, to approximate nabla sequential differential equations of fractional order, a class of discrete Liouville–Caputo fractional operators is discussed. First, some special functions are re-called that will be useful to make a connection with the proposed discrete nabla operators. These operators exhibit inherent symmetrical properties which play a crucial role in ensuring the consistency and stability of the method. Next, a formula is adopted for the solution of the discrete system via binomial coefficients and analyzing the Riemann–Liouville fractional sum operator. The symmetry in the binomial coefficients contributes to the precise approximation of the solutions. Based on this analysis, the solution of its corresponding continuous case is obtained when the step size p_0 tends to 0. The transition from discrete to continuous domains highlights the symmetrical nature of the fractional operators. Finally, an example is shown to testify the correctness of the presented theoretical results. We discuss the comparison of the solutions of the operators along with the numerical example, emphasizing the role of symmetry in the accuracy and reliability of the numerical method.

Keywords: Liouville–Caputo fractional differences; approximation methods; mixed order fractional models

MSC: 26A48; 26A51; 33B10; 39A12; 39B62

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1. Introduction

Since the 1990s, fractional calculus (in both continuous and discrete versions) has become an outstanding topic in applied mathematics research. The research work on fractional calculus has continued to deepen with the advance in technology (see earlier studies [1–6]). In the last few decades, many authors have studied fractional differential equations and fractional systems formed by Riemann–Liouville (R-L) or Liouville–Caputo (L-C) operators. In addition, these studies focus on the existence and uniqueness of their solutions including initial and boundary conditions (see [7–11]).

Discrete fractional operators arising from general fractional calculus arise in a wide range of applications and they have been of continued interest in applied mathematics for over a century. In very recent years, discrete fractional operators have been widely used to discretize fractional order models for both linear and nonlinear systems of equations. These discretizations are often expressed via fractional differences and sums (see, e.g., [5,12]). Numerous applications of discrete fractional equations may also be found in modeling mathematical issues, such as mathematical analysis [13,14], mathematical physics [15–17], uncertainty theory [18,19], stability analysis [20,21], and monotonicity and positivity analyses [22–26].

Based on the current research articles, it is known that effective and important operators for analyzing the stability, existence, and uniqueness of fractional order systems are R-L and L-C fractional operators (see, for example, [27–30]). Of relevant interest are the steady states of such systems, where non-integer orders may appear, playing an important role in the behavior of the solutions of the R-L or L-C problems (see, for example, [31–36] and the references therein).

Recently, a fractional difference technique was used by Mozyrska et al. [37] to solve a fractional differential problem of delta L-C type with specific initial value conditions:

$$\begin{aligned}\left({}^{LC}_{(\alpha-1)p_0}\Delta^{\alpha}_{p_0}\mathrm{w}\right)(n\,p_0) &= y(n\,p_0+(\beta-1)p_0),\\ \left({}^{LC}_{(\beta-1)p_0}\Delta^{\beta}_{p_0}y\right)(n\,p_0) &= f\left(n\,p_0,\mathrm{w}(n\,p_0+(\alpha-1)p_0)\right),\end{aligned}$$

with

$$\begin{aligned}\left({}^{LC}_{(\alpha-1)p_0}\Delta^{\alpha}_{p_0}\mathrm{w}\right)(0) &= \mathrm{w}_0,\\ \mathrm{w}((\alpha-1)p_0) &= \mathrm{w}_1,\end{aligned}$$

where $\alpha,\beta\in(0,1]$, $p_0>0$, w_0 and w_1 are constant vectors in $\mathbb{R}$.

However, the basic concepts of solving nabla L-C fractional problems still lack adequate research. Thus, in this article, we present a numerical solution of a nabla L-C fractional system by using the nabla fractional difference technique.

The innovative contents of this article are summarized below. Section 2 briefly reviews the RL and L-C fractional operators and presents some useful existing concepts. Section 3 contains the main contribution of this article, where an L-C approximation algorithm based on fractional derivatives is introduced, and the foundation of its solution and uniqueness is examined by using the Lipschitz condition (LiC). Numerical problems are implemented to illustrate the efficiency and accuracy of the proposed scheme in Section 4.

2. Preliminaries

In this section, we present a comprehensive introduction to R-L fractional operators of order $\alpha>0$ and some related properties. Throughout the article, we suppose that $x_1(np_0)=n\,p_0-p_0$ and $x_2(np_0)=n\,p_0+p_0$.

