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Noncommutative Multi-Parameter Subsequential Wiener–Wintner-Type Ergodic Theorem

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Abstract: This paper is devoted to the study of a multi-parameter subsequential version of the "Wiener–Wintner" ergodic theorem for the noncommutative Dunford–Schwartz system. We establish a structure to prove "Wiener–Wintner"-type convergence over a multi-parameter subsequence class Δ instead of the weight class case. In our subsequence class, every term of $\underline{\mathbf{k}} \in \Delta$ is one of the three kinds of nonzero density subsequences we consider. As key ingredients, we give the maximal ergodic inequalities of multi-parameter subsequential averages and obtain a noncommutative subsequential analogue of the Banach principle. Then, by combining the critical result of the uniform convergence for a dense subset of the noncommutative $L_p(\mathcal{M})$ space and the noncommutative Orlicz space, we immediately obtain the main theorem.

Keywords: noncommutative L_p spaces; noncommutative dynamical systems; subsequential ergodic theorem; multi-parameter individual ergodic theorem; nonzero density; uniform sequence

MSC: 46L53; 46L55; 47A35; 37A99

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1. Introduction

Analyzing the limit process of Cesàro averages along certain subsequences is a typical way of accessing the ergodic theory. Over the decades, many relevant works have appeared in this direction, gradually forming a fruitful branch. By a dynamical system $(X, \mathcal{F}, \mu, T)$, we mean that $(X, \mathcal{F}, \mu)$ is a measured space and T is a certain linear operator acting on $L_p(\mu)$ $(1 \leq p \leq \infty)$. Let $\underline{k} = \{k_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of non-negative integers, i.e., a subsequence of $\{0, 1, 2, \cdots\}$. Denote $\mathbb{N}$ as all positive integers; then, a series of discussions of the (strong, weak, or almost everywhere) convergence of

$$A_n(\underline{k}, T)f = \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f, \quad n \in \mathbb{N}$$

leads to various developments, such as the early work of Blum and Hanson [1], Baxter and Olsen [2], Bellow and Losert [3,4], Bourgain [5–8], Jones [9], Jones and Olsen [10], Wierdl [11], as well as others.

Two factors that are often introduced at the start of the discussion are the type of subsequence and the type of action in the dynamical system; either one can be fixed so that an ample argument can be made out of the other. The type of subsequence is of more interest in this paper. One way to classify the subsequences is to consider their densities. $E \subset \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is said to have a *higher (respectively, lower) density* δ if

$$\lim_{n\to\infty} \frac{\mathrm{card}(\{0, \ldots, n\} \cap E)}{n+1} = \delta \quad \text{(respectively, } \liminf_{n\to\infty} \frac{\mathrm{card}(\{0, \ldots, n\} \cap E)}{n+1} = \delta).$$

Accordingly, the above sequence $\underline{k} = \{k_i\}_{i=1}^{\infty}$ has a *higher (respectively, lower) density* δ if the set $\{k_n : n \in \mathbb{N}\}$ has a higher (respectively, lower) density δ. We have a quick connection to the weighted ergodic theory if $\{k_n : n \in \mathbb{N}\}$ has nonzero density (Proposition 1.7 [4]), which is why weighted and subsequential ergodic theorems have sometimes been studied together in the past. In this paper, we follow this line and restrict the system to only consider sequences with a density of one, uniform sequences, and block sequences; we introduce the specific descriptions later.

On the other hand, since the 1970s, noncommutative ergodic theory has been developed step-by-step based on the rapid growth of noncommutative harmonic analysis. A constructive notion of almost-uniform convergence in noncommutative L_p space was invented by Lance [12], who substituted exactly the classical almost-everywhere convergence in individual ergodic theorems. Since our discussion concerns the multi-parameter setting, we extend the concept accordingly. Fix d to be any positive integer and denote $\mathbf{n} = (n_1, n_2, \cdots, n_d) \in \mathbb{N}^d$ as the d-parameter index. Let $x_{\mathbf{n}}, x \in \mathcal{B}(\mathcal{M})$, where $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful trace τ and $\mathcal{B}(\mathcal{M})$ is any Banach space of measurable operators associated with $\mathcal{M}$ (the category will be specified in Section 2). Then by $\lim_{\mathbf{n}} x_{\mathbf{n}} = x$, we mean that, given any $\varepsilon > 0$, there is an $\mathbf{N} \in \mathbb{N}^d$ such that for all $\mathbf{n} > \mathbf{N}$ $(n_j > N_j, j = 1, 2, \cdots, d)$, $\|x_{\mathbf{n}} - x\| < \varepsilon$, and we say $x_{\mathbf{n}}$ converges to x. A multi-parameter sequence $\{x_{\mathbf{n}}\}$ is said to *converge bilaterally almost uniformly (respectively, almost uniformly)* to x if for any $\varepsilon > 0$, there exists $e \in P(\mathcal{M})$ (the lattice of projections in $\mathcal{M}$) such that

$$\tau(e^{\perp}) \leq \varepsilon \text{ and } \{e(x_{\mathbf{n}} - x)e\} \text{ (respectively, } \{(x_{\mathbf{n}} - x)e\})$$

converges to 0 in $\mathcal{M}$. Usually, we denote it as b.a.u. (respectively, a.u.) convergence. Since 2007, Junge and Xu's [13] real interpolation method to obtain the strong-type noncommutative Dunford–Schwartz maximal ergodic theorem after Yeadon's [14] weak-type $(1,1)$ inequality has been seen as a closed answer to the main problem of establishing a noncommutative individual ergodic theorem. Based on the structure, noncommutative ergodic theory has been going forward in some sophisticated directions. Generalizing a dynamical system's transformation to Lamperty operators [15] and to group actions [16] and changing the forms of Cesàro averages to obtain the weighted (even Wiener–Wintner) ergodic theorem [17–19] and the subsequential case [20,21] are some representative achievements. Inspired by [22], in this paper, we intend to give the structure of the multi-parameter subsequential ergodic theorem for a noncommutative Dunford–Schwartz system.

Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a vector of d Dunford–Schwartz operators (defined in Section 2.3). As $L_1(\mathcal{M}) + \mathcal{M} \to L_1(\mathcal{M}) + \mathcal{M}$ (the action can be uniquely extended to any noncommutative Banach space), then $\mathbf{T}$ is also called a Dunford–Schwartz function, and a noncommutative multi-parameter Dunford–Schwartz system $(\mathcal{M}, \tau, \mathbf{T})$ is given. Meanwhile, let $\underline{\mathbf{k}} = (\underline{k}^{(1)}, \cdots, \underline{k}^{(d)}) \subset \mathbb{N}_0^d$ be a vector of d sequences of strictly increasing non-negative integers, i.e., every $\underline{k}^{(j)} = \{k_i^{(j)}\}_{i=1}^{\infty}$ is a subsequence of $\{0, 1, 2, \cdots\}$, $j = 1, 2, \cdots, d$. Naturally, we can give the density δ of $\underline{\mathbf{k}}$ as the product of the densities of each $\underline{k}^{(j)}$, denoted simply as $\delta = \delta^{(1)}\delta^{(2)} \cdots \delta^{(d)}$. On the other hand, we will need the notation $M_{\underline{k}} = \sup_{n \geq 1} \frac{k_n + 1}{n}$, which is finite for every positive lower density sequence $\underline{k}$, and we denote this as $M_{\underline{\mathbf{k}}} = M_{\underline{k}^{(1)}} \cdots M_{\underline{k}^{(d)}}$ hereafter.

Similar to the classical notion, we let

$$A_n(\underline{k}^{(j)}, T_j) = \frac{1}{n} \sum_{i=1}^{n} T_j^{k_i^{(j)}}, \quad n \in \mathbb{N},\ j = 1, \cdots, d$$

be the associated subsequential averaging actions and

$$A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T}) = A_{n_1}(\underline{k}^{(1)}, T_1) \cdots A_{n_d}(\underline{k}^{(d)}, T_d), \quad \mathbf{n} \in \mathbb{N}^d$$

for the multi-parameter case.