Definition 1 (see [1,38]). *The R-L integral operator is defined by*

$$({}_aI^{\alpha}f)(t):=\frac{1}{\Gamma(\alpha)}\int_a^t(t-r)^{\alpha-1}f(r)dr\quad\text{for } a<t, \tag{1}$$

and its discrete version is given by

$$\left({}_a\nabla^{-\alpha}_{p_0}f\right)(t):=\frac{p_0}{\Gamma(\alpha)}\sum_{r=\frac{a}{p_0}+1}^{\frac{t}{p_0}}(t+p_0-rp_0)^{\overline{\alpha-1}}_{p_0}f(rp_0)\quad\text{for } t\in\mathbb{N}_{a+p_0,p_0}, \tag{2}$$

where $\mathbb{N}_{a,p_0}$ and $t^{\overline{\alpha}}_{p_0}$ are given by

$$\mathbb{N}_{a,p_0} := \{a, a+p_0, a+2p_0, \dots\}, \tag{3}$$

$$t_{p_0}^{\overline{\alpha}} = {p_0}^{\alpha} \frac{\Gamma\left(\frac{t}{p_0}+\alpha\right)}{\Gamma\left(\frac{t}{p_0}\right)} = \Gamma(\alpha+1)\binom{\frac{t}{p_0}+\alpha-1}{\frac{t}{p_0}-1} {p_0}^{\alpha}, \tag{4}$$

For $\frac{t}{p_0}, \alpha \in \mathbb{R}$ *such that neither* $\frac{t}{p_0}+\alpha$ *nor* $\frac{t}{p_0}$ *is a pole of* Γ*. Furthermore, we recall the binomial formula*

$$\binom{\beta}{\alpha} := \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)}. \tag{5}$$

Remark 1. *By considering Definition 1, it can be noted that*

$$\begin{aligned}\left({}_{a}\nabla_{p_0}^{-\alpha} f\right)(t) &= \frac{{p_0}^{\alpha}}{\Gamma(\alpha)} \sum_{s=0}^{n} \frac{\Gamma(n-s+\alpha)}{\Gamma(n-s+1)} f(a+sp_0+p_0) \\ &= {p_0}^{\alpha} \sum_{s=0}^{n} \binom{n-s-(-\alpha)-1}{n-s} f(a+sp_0+p_0) \\ &= {p_0}^{\alpha} \sum_{s_0=0}^{n} (-1)^{s_0} \binom{-\alpha}{s_0} f(a+p_0-s_0p_0),\end{aligned}$$

For $t = a+(n+\alpha)p_0$, $n \in \mathbb{N}_0$*, and it is seen that (see [39])*

$$\binom{s_0-\alpha-1}{s_0} = (-1)^{s_0}\binom{\alpha}{s_0}, \tag{6}$$

The following definitions present the R-L fractional operators associated with (1) and (2), respectively.

Definition 2 (see [1,23,40])**.** *For* $\alpha \in [0,1)$*, the R-L fractional derivative is defined by*

$$\left({}_{a}^{RL}D^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{a}^{t} (t-r)^{-\alpha} f(r)dr \quad \text{for } t > a, \tag{7}$$

and the nabla R-L fractional difference is defined by

$$\left({}_{a}^{\mathrm{RL}}\nabla_{p_0}^{\alpha} f\right)(t) = \frac{p_0}{\Gamma(-\alpha)} \sum_{r=\frac{a}{p_0}+1}^{\frac{t}{p_0}} (t+p_0-r\,p_0)_{p_0}^{\overline{-\alpha-1}} f(r\,p_0), \tag{8}$$

For $t \in \mathbb{N}_{a+m\,p_0,p_0}$*, where* $m-1<\alpha<m$*,* $m \in \mathbb{N}_1$*. It is worth noting that*

$$t_{p_0}^{\overline{\alpha}} = \frac{\Gamma\left(\frac{t}{p_0}+\alpha\right)}{\Gamma\left(\frac{t}{p_0}\right)} {p_0}^{\alpha}, \tag{9}$$

such that $t_{p_0}^{\overline{\alpha}} \to 0$ *when* $\Gamma\left(\frac{t}{p_0}\right)$ *is undefined.*

Specifically, the nabla difference operator can be expressed as follows:

$$(\nabla_{p_0} f)(t) = \frac{1}{p_0}\{f(t)-f(t-p_0)\} \quad \text{for } t \in \mathbb{N}_{a+p_0,p_0}.$$

Below, we recall the corresponding L-C derivative and difference associated with (1) and (2), respectively.

Definition 3 (see [1,23,40] [Lemma 2.3]). *Assume that $\alpha \in [0,1)$. Then, the L-C fractional derivative can be written as*

$$\left({}^{LC}_{a}D^{\alpha} f\right)(t) = \int_a^t \frac{(t-r)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{dr} f(r) dr$$

$$= \lim_{p_0 \to 0} \frac{1}{{p_0}^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{p_0}\right]+1} (-1)^j \binom{\alpha}{j} f(t - j p_0) - \frac{(t-a)^{\alpha}}{\Gamma(1-\alpha)} f(a). \tag{10}$$

And the nabla Liouville–Caputo p_0 fractional difference can be written as

$$\left({}^{LC}_{a}\nabla^{\alpha}_{p_0} f\right)(t) = \frac{p_0}{\Gamma(-\alpha)} \sum_{r=\frac{a}{p_0}+1}^{\frac{t}{p_0}} (t + p_0 - r\, p_0)_{p_0}^{\overline{-\alpha-1}} f(r\, p_0) - \frac{(t-a)_{p_0}^{\overline{-\alpha}}}{\Gamma(1-\alpha)} f(a), \tag{11}$$

for t in $\mathbb{N}_{a+p_0,p_0}$.

Lemma 1 (see [23] [Property 2.1]). *The following power difference formula can be deduced:*

$$\nabla_{p_0}\left(t_{p_0}^{\overline{\alpha}}\right) = \alpha\, t_{p_0}^{\overline{\alpha-1}},$$

for non-negative values of α, p_0, and t in $\mathbb{N}_{0,p_0}$.

Lemma 2 (see [5] [Theorem 3.93]). *For $\alpha > 0$ and $\beta \in \mathbb{R}$, one can have*

$${}_{a_0}\nabla^{-\alpha}_{p_0} (t-a_0)_{p_0}^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} (t-a_0)_{p_0}^{\overline{\beta+\alpha}}, \tag{12}$$

and

$${}^{RL}_{a_0}\nabla^{\alpha}_{p_0} (t-a_0)_{p_0}^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} (t-a_0)_{p_0}^{\overline{\beta-\alpha}}, \tag{13}$$

where $t \in \mathbb{N}_{a_0,p_0}$, and $\beta+\alpha, \beta-\alpha \geq 0$.

Lemma 3 (see [5] [Theorem 3.107]). *Let f be defined on $\mathbb{N}_{a_0+p_0,p_0}$ with $\alpha, h, \beta > 0$. Then,*

$$\left[{}_{a_0}\nabla^{-\alpha}_{p_0}\left({}_{a_0}\nabla^{-\beta}_{p_0} f\right)\right](t) = \left({}_{a_0}\nabla^{-(\alpha+\beta)}_{p_0} f\right)(t) = \left[{}_{a_0}\nabla^{-\beta}_{p_0}\left({}_{a_0}\nabla^{-\alpha}_{p_0} f\right)\right](t), \tag{14}$$

where $t \in \mathbb{N}_{a_0,p_0}$.

Lemma 4. *Suppose that f is defined on $\mathbb{N}_{a_0,p_0}$, and $0 < \alpha \leq 1$, $p_0 > 0$. Then, we have*

$$\left[{}_{a_0}\nabla^{-\alpha}_{p_0}\left({}^{LC}_{a_0}\nabla^{\alpha}_{p_0} f\right)\right](t) = f(t) - f(a_0). \tag{15}$$

Proof. This proof can be deduced from the following identity (see [38] [Proposition 6]):

$$\left[{}_{a_0}\nabla^{-\alpha}_{p_0}\left({}^{RL}_{a_0}\nabla^{\alpha}_{p_0} f\right)\right](t) = f(t),$$

using Definition (11) and identity (12). □

Next, we consider binomial special functions, which are defined in [41].

Definition 4 (see [41]). *For $\alpha, \beta > 0$, we define $\psi_{\kappa,s}$ and $\tilde{\psi}_{\kappa,s}$ as follows:*

$$\psi_{\kappa,s}(n\,p_0) := \begin{cases} \binom{n-\kappa+\alpha\kappa+\beta s}{n-\kappa} {p_0}^{\alpha\kappa+\beta s}, & \text{when } n \in \mathbb{N}_\kappa, \\ 0, & \text{when } n \notin \mathbb{N}_\kappa, \end{cases}$$

and

$$\tilde{\psi}_{\kappa,s}(n\,p_0) := \begin{cases} \binom{n+\mu-1}{n} {p_0}^{\beta}, & \text{for } n \in \mathbb{N}_0, \\ 0, & \text{for } n \notin \mathbb{N}_0, \end{cases}$$

for $n, \kappa, s \in \mathbb{N}_0$, $\alpha, \beta > 0$ *and* $\mu = \alpha\kappa + \beta s$.