Corresponding to the classical theory, we can talk about the "goodness" of the multi-parameter subsequence.

Definition 1. *Let $\mathcal{B}(\mathcal{M})$ be a Banach space constructed from $\mathcal{M}$, and let $\mathbf{S}$ be a vector of d linear actions as $\mathcal{B}(\mathcal{M}) \to \mathcal{B}(\mathcal{M})$. We say $\underline{\mathbf{k}}$ is a bilaterally good (respectively, good) subsequence in $\mathcal{B}(\mathcal{M})$ for $\mathbf{S}$ if for every $x \in \mathcal{B}(\mathcal{M})$, $\{A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{S})(x)\}_{\mathbf{n}}$ converges b.a.u. (respectively, a.u.). Moreover, $\underline{\mathbf{k}}$ is a bilaterally good universal (respectively, good universal) subsequence in $\mathcal{B}(\mathcal{M})$ if it is bilaterally good (respectively, good) for any Dunford–Schwartz operator $\mathbf{T}$ on $\mathcal{B}(\mathcal{M})$.*

Consequently, given $\mathcal{B}(\mathcal{M})$, it is natural to ask: What kind of subsequences are bilaterally good universal (respectively, good universal)? In this paper, we focus on a type of subsequence that we denote as Δ. We say that $\underline{\mathbf{k}} \in \Delta$ if every element $\underline{k}^{(j)}$ $(j = 1, 2, \cdots, d)$ of $\underline{\mathbf{k}}$ is one of the following:

(1) A sequence with a density of one;
(2) A recurring uniform sequence;
(3) A block sequence with a positive lower density such that $\lim_{n\to\infty} \frac{N_{\mathcal{I}}(n)}{n} \to 0$.

It will be shown as a corollary of our main result that every $\underline{\mathbf{k}} \in \Delta$ is bilaterally good universal in $L_p(\mathcal{M})$ if $1 < p < \infty$ and good universal in $L_p(\mathcal{M})$ if $2 < p < \infty$; moreover, if $(\mathcal{M}, \tau)$ is a noncommutative probability space, then $\underline{\mathbf{k}}$ is bilaterally good universal in $L_1 \log^{2(d-1)} L(\mathcal{M})$ and good universal in $L_2 \log^{2(d-1)} L(\mathcal{M})$.

The single-parameter case of the above question was firstly studied by Litvinov and Mukhamedov in [20], and their result has been recently extended by O'Brien in [21], while the original motive comes from commutative works by Brunel and Keane [23] and Sato [24,25]. In Litvinov and O'Brien's works, they mainly apply the language of "uniform and bilaterally uniform equicontinuity in measure (in short, u.e.m. and b.u.e.m.) at zero" to treat the a.u. and b.a.u. convergence problem in $L_p(\mathcal{M})$ (p takes values in $[1, +\infty]$ accordingly). However, in this paper, we seek a more "Littlewood–Paley" path, as in [13], and establish a "maximal" to "individual" procedure that can be expanded in future developments.

Actually, we push the question to a relatively blank region for the subsequential theory, and thus, the above solution can be naturally included. The idea is to consider a "Wiener–Wintner"-type convergence for a certain set of subsequences as follows.

Definition 2. *A set $\mathbf{K}$ of bilaterally good universal (respectively, good universal) subsequences of $\mathcal{B}(\mathcal{M})$ is said to be of $\mathcal{B}$ bilaterally subsequential Wiener–Wintner-type (respectively, $\mathcal{B}$ subsequential Wiener–Wintner-type)—in short, of $\mathcal{B}$-bsWW (respectively, $\mathcal{B}$-sWW) type—if for any $x \in \mathcal{B}(\mathcal{M})$ and any $\varepsilon > 0$, there exists $e \in P(\mathcal{M})$ such that $\tau(e^{\perp}) \leq \varepsilon$ and*

$$\left\{e\left(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x)\right)e\right\}_{\mathbf{n}} \left(\textit{respectively, } \left\{\left(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x)\right)e\right\}_{\mathbf{n}}\right) \textit{ converges for all } \underline{\mathbf{k}} \in \mathbf{K}.$$

In fact, if every $\underline{\mathbf{k}} \in \mathbf{K}$ has a density $\delta > 0$, by a characteristic function argument

$$\mathbf{1}_{\underline{\mathbf{k}}}(\mathbf{n}) = \mathbf{1}_{\{(k^{(1)}_{i_1}, k^{(2)}_{i_2}, \cdots, k^{(d)}_{i_d}) : i_1, i_2, \ldots, i_d \in \mathbb{N}\}}(n_1, n_2, \cdots, n_d) = \mathbf{1}_{\underline{k}^{(1)}}(n_1)\mathbf{1}_{\underline{k}^{(2)}}(n_2) \cdots \mathbf{1}_{\underline{k}^{(d)}}(n_d),\ \mathbf{n} \in \mathbb{N}_0^d,$$

and we have a transfer

$$\begin{aligned}
A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) =& A_{n_1}(\underline{k}^{(1)}, T_1) \cdots A_{n_d}(\underline{k}^{(d)}, T_d)(x) = \frac{1}{n_1} \sum_{i_1=1}^{n_1} T_1^{k^{(1)}_{i_1}} \frac{1}{n_2} \sum_{i_2=1}^{n_2} T_2^{k^{(2)}_{i_2}} \cdots \frac{1}{n_d} \sum_{i_d=1}^{n_d} T_d^{k^{(d)}_{i_d}}(x) \\
=& \frac{k^{(1)}_{n_1} + 1}{n_1} \frac{1}{k^{(1)}_{n_1} + 1} \sum_{i_1=0}^{k^{(1)}_{n_1}} \mathbf{1}_{\underline{k}^{(1)}}(i_1) T_1^{i_1} \cdots \frac{k^{(d)}_{n_d} + 1}{n_d} \frac{1}{k^{(d)}_{n_d} + 1} \sum_{i_d=0}^{k^{(d)}_{n_d}} \mathbf{1}_{\underline{k}^{(d)}}(i_d) T_d^{i_d}(x) \\
\sim& \frac{1}{\delta^{(1)}} \cdots \frac{1}{\delta^{(d)}} \frac{1}{k^{(1)}_{n_1} + 1} \cdots \frac{1}{k^{(d)}_{n_d} + 1} \sum_{i_1=0}^{k^{(1)}_{n_1}} \cdots \sum_{i_d=0}^{k^{(d)}_{n_d}} \mathbf{1}_{\underline{\mathbf{k}}}(i_1, \cdots, i_d) T_1^{i_1} \cdots T_d^{i_d}(x) \\
=& \frac{1}{\delta} M_{(k^{(1)}_{n_1}, \cdots, k^{(d)}_{n_d})}(x, \mathbf{1}_{\underline{\mathbf{k}}}(\cdot)).
\end{aligned}$$

The last term of the above equality is a multi-parameter weighted ergodic average. It shows that whether $\mathbf{K}$ is of $\mathcal{B}$-bsWW type is closely relevant to the question of whether the weight function set $\{\mathbf{1}_{\underline{\mathbf{k}}}(\cdot) : \underline{\mathbf{k}} \in \mathbf{K}\}$ is of $\mathcal{B}$-NCbWW ($\mathcal{B}$-noncommutative bilateral Wiener–Wintner) type, which concerns the topic in [17].

In addition, we give an independent construction and prove that Δ is of L_p-bsWW type for $1 < p < \infty$ and L_p-sWW type for $2 < p < \infty$. Moreover, if τ is finite, Δ is of $L_1 \log^{2(d-1)} L$-bsWW type and $L_2 \log^{2(d-1)} L$-sWW type.

Nevertheless, for the bigger sets of nonzero density subsequences or even zero density subsequences, the question seems quite sensible and needs more investigation in the future.