Remark 2 (see [41]). *The above special function has some major properties:*

(i) $\psi_{0,0}(n\,p_0) = 1$;

(ii) $\psi_{1,0}(n\,p_0) = \left({}_0\nabla_{p_0}^{-\alpha}\, 1\right)(n\,p_0) = \binom{n+\alpha-1}{n-1} {p_0}^{\alpha}$
and $\psi_{0,1}(n\,p_0) = \left({}_0\nabla_{p_0}^{-\beta}\, 1\right)(n\,p_0) = \binom{n+\beta-1}{n-1} {p_0}^{\beta}$;

(iii) $\psi_{2,0}(n\,p_0) = \left({}_0\nabla_{p_0}^{-\alpha}\, 1\right)(n\,p_0 - p_0) = \binom{n+\alpha-2}{n-2} {p_0}^{\alpha}$;

(iv) $\psi_{\kappa,s}(n\,p_0) = \dfrac{((n-\kappa+1)p_0)^{\overline{\alpha\kappa+\beta s}}}{\Gamma(\alpha\kappa+\beta s+1)}$.

(v) *For* $0 < \alpha, \beta \leq 1$, *and* $p_0 > 0$, *we have*

$$\left({}_0\nabla_{p_0}^{-\alpha}\psi_{\kappa,s}\right)(x_1(np_0)) = \psi_{\kappa+1,s}(n\,p_0), \tag{16}$$

and

$$\left({}_0\nabla_{p_0}^{-\beta}\psi_{\kappa,s}\right)(n\,p_0) = \psi_{\kappa,s+1}(n\,p_0). \tag{17}$$

(vi) *It can be noted, for* $n = 0$, *that*

$$\psi_{\kappa,s}(0) = 0,$$

for $\kappa > 0$.

3. Solution of Difference Systems

In this section, our target is to approximate the following system of difference equations:

$$\begin{aligned} \left({}^{LC}_{0}\nabla_{p_0}^{\alpha}\mathrm{w}\right)(x_1(np_0)) &= y(x_2(np_0)), \\ \left({}^{LC}_{0}\nabla_{p_0}^{\beta}y\right)(n\,p_0) &= f\big(x_2(np_0), \mathrm{w}(x_1(np_0))\big), \end{aligned} \tag{18}$$

subject to the conditions

$$\begin{aligned} \left({}^{LC}_{0}\nabla_{p_0}^{\alpha}\mathrm{w}\right)(-2p_0) &= \mathrm{w}_0, \\ \mathrm{w}(-p_0) &= \mathrm{w}_1, \end{aligned} \tag{19}$$

where $\alpha, \beta \in (0,1]$, $p_0 > 0$, w_0 and w_1 are constant vectors in $\mathbb{R}$.

Theorem 1. *System* (18) *with* (19) *has the solution*

$$\mathrm{w}(x_1(np_0)) = \mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\alpha}g\right)(x_1(np_0)), \tag{20}$$

where $g(n\,p_0) = \left({}_0\nabla_{p_0}^{-\beta} F\right)(x_2(np_0))$ *and* $F(x_2(np_0)) = f\big(x_2(np_0), \mathrm{w}(x_1(np_0))\big)$*, for* $n \in \mathbb{N}_0$*.*

Proof. By considering Lemma 4 with $t = x_2(np_0), x_1(np_0)$, one can have

$$\begin{aligned}\left[{}_0\nabla_{p_0}^{-\beta}\left({}_0^{LC}\nabla_{p_0}^{\beta} y\right)\right](x_2(np_0)) &= y(x_2(np_0)) - y(0)\\ &= \left({}_0^{LC}\nabla_{p_0}^{\alpha} \mathrm{w}\right)(x_1(np_0)) - \mathrm{w}_0,\end{aligned} \tag{21}$$

and

$$\left[{}_0\nabla_{p_0}^{-\alpha}\left({}_0^{LC}\nabla_{p_0}^{\alpha} \mathrm{w}\right)\right](x_1(np_0)) = \mathrm{w}(x_1(np_0)) - \mathrm{w}_1. \tag{22}$$

It follows from (21) that

$$\left({}_0^{LC}\nabla_{p_0}^{\alpha} \mathrm{w}\right)(x_1(np_0)) = \mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\beta} F\right)(x_2(np_0)),$$

and it follows from this, Remark 2 (ii), and (22) that

$$\begin{aligned}\mathrm{w}(x_1(np_0)) &= \mathrm{w}_1 + \left({}_0\nabla_{p_0}^{-\alpha}\left[\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\beta} F\right)(x_2(np_0))\right]\right)(x_1(np_0))\\ &= \mathrm{w}_1 + \left({}_0\nabla_{p_0}^{-\alpha} 1\right)(x_1(np_0))\,\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\alpha} g\right)(x_1(np_0))\\ &= \mathrm{w}_1 + \binom{n-2+\alpha}{n-2} {p_0}^{\alpha}\,\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\alpha} g\right)(n\,p_0)\\ &= \mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\alpha} g\right)(x_1(np_0)),\end{aligned}$$

where $g(n\,p_0) = \left({}_0\nabla_{p_0}^{-\beta} F\right)(x_2(np_0))$ and $F(x_2(np_0)) = f\big(x_2(np_0), \mathrm{w}(x_1(np_0))\big)$, for $n \in \mathbb{N}_0$. Hence, the proof is completed. □

Remark 3. *Following* [42] [Theorem 4.1], *we can deduce that system* (18) *has a unique solution (i.e.,* (20) *is unique) such that* y *and* f *satisfy the Lipschitz condition, i.e.,*

$$\Big|y(n_1) - y(n_2)\Big| \le L_1|n_1 - n_2|,$$

and

$$\Big|f(n,\mathrm{w}) - f(n_2,\tilde{\mathrm{w}})\Big| \le L_2|\mathrm{w} - \tilde{\mathrm{w}}|,$$

for some positive constant L_1 *and* L_2*.*

In the following theorem, we examine the solution of a particular case of system (18).

Theorem 2. *Systems* (18) *and* (19) *have the following solution:*

$$\mathrm{w}(x_1(np_0)) = \sum_{\kappa=0}^{\infty} A^{\kappa}\left(\psi_{\kappa,\kappa}\,\mathrm{w}_1 + \psi_{\kappa+2,\kappa}\,\mathrm{w}_0\right)(n\,p_0), \tag{23}$$

where $f\big(x_1(np_0), \mathrm{w}(x_1(np_0))\big) = A\,\mathrm{w}(x_1(np_0))$*, and* $n \in \mathbb{N}_0$*.*

Proof. First, we try to investigate (23). In the case of $n = 0$, it follows from (20) that

$$\mathrm{w}(-p_0) = \mathrm{w}_1 + \psi_{2,0}(0)\,\mathrm{w}_0 + \left({}_0\nabla_{p_0}^{-\alpha} g\right)(-p_0) = \mathrm{w}_1,$$

which is equivalent to

$$\mathrm{w}(-p_0) = \big(\psi_{0,0}\,\mathrm{w}_1 + \psi_{2,0}\,\mathrm{w}_0\big)(0) + \underbrace{\sum_{\kappa=1}^{\infty} A^{\kappa}\,\big(\psi_{\kappa,\kappa}\,\mathrm{w}_1 + \psi_{\kappa+1,\kappa}\,\mathrm{w}_0\big)(0)}_{=0},$$

according to Remark 2 (vi) and the convention that

$$\Big({}_0\nabla_{p_0}^{-\alpha} g\Big)(-p_0) = \frac{p_0}{\Gamma(\alpha)} \sum_{r=1}^{-1} (-rp_0)_{p_0}^{\overline{\alpha-1}} g(rp_0) = 0.$$

Next, for $n > 0$, we define

$$\mathrm{w}_{m+1}(x_1(np_0)) = \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + \Big({}_0\nabla_{p_0}^{-\alpha} g_m\Big)(n\,p_0),$$

where $g_m(n\,p_0) = \Big({}_0\nabla_{p_0}^{-\beta} F_m\Big)(x_2(np_0))$ and $F_m(x_2(np_0)) = A\,\mathrm{w}_m(x_1(np_0))$, for $m \in \mathbb{N}_0$.
Then, we see that

$$\begin{aligned}
\mathrm{w}_1(x_1(np_0)) &= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + \Big({}_0\nabla_{p_0}^{-\alpha} g_0\Big)(x_1(np_0)) \\
&= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 \\
&\qquad + A\,\mathrm{w}_1 \Big[{}_0\nabla_{p_0}^{-\alpha} \Big({}_0\nabla_{p_0}^{-\beta} 1\Big)(n\,p_0)\Big](x_1(np_0)) \\
&= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + A\,\mathrm{w}_1 \Big({}_0\nabla_{p_0}^{-\alpha}\, \psi_{0,1}\Big)(x_1(np_0)) \\
&= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + A\,\psi_{1,1}(n\,p_0)\,\mathrm{w}_1,
\end{aligned}$$

where we use Remark 2 (iii), (v), and

$$\begin{aligned}
g_0(n\,p_0) = \Big({}_0\nabla_{p_0}^{-\beta} F_0\Big)(x_2(np_0)) &= A\,\mathrm{w}_0(x_1(np_0))\,\Big({}_0\nabla_{p_0}^{-\beta} 1\Big)(x_2(np_0)) \\
&= A\,\mathrm{w}_1\,\psi_{0,1}(n\,p_0).
\end{aligned}$$