2. Preliminaries

2.1. Noncommutative Vector-Valued L_p Spaces

We use standard notions from the theory of noncommutative L_p spaces. Our main references are [26]. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $L_0(\mathcal{M})$ be the space of measurable operators associated with $(\mathcal{M}, \tau)$. For a measurable operator x, its generalized singular number is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \tau\big(\mathbb{1}_{(\lambda,\infty)}(|x|)\big) \leq t\}, \quad t > 0.$$

The trace τ can be extended to the positive cone $L_0^+(\mathcal{M})$ of $L_0(\mathcal{M})$, still denoted by τ, by setting

$$\tau(x) = \int_0^\infty \mu_t(x)dt, \quad x \in L_0^+(\mathcal{M}).$$

Given $0 < p < \infty$, let

$$L_p(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \tau(|x|^p) < \infty\},$$

and for $x \in L_p(\mathcal{M})$,

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} = \Big(\int_0^\infty (\mu_t(x))^p dt\Big)^{\frac{1}{p}}.$$

Then $(L_p(\mathcal{M}), \|\cdot\|_p)$ is a Banach space (or quasi-Banach space when $p < 1$). This is the noncommutative L_p space associated with $(\mathcal{M}, \tau)$, denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm.

Noncommutative Orlicz spaces are defined similarly to commutative ones. Given an Orlicz function Φ, the Orlicz space $L_\Phi(\mathcal{M})$ is defined as the set of all measurable operators x such that $\Phi(\frac{|x|}{\lambda}) \in L_1(\mathcal{M})$ for some $\lambda > 0$. Equipped with the norm

$$\|x\|_\Phi = \inf\Big\{\lambda > 0 : \tau\Big[\Phi\Big(\frac{|x|}{\lambda}\Big)\Big] \leq 1\Big\},$$

$L_\Phi(M)$ is a Banach space. When $\Phi(t) = t^p$, with $1 \leq p < \infty$, the space $L_\Phi(\mathcal{M})$ coincides with $L_p(\mathcal{M})$. If $\Phi(t) = t^p(1 + \log^+ t)^r$ for $1 \leq p < \infty$ and $r > 0$, we have the space $L_p \log^r L(\mathcal{M})$. From the definition, if, moreover, the trace τ is finite, i.e., $(\mathcal{M}, \tau)$ is a noncommutative probability space, it is easy to check that

$$L_q(\mathcal{M}) \subset L_p \log^r L(\mathcal{M}) \subset L_s(\mathcal{M})$$

for $q > p \geq s \geq 1$.

The spaces $L_p(\mathcal{M}; \ell_\infty)$ and $L_p(\mathcal{M}; \ell_\infty^c)$ are important in the formulation of noncommutative maximal inequalities. In the following, we give a more general description of such spaces in the multi-parameter case. A d parameter sequence $\{x_{\mathbf{n}}\}_{\mathbf{n}} \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty(\mathbb{N}^d))$ if and only if it can be factored as $x_{\mathbf{n}} = ay_{\mathbf{n}}b$ with $a,b \in L_{2p}(\mathcal{M})$ and $\{y_{\mathbf{n}}\}_{\mathbf{n}} \subset L_\infty(\mathcal{M})$ is a bounded sequence. We then define

$$\|\{x_{\mathbf{n}}\}_{\mathbf{n}}\|_{L_p(\ell_\infty(\mathbb{N}^d))} = \inf_{x_{\mathbf{n}}=ay_{\mathbf{n}}b} \{\|a\|_{2p} \sup_{\mathbf{n}} \|y_{\mathbf{n}}\|_\infty \|b\|_{2p}\}.$$

Following [13], this norm is symbolically denoted by $\|\sup_{\mathbf{n}}{}^{+} x_{\mathbf{n}}\|_p$, and for a positive sequence $\{x_{\mathbf{n}}\}_{\mathbf{n}}$, it is equivalent to write

$$\|\sup_{\mathbf{n}}{}^{+} x_{\mathbf{n}}\|_p = \inf\{\|a\|_p \ : \ a \in L_p^+(\mathcal{M}) \text{ s.t. } x_{\mathbf{n}} \le a, \ \forall\, \mathbf{n} \in \mathbb{N}^d\}.$$

Here and in the rest of the paper, $L_p^+(\mathcal{M})$ denotes the positive cone of $L_p(\mathcal{M})$. The space $L_p(\mathcal{M};\ell_\infty^c(\mathbb{N}^d))$ is defined as the set of sequences $\{x_{\mathbf{n}}\}_{\mathbf{n}}$ for which $\{|x_{\mathbf{n}}|^2\}_{\mathbf{n}}$ belongs to $L_{p/2}(\mathcal{M};\ell_\infty(\mathbb{N}^d))$ equipped with the (quasi) norm

$$\|\{x_{\mathbf{n}}\}_{\mathbf{n}}\|_{L_p(\ell_\infty^c(\mathbb{N}^d))} = \|\{|x_{\mathbf{n}}|^2\}_{\mathbf{n}}\|^{\frac{1}{2}}_{L_{\frac{p}{2}}(\ell_\infty(\mathbb{N}^d))}.$$

We refer to [13,27,28] for more information on these spaces and for facts related to the one-parameter case.

Vector-valued Orlicz spaces $L_p \log^r L(\mathcal{M};\ell_\infty)$ $(1 \le p < \infty,\ r > 0)$ were first introduced by Bekjan et al. in [29]. It is observed that the $L_p(\ell_\infty)$-norm has an equivalent formulation:

$$\|\{x_n\}_n\|_{L_p(\ell_\infty)} = \inf\{\frac{1}{2}(\|a\|_{2p}^2 + \|b\|_{2p}^2) \sup_n \|y_n\|_\infty\},$$

where the infimum is taken over the same parameter. Given an Orlicz function Φ, let $\{x_{\mathbf{n}}\}_{\mathbf{n}}$ be a multi-parameter sequence of operators in $L_\Phi(\mathcal{M})$. We define

$$\tau(\Phi(\sup_{\mathbf{n}}{}^{+} x_{\mathbf{n}})) = \inf\{\frac{1}{2}(\tau(\Phi(|a|^2)) + \tau(\Phi(|b|^2))) \sup_{\mathbf{n}} \|y_{\mathbf{n}}\|_\infty\},$$

where the infimum is taken over all the decompositions $x_{\mathbf{n}} = ay_{\mathbf{n}}b$ for $a,b \in L_0(\mathcal{M})$ and $y_{\mathbf{n}} \in L_\infty(\mathcal{M})$, with $|a|^2, |b|^2 \in L_\Phi(\mathcal{M})$ and $\sup_{\mathbf{n}} \|y_{\mathbf{n}}\|_\infty \le 1$. Then $L_\Phi(\mathcal{M};\ell_\infty(\mathbb{N}^d))$ is defined to be the set of sequences $\{x_{\mathbf{n}}\}_{\mathbf{n}} \subset L_\Phi(\mathcal{M})$ such that there exists one $\lambda > 0$ satisfying

$$\tau(\Phi(\sup_{\mathbf{n}}{}^{+} \frac{x_{\mathbf{n}}}{\lambda})) < \infty$$

equipped with the norm

$$\|\{x_{\mathbf{n}}\}_{\mathbf{n}}\|_{L_\Phi(\ell_\infty(\mathbb{N}^d))} = \inf\Big\{\lambda > 0 : \tau(\Phi(\sup_{\mathbf{n}}{}^{+} \frac{x_{\mathbf{n}}}{\lambda})) < 1\Big\}.$$

Then $(L_\Phi(\mathcal{M};\ell_\infty(\mathbb{N}^d)), \|\cdot\|_{L_\Phi(\ell_\infty(\mathbb{N}^d))})$ is a Banach space. A similar characterization holds for sequences of positive operators:

$$\tau(\Phi(\sup_{\mathbf{n}}{}^{+} x_{\mathbf{n}})) \approx \inf\{\tau(\Phi(a)) \ : \ a \in L_\Phi^+(\mathcal{M}) \text{ s.t. } x_{\mathbf{n}} \le a, \ \forall\, \mathbf{n} \in \mathbb{N}^d\},$$

which implies a similar characterization for the norm

$$\|\{x_{\mathbf{n}}\}_{\mathbf{n}}\|_{L_\Phi(\ell_\infty(\mathbb{N}^d))} \approx \inf\{\|a\|_\Phi \ : \ a \in L_\Phi^+(\mathcal{M}) \text{ s.t. } x_{\mathbf{n}} \le a, \ \forall\, \mathbf{n} \in \mathbb{N}^d\}.$$

For the same reason, whenever $(\mathcal{M},\tau)$ is a probability space, we have

$$L_q(\mathcal{M};\ell_\infty(\mathbb{N}^d)) \subset L_p \log^r L(\mathcal{M};\ell_\infty(\mathbb{N}^d)) \subset L_s(\mathcal{M};\ell_\infty(\mathbb{N}^d))$$

for $q > p \geq s \geq 1$. We refer the reader to [29] for more information on vector-valued Orlicz spaces.

Furthermore, we denote by $L_p(\mathcal{M}; c_0(\mathbb{N}^d))$ the closure of finite sequences in $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^d))$ for $1 \leq p < \infty$, and we define $L_p(\mathcal{M}; c_0^c(\mathbb{N}^d))$ as the closure of finite sequences in $L_p(\mathcal{M}; \ell_\infty^c(\mathbb{N}^d))$. Similarly, we have denotations $L_\Phi(\mathcal{M}; c_0(\mathbb{N}^d))$ and $L_\Phi(\mathcal{M}; c_0^c(\mathbb{N}^d))$.

The following complex interpolation theorem of these noncommutative vector-valued L_p spaces is useful later; it originated from Proposition 2.5 of [13].

Proposition 1. *Let $1 \leq p_0 < p_1 \leq \infty$ and $0 < \theta < 1$. Then, we have isometrically*

$$L_p(\mathcal{M}; \ell_\infty) = (L_{p_0}(\mathcal{M}; \ell_\infty), L_{p_1}(\mathcal{M}; \ell_\infty))_\theta,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If additionally $p_0 \geq 2$, then we have isometrically

$$L_p(\mathcal{M}; \ell_\infty^c) = (L_{p_0}(\mathcal{M}; \ell_\infty^c), L_{p_1}(\mathcal{M}; \ell_\infty^c))_\theta,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

2.2. Nonzero Density Subsequences

For the main result of this paper, we restrain the discussion within three kinds of nonzero density subsequences: the first kind is simply those with a density of one, the second and third kinds are uniform sequences and block sequences, respectively.

We give a brief review of the uniform sequences in the following; they were originally generalized in the work of Brunel and Kean [23] and were first introduced by Sato [25].

Let Ω be a compact Hausdorff space, and let φ be a continuous map of Ω into itself such that the family $\{\varphi^n\}_{n\geq 0}$ is equicontinuous. The system (Ω, φ) is called strictly ergodic if there exists a unique φ-invariant measure μ on $(\Omega, \mathcal{B})$, with $supp(\mu) = \Omega$, where $\mathcal{B}$ stands for the σ algebra of all Borel subsets of Ω, such that for any $\omega \in \Omega$ and $f \in C(\Omega)$,

$$\int_\Omega f d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k \omega)$$

with respect to the uniform norm in $C(\Omega)$.

Definition 3. *A sequence $\underline{k} = \{k_i\}_{i=1}^\infty$ of non-negative integers is said to be uniform if there exist*

(i) A strictly ergodic system $(\Omega, \mathcal{B}, \mu, \varphi)$;

(ii) A set $Y \in \mathcal{B}$ with $\mu(Y) > 0 = \mu(\partial Y)$, where ∂Y denotes the boundary of Y;

(iii) A point $\omega_0 \in \Omega$ such that

$$k_1 = min\{k \geqslant 0 : \varphi^k \omega_0 \in Y\};$$

$$k_n = min\{k > k_{n-1} : \varphi^k \omega_0 \in Y\}, n \geqslant 2$$

so that $\underline{k}$ is a strictly increasing sequence of non-negative integers.

The triplet $(\Omega, \mathcal{B}, \mu, \varphi)$, Y, and ω_0 will be called the *apparatus* for $\underline{k}$. The following two lemmas can be found in [25].

Lemma 1. *If $\underline{k}$ is a uniform sequence as above, then*

$$\lim_{n\to\infty} n/k_n = \mu(Y),$$

so that $\underline{k}$ has positive density.

Lemma 2. *If $\underline{k}$ is a uniform sequence as above, then for any $\epsilon > 0$, there exist open subsets Y_1, Y_2 and W of Ω such that*

(i) $Y_1 \subset Y \subset Y_2, \mu(Y_2 - Y_1) \leq \epsilon$ *and* $\mu(\partial Y_1) = 0 = \mu(\partial Y_2)$;
(ii) $\omega_0 \in W$, *and for every* $\omega \in W$ *and all* $k \geqslant 0$,

$$\chi_{Y_1}(\varphi^k \omega) \leq \chi_Y(\varphi^k \omega_0) \leq \chi_{Y_2}(\varphi^k \omega),$$

where $\chi_E(\omega)$ *is the characteristic function of a set* E.

Restricted by the technique and the final target we are trying to reach, it is still difficult to handle the uniform sequence in the general case. However, we can make progress in a special case that we call the recurring uniform sequence. It is in a way inspired by the Poincaré recurrence theorem and is defined as follows.

Definition 4. *Let* $\underline{k}$ *be a uniform sequence as above. If for any neighborhood* W *of* ω_0 *there exists a point* $\omega_1 \in W$ *and a non-negative integer* $i_0 \in \mathbb{N}$ *such that*

$$\varphi^{i_0} \omega_1 \in W \text{ and } \varphi^{i_0+1} \omega_1 \in W,$$

then $\underline{k}$ *is called a recurring uniform sequence.*

Next, we introduce block sequences with positive lower densities.

Let $\mathcal{I} = \{I_n = [a_n, b_n]\}_n^\infty = 1$ be a sequence of intervals in $[0, \infty)$ whose endpoints are in $\mathbb{N}_0$ such that $b_n < a_{n+1}$ for every $n \geq 1$.

Definition 5. *A sequence* $\underline{k} = \{k_n\}_{n=1}^\infty$ *is called a block sequence if it is determined by a strictly increasing enumeration of* $\bigcup_n I_n \cap \mathbb{N}_0$. *Denote the function* $N_{\mathcal{I}} : \mathbb{N}_0 \to \mathbb{N}_0$ *to be such that*

$$k_n \in [a_{N_{\mathcal{I}}(n)}, b_{N_{\mathcal{I}}(n)}].$$

It is easily seen that $N_{\mathcal{I}}(n)$ *stands for the number of intervals one has to skip before finding the interval that* k_n *belongs to.*

2.3. Dunford–Schwartz Operators

The concept of absolute contraction in the noncommutative setting was first considered in [14]. It is a positive linear map $T : L_1(\mathcal{M}) \to L_1(\mathcal{M})$ satisfying $T(I) \leq I$ and $\tau(T(x)) \leq \tau(x)$ for every $x \geq 0$. In general, it was pointed out in [18] that there exists a unique extension such that for $x \in L_p(\mathcal{M})$ $(1 \leq p \leq \infty)$ and $k \in \mathbb{N}$, we have $\|T^k(x)\|_p \leq 2\|x\|_p$. Specifically, according to the well-known classical narrative, a linear map $T : L_1 + \mathcal{M} \to L_1 + \mathcal{M}$ satisfying

$$\|T(x)\|_1 \leq \|x\|_1 \ \forall\, x \in L_1 \ \text{ and } \ \|T(x)\|_\infty \leq \|x\|_\infty \ \forall\, x \in \mathcal{M}$$

is called a *Dunford–Schwartz operator*. Moreover, if $T(x) \geq 0$ whenever $x \geq 0$, then T is called a *positive Dunford–Schwartz operator*, and we write $T \in DS^+(\mathcal{M}, \tau)$ or just $T \in DS^+$. It was practically shown in Proposition 1.1 of [30] that absolute contractions can be uniquely extended to positive Dunford–Schwartz operators. Also, it was shown in Lemma 1.1 of [13] that any $T \in DS^+$ can be extended uniquely to a positive linear contraction on L_p for each $1 < p < \infty$. Thus, denoting these extensions by T, we have $\|T(x)\|_p \leq \|x\|_p$ for all $x \in L_p$.