Using the same technique together with using Remark 2 (iii) and (v), we can deduce

$$\begin{aligned}
\mathrm{w}_2(x_1(np_0)) &= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + \Big({}_0\nabla_{p_0}^{-\alpha} g_1\Big)(x_1(np_0)) \\
&= \psi_{0,0}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,0}(n\,p_0)\,\mathrm{w}_0 + A\,[\psi_{1,1}(n\,p_0)\,\mathrm{w}_1 + \psi_{3,1}(n\,p_0)\,\mathrm{w}_0] \\
&\qquad + A^2\,\psi_{2,2}(n\,p_0)\,\mathrm{w}_1,
\end{aligned}$$

where

$$\begin{aligned}
g_1(n\,p_0) &= \Big({}_0\nabla_{p_0}^{-\beta} F_1\Big)(x_2(np_0)) \\
&= A\Big[\mathrm{w}_1\,\Big({}_0\nabla_{p_0}^{-\beta}\psi_{0,0}\Big)(n\,p_0) + \mathrm{w}_0\,\Big({}_0\nabla_{p_0}^{-\beta}\psi_{2,0}\Big)(n\,p_0) + A\,\mathrm{w}_1\,\Big({}_0\nabla_{p_0}^{-\beta}\psi_{1,1}\Big)(n\,p_0)\Big] \\
&= A\,\big[\psi_{0,1}(n\,p_0)\,\mathrm{w}_1 + \psi_{2,1}(n\,p_0)\,\mathrm{w}_0 + A\,\psi_{1,2}(n\,p_0)\,\mathrm{w}_1\big].
\end{aligned}$$

By the same process, we can obtain the desired result in (23). □

The following lemmas are useful to obtain our main result.

Lemma 5. *If $0 < \alpha \le 1$, then the following summation formula can be obtained:*

$$\lim_{p_0 \to 0} (t_{p_0} - a)^{\overline{-\alpha}} = (t - a)^{-\alpha},$$

for $p_0 > 0$, $t > a$, *where* $a \in \mathbb{R}$, $t_{p_0} = a + p_0 + n\,p_0$, *and* $n = \left[\frac{t-a}{p_0}\right] + 1$.

Proof. Consider

$$\begin{aligned}(t_{p_0} - a)^{\overline{-\alpha}} &= \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} {p_0}^{-\alpha} \\ &= \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} (t_{p_0} - a)^{-\alpha} (n+1)^{\alpha}.\end{aligned}$$

As $p_0 \to 0$ (or $n \to \infty$), it follows that

$$\begin{aligned}\lim_{p_0 \to 0} (t_{p_0} - a)^{\overline{-\alpha}} &= \lim_{p_0 \to 0} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} (t_{p_0} - a)^{-\alpha} (n+1)^{\alpha} \\ &= \lim_{p_0 \to 0} (t_{p_0} - a)^{-\alpha} \cdot \lim_{n \to \infty} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} (n+1)^{\alpha} \\ &\approx (t-a)^{-\alpha},\end{aligned}$$

where it is seen that

$$\lim_{p_0 \to 0} (t_{p_0} - a) = \lim_{p_0 \to 0} (p_0 + (t-a) + p_0) = t - a,$$

and the asymptotic formula is (see [39])

$$\lim_{n \to \infty} \frac{\Gamma(n+1-\alpha)}{\Gamma(n+1)} (n+1)^{\alpha} = 1 + O\left(\frac{1}{n+1}\right) \approx 1. \tag{24}$$

This proves the result. □

Lemma 6. *If* $\alpha \in (0,1]$, *then the following summation formula holds:*

$$\left({}^{RL}_{a}\nabla^{\alpha}_{p_0} \mathrm{w}\right)(t_{p_0}) = {p_0}^{-\alpha} \sum_{\jmath=0}^{n} (-1)^{\jmath} \binom{\alpha}{\jmath} \mathrm{w}(t_{p_0} - \jmath p_0), \tag{25}$$

for $t > a$ *and* $p_0 > 0$ *with* $a \in \mathbb{R}$, $t_{p_0} = a + p_0 + n\,p_0$, *and* $n \in \mathbb{N}_0$.

Proof. Considering Definition (8), one can have

$$\begin{aligned}\left({}^{RL}_{a}\nabla^{\alpha}_{p_0} \mathrm{w}\right)(t_{p_0}) &= \frac{1}{\Gamma(-\alpha)} \sum_{m=0}^{\frac{t-a}{p_0}-1} (t_{p_0} - a - m p_0)_{p_0}^{\overline{-\alpha-1}}\, p_0\, \mathrm{w}(a + p_0 + m p_0) \\ &= p_0 \sum_{m=0}^{n} \frac{(n p_0 + p_0 - m p_0)_{p_0}^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \mathrm{w}(a + p_0 + m p_0) \\ &= {p_0}^{-\alpha} \sum_{m=0}^{n} \frac{\Gamma(n-m-\alpha)}{\Gamma(n-m+1)\,\Gamma(-\alpha)} \mathrm{w}(a + p_0 + m p_0) \\ &= {p_0}^{-\alpha} \sum_{m=0}^{n} \binom{n-m-\alpha-1}{n-m} \mathrm{w}(a + p_0 + m p_0) \\ &= {p_0}^{-\alpha} \sum_{\jmath=0}^{n} \binom{\jmath-\alpha-1}{\jmath} \mathrm{w}(a + x_2(n p_0) - \jmath p_0)\end{aligned}$$