Each $T \in DS^+$ induces canonical splitting of $L_p(\mathcal{M})$ for $1 < p < \infty$:

$$L_p(\mathcal{M}) = \mathcal{F}_p(T) \oplus \mathcal{F}_p(T)^\perp,$$

where $\mathcal{F}_p(T) = \{x \in L_p(\mathcal{M}) : T(x) = x\}$ and $\mathcal{F}_p(T)^\perp$ is the closure of the image $(I - T)(L_p(\mathcal{M}))$. Usually, F is denoted as the projection from $L_p(\mathcal{M})$ onto $\mathcal{F}_p(T)$.

The following result by Yeadon [14] plays a fundamental role in this paper. $P(\mathcal{M})$ denotes the lattice of projections in $\mathcal{M}$. Given $e \in P(\mathcal{M})$, we set $e^\perp = 1 - e$.

Lemma 3. *Let $T \in DS^+$ and $x \in L_1^+(\mathcal{M})$. Denote $A_n(T)(x) = \frac{1}{n+1}\sum_{k=0}^{n} T^k(x)$. Then for any ł > 0, there is $e \in P(\mathcal{M})$ such that*

$$\tau(e^{\perp}) \leq \frac{\|x\|_1}{ł} \quad \textit{and} \quad \sup_{n\geq 0} \|e\, A_n(T)(x)\, e\|_\infty \leq ł.$$

3. Noncommutative Maximal Ergodic Inequalities

Usually when proving individual ergodic theorems, e.g., in [13,17], maximal inequalities are established as a primary part. In the following, similar to several preparatory works, we give one-parameter estimates of ergodic averages along nonzero density subsequences; then, by a routine iteration argument, multi-parameter maximal ergodic inequalities are obtained. In the process, we also see some close connections between our subsequential case and the weighted case.

Proposition 2. *Given $1 \leq p \leq \infty$, let $x_n, y_n, z \in L_p^+(\mathcal{M})$ and suppose $x_n \leq y_n \leq z$ for every $n \in \mathbb{N}_0$; then $\|\sup_n^+ x_n\|_p \leq \|\sup_n^+ y_n\|_p$.*

Proof. By the given condition, we know that

$$\left\{\|a\|_p \ : \ a \in L_p^+(\mathcal{M}) \text{ s.t. } y_n \leq a,\ \forall\, n \geq 0\right\} \subset \left\{\|a\|_p \ : \ a \in L_p^+(\mathcal{M}) \text{ s.t. } x_n \leq a,\ \forall\, n \geq 0\right\};$$

thus, we know

$$\inf\left\{\|a\|_p \ : \ a \in L_p^+(\mathcal{M}) \text{ s.t. } x_n \leq a,\ \forall\, n \geq 0\right\} \leq \inf\left\{\|a\|_p \ : \ a \in L_p^+(\mathcal{M}) \text{ s.t. } y_n \leq a,\ \forall\, n \geq 0\right\},$$

which completes the proof. □

We extract the following maximal inequality from the proof of Theorem 3.5 in [17], and the same argument is valid when T extends to Dunford–Schwartz operators.

Lemma 4. *Let $T \in DS^+$ be associated with a noncommutative probability space $(\mathcal{M}, \tau)$, and denote $A_n(T) = \frac{1}{n+1}\sum_{k=0}^{n} T^k$; then for any $s \geq 0$, there is a constant C such that*

$$\|\{A_n(T)(x)\}_n\|_{L_1 \log^s L(\ell_\infty)} \leq C\|x\|_{L_1 \log^{s+2} L}$$

holds for any $x \in L_1 \log^{s+2} L(\mathcal{M})$.

Lemma 5. *Let $T \in DS^+$ and $\beta = \{\beta_k\}_{k=0}^{\infty}$ be a bounded sequence of complex numbers, i.e., $\beta \in \ell_\infty(\mathbb{N}_0)$ and $|\beta_k| \leq \|\beta\|_\infty < \infty$ for every k.*
Denote

$$A_{\beta,n} = \frac{1}{n+1}\sum_{k=0}^{n} \beta_k T^k;$$

then for every $1 < p \leq \infty$, there exists a constant C_p such that

$$\left\|\sup_n{}^+ A_{\beta,n}(x)\right\|_p \leq C_p \|\beta\|_\infty \|x\|_p, \quad \forall x \in L_p(\mathcal{M});$$

for any $x \in L_1(\mathcal{M})$ and any $\lambda > 0$, there is a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^{\perp}) \leq 32\frac{\|x\|_1}{\lambda} \quad \textit{and} \quad \sup_n \left\|e(A_{\beta,n}(x))e\right\|_\infty \leq \|\beta\|_\infty \lambda.$$

Proof. (i) For every $x \in L_p(\mathcal{M})$, it has a linear decomposition $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_j \in L_p^+(\mathcal{M}),\ j = 1, 2, 3, 4$. By the triangle inequality and with $\|x_j\|_p \leq \|x\|_p$ for each j, we know that there is no loss of generality if we add a restriction by considering $x \in L_p^+(\mathcal{M})$. We decompose the mean $A_{\beta,n}$ as follows:

$$
\begin{aligned}
A_{\beta,n} &= \frac{1}{n+1}\sum_{k=0}^{n}\beta_k T^k \\
&= \frac{1}{n+1}\sum_{k=0}^{n}(Re[\beta_k]+\|\beta\|_\infty)T^k + i\frac{1}{n+1}\sum_{k=0}^{n}(Im[\beta_k]+\|\beta\|_\infty)T^k - \|\beta\|_\infty(1+i)\frac{1}{n+1}\sum_{k=0}^{n}T^k,
\end{aligned}
$$

where $0 \le Re[\beta_k] + \|\beta\|_\infty \le 2\|\beta\|_\infty$ and $0 \le Im[\beta_k] + \|\beta\|_\infty \le 2\|\beta\|_\infty$. Hence, we have

$$
0 \le \frac{1}{n+1}\sum_{k=0}^{n}(Re[\beta_k]+\|\beta\|_\infty)T^k(x) \le 2\|\beta\|_\infty \frac{1}{n+1}\sum_{k=0}^{n}T^k(x), \text{ for } x \in L_p^+(\mathcal{M}).
$$

By Proposition 2, we have

$$
\Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}(Re[\beta_k]+\|\beta\|_\infty)T^k(x)\Big\|_p \le 2\|\beta\|_\infty \Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}T^k(x)\Big\|_p, \quad \forall x \in L_p^+(\mathcal{M}).
$$

Similarly, we have

$$
\Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}(Im[\beta_k]+\|\beta\|_\infty)T^k(x)\Big\|_p \le 2\|\beta\|_\infty \Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}T^k(x)\Big\|_p, \quad \forall x \in L_p^+(\mathcal{M}).
$$

From Theorem 4.1 in [13], we know there exists a constant C_p' such that

$$
\Big\|{\sup_n}^+ \frac{1}{n+1}\sum_{k=0}^{n}T^k(x)\Big\|_p \le C_p'\|x\|_p, \quad \forall x \in L_p(\mathcal{M}).
$$

Then, by applying the triangle inequality to the norm several times, we obtain

$$
\big\|{\sup_n}^+ A_{\beta,n}(x)\big\|_p \le C_p\|\beta\|_\infty\|x\|_p,
$$

where $C_p = 6C_p'$.

(ii) Firstly we take the decomposition $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_j \in L_1^+(\mathcal{M})$ and $\|x_j\|_1 \le \|x\|_1,\ j = 1,2,3,4$. Applying Lemma 3 to each x_j, we know there is $e_j \in P(\mathcal{M})$ such that

$$
\tau(e_j^\perp) \le 8\frac{\|x_j\|_1}{\lambda} \quad \text{and} \quad \sup_n \|e_j(A_n(T)(x_j))e_j\|_\infty \le \frac{\lambda}{8}.
$$

Taking $e = \wedge_{j=1}^4 e_j$, we have

$$
\tau(e^\perp) \le 32\frac{\|x\|_1}{\lambda}
$$

and

$$
\begin{aligned}
&\sup_n \|e(A_{\beta,n}(x_j))e\|_\infty = \sup_n \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}\beta_k T^k(x_j)\right)e\right\|_\infty \\
&= \sup_n \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}Re[\beta_k]T^k(x_j)\right)e + e\left(\frac{1}{n+1}\sum_{k=0}^{n}iIm[\beta_k]T^k(x_j)\right)e\right\|_\infty \\
&\le \sup_n \left\{ \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}Re[\beta_k]T^k(x_j)\right)e\right\|_\infty + \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n+1}iIm[\beta_k]T^k(x_j)\right)e\right\|_\infty \right\} \\
&\le \sup_n \sup_{0\le k\le n}|\beta_k| \left\{ \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}T^k(x_j)\right)e\right\|_\infty + \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}T^k(x_j)\right)e\right\|_\infty \right\} \\
&\le \sup_n 2\|\beta\|_\infty \left\| e\left(\frac{1}{n+1}\sum_{k=0}^{n}T^k(x_j)\right)e\right\|_\infty \le \|\beta\|_\infty\frac{\lambda}{4}.
\end{aligned}
$$

Thus, by using the triangle inequality again,

$$\sup_n \|e(A_{\beta,n}(x))e\|_\infty$$

$$= \sup_n \|e(A_{\beta,n}(x_1) - A_{\beta,n}(x_2) + iA_{\beta,n}(x_3) - iA_{\beta,n}(x_4)e\|_\infty \le \|\beta\|_\infty \lambda.$$

□

Theorem 1. *Let $T \in DS^+$, and suppose that the subsequence $\underline{k} = \{k_i\}_{i=1}^\infty$ has a positive lower density. Then for any $1 < p \le \infty$, there exists a constant C_p such that*

$$\big\| \sup_n{}^+ A_n(\underline{k}, T)(x) \big\|_p \le C_p M_{\underline{k}} \|x\|_p\,, \quad \forall\, x \in L_p(\mathcal{M}).$$

Also, for any $x \in L_1(\mathcal{M})$ and any $\lambda > 0$, there exists a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^\perp) \le 32\frac{\|x\|_1}{\lambda} \quad \text{and} \quad \sup_n \big\| e(A_n(\underline{k}, T)(x))e \big\|_\infty \le M_{\underline{k}} \lambda.$$

Moreover, if the trace τ is finite, i.e., $(\mathcal{M}, \tau)$ is a noncommutative probability space, then for any $s \ge 0$, there exists a constant C such that

$$\big\| \{A_n(\underline{k}, T)(x)\}_n \big\|_{L_1 \log^s L(\ell_\infty)} \le C M_{\underline{k}} \|x\|_{L_1 \log^{s+2} L}\,, \quad \forall\, x \in L_1 \log^{s+2} L(\mathcal{M}).$$

Proof. (i) For every $x \in L_p(\mathcal{M})$, it has a linear decomposition $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_i \in L_p^+(\mathcal{M}), i = 1, 2, 3, 4$. Hence, we consider only $x \in L_p^+(\mathcal{M})$. Since

$$\begin{aligned} A_n(\underline{k}, T)(x) &= \frac{1}{n} \sum_{i=1}^n T^{k_i}(x) \\ &= \frac{1}{n} \sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i) T^i(x) \\ &= \frac{k_n + 1}{n} \cdot \frac{1}{k_n + 1} \sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i) T^i(x); \end{aligned}$$

then, we have

$$\big\|\sup_n{}^+ A_n(\underline{k}, T)(x) \big\|_p \le M_{\underline{k}} \Big\| \sup_n{}^+ \frac{1}{k_n + 1} \sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i) T^i(x) \Big\|_p.$$

By the definition of the norm of $L_p(\mathcal{M}; \ell_\infty)$, we know

$$\Big\|\sup_n{}^+ \frac{1}{k_n + 1} \sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i) T^i(x) \Big\|_p \le \Big\| \sup_n{}^+ \frac{1}{n + 1} \sum_{i=0}^{n} \mathbf{1}_{\underline{k}}(i) T^i(x) \Big\|_p.$$

Since $\mathbf{1}_{\underline{k}}(i) \le 1$ for all i, by Lemma 5, we have

$$\Big\|\sup_n{}^+ \frac{1}{n + 1} \sum_{i=0}^{n} \mathbf{1}_{\underline{k}}(i) T^i(x) \Big\|_p \le C_p \|x\|_p,$$

and then,

$$\big\|\sup_n{}^+ A_n(\underline{k}, T)(x) \big\|_p \le C_p M_{\underline{k}} \|x\|_p.$$

(ii) Since we have

$$\sup_n \|e(A_n(\underline{k},T)(x))e\|_\infty = \sup_n \left\| e\left(\frac{k_n+1}{n}\cdot\frac{1}{k_n+1}\sum_{i=0}^{k_n}\mathbf{1}_{\underline{k}}(i)T^i(x)\right)e\right\|_\infty$$
$$\leq M_{\underline{k}} \sup_n \left\| e\left(\frac{1}{k_n+1}\sum_{i=0}^{k_n}\mathbf{1}_{\underline{k}}(i)T^i(x)\right)e\right\|_\infty,$$

we can apply the weak-type result in Lemma 5, and we have the weak-type inequality for the subsequential case.

(iii) For the τ finite noncommutative Orlicz space case, a similar argument applies (with a minor change in notation in the proof of Lemma 5 and part (i) above); then, by Lemma 4, we finish the proof. □

Theorem 2. *Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a vector of d Dunford–Schwartz operators, and let $\underline{\mathbf{k}} = (\underline{k}^{(1)}, \cdots, \underline{k}^{(d)}) \subset \mathbb{N}_0^d$ be a vector of d sequences of strictly increasing non-negative integers, with every $\underline{k}^{(j)}$ having a positive lower density, where $j = 1, 2, \cdots, d$.*
Then, for any $1 < p \leq \infty$, there exists a constant C_p (inherited from Theorem 1) such that

$$\left\| {\sup_{\mathbf{n}}}^+ A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x)\right\|_p \leq C_p^d M_{\underline{\mathbf{k}}} \|x\|_p, \quad \forall\, x \in L_p(\mathcal{M});$$

if $2 < p \leq \infty$, we have

$$\left\|\{A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x)\}_{\mathbf{n}}\right\|_{L_p(\mathcal{M};\ell_\infty^c(\mathbb{N}^d))} \leq \sqrt{C_{p/2}M_{\underline{\mathbf{k}}}}\|x\|_p, \quad \forall\, x \in L_p(\mathcal{M}).$$

Moreover, if the trace τ is finite, i.e., $(\mathcal{M},\tau)$ is a noncommutative probability space, given $x \in L_1\log^{2(d-1)}L(\mathcal{M})$, for any $\lambda > 0$, there are a positive constant C and a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^\perp) \leq C\frac{\|x\|_{L_1\log^{2(d-1)}L}}{\lambda} \quad \text{and} \quad \sup_{\mathbf{n}} \|e(A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x))e\|_\infty \leq M_{\underline{\mathbf{k}}}\lambda;$$

for $x \in L_2\log^{2(d-1)}L(\mathcal{M})$, we have the following estimates:

$$\tau(e^\perp) \leq \left(C\frac{\|x\|_{L_2\log^{2(d-1)}L}}{\lambda}\right)^2 \quad \text{and} \quad \sup_{\mathbf{n}} \|(A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x))e\|_\infty \leq M_{\underline{\mathbf{k}}}\lambda.$$