$$= {p_0}^{-\alpha} \sum_{\jmath=0}^{n} (-1)^{\jmath} \binom{\alpha}{\jmath} \mathrm{w}(a + x_2(np_0) - \jmath p_0)$$
$$= {p_0}^{-\alpha} \sum_{\jmath=0}^{n} (-1)^{\jmath} \binom{\alpha}{\jmath} \mathrm{w}(t_{p_0} - \jmath p_0),$$

where we used (6). Thus, the proof is completed. □

Theorem 3. *If* $0 < \alpha \leq 1$, w *is a continuous function, and* w' *is integrable on* $[a, \Lambda]$ *with* $\Lambda > 0$, *then, for* $p_0 > 0$ *and* $t > a$ *with* $a \in \mathbb{R}$, *one can have*

$$\left({}^{LC}_{a}D^{\alpha}\mathrm{w}\right)(t) = \lim_{p_0 \to 0} \left({}^{LC}_{a}\nabla^{\alpha}_{p_0}\mathrm{w}\right)(t_{p_0}),$$

where $t_{p_0} = a + p_0 + n\, p_0$ *and* $n \in \mathbb{N}_0$.

Proof. By using (25) in (11), we can deduce

$$\left({}^{LC}_{a}\nabla^{\alpha}_{p_0}\mathrm{w}\right)(t_{p_0}) = {p_0}^{-\alpha} \sum_{\jmath=0}^{n} (-1)^{\jmath} \binom{\alpha}{\jmath} \mathrm{w}(t_{p_0} - \jmath p_0) - \frac{(t_{p_0} - a)_{p_0}^{\overline{-\alpha}}}{\Gamma(1-\alpha)} \mathrm{w}(a), \tag{26}$$

for $t_{p_0} = a + p_0 + n\, p_0$. As $p_0 \to 0$ on both sides of (26), we obtain the result by using Lemma 5. □

Proposition 1. *If* $\alpha \in (0, 1]$ *and* $p_0 > 0$, *then the solution of*

$$\left({}^{LC}_{0}D^{\alpha}\mathrm{w}\right)(t) = f(t, \mathrm{w}(t)),$$
$$\mathrm{w}(0) = \mathrm{w}_0,$$

can be estimated with the solution of

$$\left({}^{LC}_{a}\nabla^{\alpha}\tilde{\mathrm{w}}\right)(t) = f(t, \tilde{\mathrm{w}}(t)),$$
$$\tilde{\mathrm{w}}(a) = \mathrm{w}_0,$$

via the following limit:

$$\lim_{p_0 \to 0} \tilde{\mathrm{w}}(t_{p_0}) = \mathrm{w}(t),$$

where $t_{p_0} = a + p_0 + n\, p_0$ *and* $n = [(t - p_0)/p_0] + 1$.

Proposition 2. *If* $\alpha \in (0, 1]$ *and* $p_0 > 0$, *then the solution of*

$$\left[{}^{LC}_{0}D^{\beta}\left({}^{LC}_{0}D^{\alpha}\mathrm{w}\right)\right](t) = f(t, \mathrm{w}(t)),$$
$$\mathrm{w}(0) = \mathrm{w}_0,$$

can be approximated with the solutions of (18) *and* (19) *via the limit*

$$\lim_{p_0 \to 0} \tilde{\mathrm{w}}(t_{p_0}) = \mathrm{w}(t),$$

where n and t_{p_0} *are given in Proposition 1.*

4. Numerical Tests

In this section, we will present the result of the numerical experiment in [40] to demonstrate the effectiveness of the proposed technique.

For this reason, we consider

$$\left({}^{LC}_{\ 0}D^{\alpha}\mathrm{w}\right)(t) = \mathrm{w}(t) \quad \text{with} \quad \mathrm{w}(0) = 1,$$

where its solution is given in [40] by

$$\mathrm{w}(t) = E(t^{\alpha}) := \sum_{\kappa=0}^{\infty} \frac{t^{\alpha\,\kappa}}{\Gamma(\alpha\,\kappa + 1)}.$$

To compare the efficiency of the methods, we report, in Figures 1 and 2, the actual computational results (w) compared with the numerical results ($\tilde{\mathrm{w}}$) in the cases of $\alpha = \frac{1}{2}$ and $t = 0, 0.1, \ldots, 1$ for different values of p_0 and Λ.

(**a**) For $p_0 = \frac{1}{10}$ and $\Lambda = 10$. (**b**) For $p_0 = \frac{1}{20}$ and $\Lambda = 20$.