Proof. For the same reason, we can restrict our consideration to $x \in L_p^+(\mathcal{M})$ and apply Theorem 1 with its equivalent formulations, e.g.,

$$\left\| {\sup_n}^+ A_n(\underline{k},T)(x)\right\|_p \leq C_p M_{\underline{k}} \|x\|_p$$

is equivalent to saying there is $a \in L_p^+(\mathcal{M})$ satisfying

$$\|a\|_p \leq C_p M_{\underline{k}} \|x\|_p \quad \text{and} \quad A_n(\underline{k},T)(x) \leq a, \ \forall\, n \geq 0.$$

Thus, the multi-parameter maximal inequalities can be obtained from iterations of the single-parameter case, and the "$p = 2$" cases are similar to Corollary 4.4 [13] and Theorem 3.5 [17]; this can be proved using the same arguments. □

Theorem 3. *Let $1 \leq p < \infty$ and $T \in DS^+$, and let $\underline{k} = \{k_i\}_{i=1}^{\infty}$ be a sequence of strictly increasing non-negative integers: $\underline{k}$ has a positive lower density. Then for any $x \in L_p(\mathcal{M})$ and any $\lambda > 0$, there is a projection $e \in P(\mathcal{M})$ such that*

$$\tau(e^{\perp}) \leq 4\left(\frac{\|x\|_p}{\lambda}\right)^p \quad \text{and} \quad \sup_n \left\|e(A_n(\underline{k}, T)(x))e\right\|_\infty \leq 8M_{\underline{k}}\lambda.$$

Proof. Since a weighted-version weak-type (p, p) inequality is obtained in Theorem 2.1 [30], by the same argument,

$$\begin{aligned}\sup_n \left\|e(A_n(\underline{k}, T)(x))e\right\|_\infty &= \sup_n \left\|e\left(\frac{k_n+1}{n} \cdot \frac{1}{k_n+1}\sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i)T^i(x)\right)e\right\|_\infty \\ &\leq M_{\underline{k}} \sup_n \left\|e\left(\frac{1}{k_n+1}\sum_{i=0}^{k_n} \mathbf{1}_{\underline{k}}(i)T^i(x)\right)e\right\|_\infty,\end{aligned}$$

and we immediately obtain the result. □

Remark 1. *We point out here that this subsequential version weak-type (p, p) inequality is a mere induction from Yeadon's weak-type $(1, 1)$ inequality (check the proof of Theorem 2.1 [30]) plus a "subsequential" argument; thus, it is independent of our "strong-type" result and has a better universal constant. Moreover, it is implied in the proof of Theorem 2.1 [30] that for $x \in L_p^+(\mathcal{M})$ and any $\lambda > 0$, there is a projection $e \in P(\mathcal{M})$ such that*

$$\tau(e^{\perp}) \leq \left(\frac{\|x\|_p}{\lambda}\right)^p \quad \text{and} \quad \sup_n \left\|e(A_n(\underline{k}, T)(x))e\right\|_\infty \leq 2M_{\underline{k}}\lambda.$$

Theorem 4. *Let $1 < p < \infty$, let $\mathbf{T} = (T_1, \cdots, T_d)$ be a vector of d Dunford–Schwartz operators, and let $\underline{\mathbf{k}} = (\underline{k}^{(1)}, \cdots, \underline{k}^{(d)}) \subset \mathbb{N}_0^d$ be a vector of d sequences of strictly increasing non-negative integers: every $\underline{k}^{(j)}$ has a positive lower density, where $j = 1, 2, \cdots, d$. There exists a positive constant C_p (inherited from Theorem 1) such that for any $x \in L_p(\mathcal{M})$ and any $\lambda > 0$, there is a projection $e \in P(\mathcal{M})$ such that*

$$\tau(e^{\perp}) \leq 4\left(C_p^{d-1}\frac{\|x\|_p}{\lambda}\right)^p \quad \text{and} \quad \sup_{\mathbf{n}} \left\|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\right\|_\infty \leq 8M_{\underline{\mathbf{k}}}\lambda.$$

Moreover, for $2 < p < \infty$, we have the following estimates:

$$\tau(e^{\perp}) \leq 2\left(C_{\frac{p}{2}}^{\frac{d-1}{2}}\frac{\|x\|_p}{\lambda}\right)^p \quad \text{and} \quad \sup_{\mathbf{n}} \left\|(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\right\|_\infty \leq 2\sqrt{2M_{\underline{\mathbf{k}}}}\lambda.$$

Proof. Let $1 < p < \infty$. By Theorem 1, we have

$$\left\|{\sup_n}^+ A_n(\underline{k}, T)(x)\right\|_p \leq C_p M_{\underline{k}}\|x\|_p, \text{ for } x \in L_p^+(\mathcal{M}).$$

We use an equivalent formulation: there exists an operator $a \in L_p^+(\mathcal{M})$ such that

$$A_n(\underline{k}, T)(x) \leq a, \ \forall n \quad \text{and} \quad \|a\|_p \leq C_p M_{\underline{k}}\|x\|_p.$$

Applying the previous formulation $d-1$ times, there exists an operator $x_{d-1} \in L_p^+(\mathcal{M})$ such that

$$A_{n_2}(\underline{k}^{(2)}, T_2)\cdots A_{n_d}(\underline{k}^{(d)}, T_d)(x) \leq x_{d-1}, \ \forall (n_2, \cdots, n_d) \quad \text{and} \quad \|x_{d-1}\|_p \leq C_p^{d-1}M_{\underline{k}^{(2)}}\cdots M_{\underline{k}^{(d)}}\|x\|_p.$$

Then by the previous theorem and Remark 1, this implies that for $x \in L_p^+(\mathcal{M})$ and any $\lambda > 0$, there is a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^\perp) \leq (C_p^{d-1} M_{\underline{k}^{(2)}} \cdots M_{\underline{k}^{(d)}} \frac{\|x\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty \leq 2M_{\underline{k}^{(1)}}\lambda.$$

It is equivalent to say that for any $\lambda > 0$ there is a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^\perp) \leq (C_p^{d-1} \frac{\|x\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty \leq 2M_{\underline{k}^{(1)}} M_{\underline{k}^{(2)}} \cdots M_{\underline{k}^{(d)}}\lambda.$$

Given $x \in L_p(\mathcal{M})$, we have $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_j \in L_p^+(\mathcal{M})$ and $\|x_j\|_p \leq \|x\|_p$ for each $j = 1, 2, 3, 4$. Hence, we have that for any $\lambda > 0$, there are $e_j \in P(\mathcal{M})$, $j = 1, 2, 3, 4$, satisfying

$$\tau(e_j^\perp) \leq (C_p^{d-1} \frac{\|x_j\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|e_j(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x_j))e_j\|_\infty \leq 2M_{\underline{k}^{(1)}} M_{\underline{k}^{(2)}} \cdots M_{\underline{k}^{(d)}}\lambda.$$

Taking $e = \wedge_{j=1}^4 e_j$, we have

$$\tau(e^\perp) \leq 4(C_p^{d-1} \frac{\|x\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty \leq 8M_{\underline{\mathbf{k}}}\lambda.$$

For $2 < p < \infty$ and $x \in L_p^h(\mathcal{M})$ (all self-adjoint operators in L_p), we apply the previous estimate to $|x|^2 \in L_{\frac{p}{2}}^+(\mathcal{M})$. That is, for any $\eta > 0$, there is a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^\perp) \leq (C_{\frac{p}{2}}^{d-1} \frac{\||x|^2\|_{\frac{p}{2}}}{\eta})^{\frac{p}{2}} \quad \text{and} \quad \sup_{\mathbf{n}} \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(|x|^2))e\|_\infty \leq 2M_{\underline{\mathbf{k}}}\eta.$$