Figure 1. Numerical results for different values of p_0.

Furthermore, we know that

$$\begin{aligned} \left[{}^{LC}_{\ 0}D^{\alpha}\left({}^{LC}_{\ 0}D^{\alpha}\mathrm{w}\right)\right](t) &= \mathrm{w}(t), \\ \left({}^{LC}_{\ 0}D^{\alpha}\mathrm{w}\right)(0) &= 1, \\ \mathrm{w}(0) &= 1, \end{aligned}$$

where its exact solution is $\mathrm{w}(t) = E(t^{\alpha})$.

(**a**) $p_0 = \frac{1}{100}$ and $\Lambda = 100$. (**b**) $p_0 = \frac{1}{200}$ and $\Lambda = 200$.

Figure 2. Numerical results for different values of p_0.

As a result, by considering Figures 1 and 2 and Tables 1 and 2, we conclude that the exact solution can be obtained from $\tilde{\mathrm{w}}$ such that p_0 is sufficiently small. This implies the applicability of Theorem 3 where $p_0 \to 0$.

Table 1. Numerical results in case of $p_0 = 0.1$.

$E\left(t^{\frac{1}{2}}\right)$	**Numerical Result**	**|Error|**
1.000000000000000	1.003804458469754	0.003804458469754
1.486763397673679	1.492441614080931	0.005678216407252
1.799017244188177	1.799775787083807	0.000758542895631
2.107699203837270	2.108238705023936	0.000539501186666
2.430043141497662	2.435351117027752	0.005307975530090
2.774285957670009	2.782077629971030	0.007791672301020
3.146213032210335	3.155553139052627	0.009340106842292
3.550802683646889	3.552101745731627	0.001299062084737
3.992835834192708	3.998524070801430	0.005688236608722
4.477184810795735	4.481878717206317	0.004693906410582
5.008980080762283	5.009099101457296	0.000119020695013

Table 2. Numerical results in case of $p_0 = 0.01$.

$E\left(t^{\frac{1}{2}}\right)$	**Numerical Result**	**|Error|**
1.000000000000000	1.000711215780434	0.000711215780434
1.486763397673679	1.486985144407696	0.000221746734017
1.799017244188177	1.799134661839032	0.000117417650856
2.107699203837270	2.107995879710488	0.000296675873218
2.430043141497662	2.430361919799588	0.000318778301926
2.774285957670009	2.774710124429723	0.000424166759714
3.146213032210335	3.146720890494996	0.000507858284661
3.550802683646889	3.550888199443980	0.000085515797090
3.992835834192708	3.993098316427406	0.000262482234698
4.477184810795735	4.477985825418505	0.000801014622770
5.008980080762283	5.009009301039845	0.000029220277562

5. Concluding Remarks and Future Works

To summarize, we investigated the complete solution of a sequential fractional differential problem of Liouville–Caputo type including certain initial value conditions,

$$\left[{}^{LC}_{0}D^{\beta}\left({}^{LC}_{0}D^{\alpha}\mathrm{w}\right)\right](t) = f\left(t, \mathrm{w}(t)\right),$$

with the condition $\mathrm{w}(0) = \mathrm{w}_1$, by analyzing the corresponding system of the fractional difference,

$$\begin{aligned}\left({}^{LC}_{0}\nabla^{\alpha}_{p_0}\tilde{\mathrm{w}}\right)(x_1(np_0)) &= y(x_2(np_0)),\\ \left({}^{LC}_{0}\nabla^{\alpha}_{p_0}y\right)(n\,h + h) &= f\left(x_1(np_0), \tilde{\mathrm{w}}(x_1(np_0))\right),\end{aligned}$$

subject to the conditions

$$\begin{aligned}\left({}^{LC}_{0}\nabla^{\alpha}_{p_0}\tilde{\mathrm{w}}\right)(-p_0) &= \tilde{\mathrm{w}}_0,\\ \tilde{\mathrm{w}}(-2p_0) &= \tilde{\mathrm{w}}_a,\end{aligned}$$

via the following limitation:

$$\lim_{p_0 \to 0} \tilde{\mathrm{w}}\left(t_{p_0}\right) = \mathrm{w}(t),$$

where t_{p_0} and n are defined in Proposition 1. We used the expressions $\psi_{\kappa,s}$ and $\tilde{\psi}_{\kappa,s}$ in the solution of the discrete fractional difference system as stated in Definition 4.

Furthermore, the numerical example shows that the new reconstruction method can obtain a very high order of accuracy; when the value of p_0 is small enough (or Λ is large enough), the approximation results are in good agreement with the exact one.

Concerning future works, we plan to extend the numerical schemes presented here to apply them to other types of discrete fractional systems including exponential and Mittag–Leffler systems in kernels to establish similar results (visit [22,38] to see these operators).

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