Then for any $\lambda > 0$, taking $\eta = \lambda^2$, by the Kadison–Schwarz inequality

$$\|(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty^2 \leq \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(|x|^2))e\|_\infty \leq 2M_{\underline{\mathbf{k}}}\lambda^2,$$

we obtain

$$\tau(e^\perp) \leq (C_{\frac{p}{2}}^{\frac{d-1}{2}} \frac{\|x\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty \leq \sqrt{2M_{\underline{\mathbf{k}}}}\lambda.$$

Now, given $x \in L_p(\mathcal{M})$, $x = x_1 + ix_2$, where $x_j \in L_p^h(\mathcal{M})$ and $\|x_j\|_p \leq \|x\|_p$, $j = 1, 2$. Hence, for any $\lambda > 0$, there are $e_j \in P(\mathcal{M})$ such that

$$\tau(e_j^\perp) \leq (C_{\frac{p}{2}}^{\frac{d-1}{2}} \frac{\|x_j\|_p}{\lambda})^p \quad \text{and} \quad \sup_{\mathbf{n}} \|(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x_j))e_j\|_\infty \leq \sqrt{2M_{\underline{\mathbf{k}}}}\lambda, \ j = 1, 2.$$

By taking $e = e_1 \wedge e_2$, we obtain the final result. □

4. Noncommutative Wiener–Wintner-Type Subsequential Ergodic Theorems

We give in the following a result that acts as the Banach principle in the theory.

Lemma 6. *Let $\mathbf{T} = (T_1, \cdots, T_d)$ be a vector of d Dunford–Schwartz operators, and let $\mathbf{K}$ be a family of multi-parameter subsequences satisfying that each $\underline{\mathbf{k}} = (\underline{k}^{(1)}, \cdots, \underline{k}^{(d)}) \in \mathbf{K}$ is a vector of d sequences of strictly increasing non-negative integers and every $\underline{k}^{(j)}$ has a positive lower density, where $j = 1, 2, \cdots, d$.*

Let $1 < p < \infty$. If for a dense subset X of $L_p(\mathcal{M})$ we have $\forall x \in X$ and $\forall \varepsilon > 0$, there exists a projection $e \in P(\mathcal{M})$ such that

$$\tau(e^{\perp}) \leq \varepsilon \quad \textit{and} \quad \{(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\}$$

converges in $\mathcal{M}$ for all $\underline{\mathbf{k}} \in \mathbf{K}$, then $\mathbf{K}$ is of L_p-bsWW type. If $2 < p < \infty$, $\mathbf{K}$ is of L_p-sWW type.

Moreover, if the trace τ is finite, i.e., $(\mathcal{M}, \tau)$ is a noncommutative probability space, and X with the above property is dense in $L_1 \log^{2(d-1)} L(\mathcal{M})$, then $\mathbf{K}$ is of $L_1 \log^{2(d-1)} L$-bsWW type and of $L_2 \log^{2(d-1)} L$-sWW type.

Proof. For $1 < p < \infty$, taking any $x \in L_p(\mathcal{M})$, any $\varepsilon > 0$, and any $\lambda > 0$, since X is dense in $L_p(\mathcal{M})$, we can always find one $y \in X$ such that $\|x - y\|_p \leq \frac{\lambda(\frac{\varepsilon}{8})^{\frac{1}{p}}}{C_p^{d-1}}$, where C_p^{d-1} comes from the application of the first maximal inequality in Theorem 4 to the element $x - y$: there is a projection $e_1 \in P(\mathcal{M})$ such that

$$\tau(e_1^{\perp}) \leq 4(C_p^{d-1}\frac{\|x - y\|_p}{\lambda})^p \leq \frac{\varepsilon}{2} \text{ and } \sup_{\mathbf{n}} \|e_1(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x - y))e_1\|_\infty \leq 8M_{\underline{\mathbf{k}}}\lambda, \ \ \underline{\mathbf{k}} \in \mathbf{K}.$$

On the other hand, by the assumption, there is a projection $e_2 \in P(\mathcal{M})$ such that

$$\tau(e_2^{\perp}) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \{e_2(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y))e_2\} \text{ converges in } \mathcal{M},$$

which means that there exists $\mathbf{N} \in \mathbb{N}_0^d$ so that whenever $\mathbf{m}, \mathbf{n} \geq \mathbf{N}$, we have

$$\|e_2(A_{\mathbf{m}}(\underline{\mathbf{k}}, \mathbf{T})(y))e_2 - e_2(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y))e_2\| \leq \lambda.$$

Now, take $e = e_1 \wedge e_2$. Then we have $\tau(e^{\perp}) \leq \varepsilon$ and

$$\begin{aligned}
&\|e(A_{\mathbf{m}}(\underline{\mathbf{k}}, \mathbf{T})(x))e - e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\|_\infty \\
&\leq \|e(A_{\mathbf{m}}(\underline{\mathbf{k}}, \mathbf{T})(x - y))e\|_\infty + \|e(A_{\mathbf{m}}(\underline{\mathbf{k}}, \mathbf{T})(y))e - e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y))e\|_\infty + \|e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y - x))e\|_\infty \\
&\leq (16M_{\underline{\mathbf{k}}} + 1)\lambda.
\end{aligned}$$

This means that $\{e(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x))e\}$ is a Cauchy sequence and thus converges in $\mathcal{M}$ for all $\underline{\mathbf{k}} \in \mathbf{K}$. Therefore, we conclude that $\mathbf{K}$ is of L_p-bsWW type.

The rest can be shown by similar arguments with the use of the corresponding maximal inequalities in Theorem 4 and Theorem 2. □

As the main result of this paper, we give here the subsequential Wiener–Wintner-type ergodic theorem.

Theorem 5. *The subsequence class Δ is of L_p-bsWW type for $1 < p < \infty$ and L_p-sWW type for $2 < p < \infty$. Moreover, if τ is finite, Δ is of $L_1 \log^{2(d-1)} L$-bsWW type and $L_2 \log^{2(d-1)} L$-sWW type.*

Proof. We know that

$$\begin{aligned}
\Delta = \{&\underline{\mathbf{k}} = (\underline{k}^{(1)}, \cdots, \underline{k}^{(d)}) : \underline{k}^{(j)} \text{ has a density of one, a recurring uniform sequence, or a block} \\
&\text{sequence with positive lower density such that } \lim_{n \to \infty} \frac{N_{\mathcal{I}}(n)}{n} \to 0, \ \ j = 1, 2, \cdots, d\}.
\end{aligned}$$

Let $1 < p < \infty$. $T_j (j = 1, \cdots, d)$ induces a canonical splitting on $L_p(\mathcal{M})$; that is, $L_p(\mathcal{M}) = \mathcal{F}_p(T_j) \oplus \mathcal{F}_p(T_j)^{\perp} = \{x \in L_p(\mathcal{M}) : T_j(x) = x\} \oplus \overline{(I - T_j)(L_p(\mathcal{M}))}$. According to the decomposition, it is sufficient that we discuss x in each subset separately.

For $x \in \mathcal{F}_p(T_d)$, here we consider only the typical case $d = 2$; then $x \in \mathcal{F}_p(T_1) \cap \mathcal{F}_p(T_2)$ or $x \in \mathcal{F}_p(T_1)^\perp \cap \mathcal{F}_p(T_2)$.

When $x \in \mathcal{F}_p(T_1) \cap \mathcal{F}_p(T_2)$, the average $A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) = x$; hence, there is a projection $e_1 \in P(\mathcal{M})$ such that

$$\tau(e_1^\perp) \leq \varepsilon \text{ and } \|e_1(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - x)e_1\|_\infty \to 0 \text{ for all } \underline{\mathbf{k}} \in \Delta.$$

When $x \in \mathcal{F}_p(T_1)^\perp \cap \mathcal{F}_p(T_2)$, the average $A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) = A_{n_1}(\underline{k}^{(1)}, T_1)(x)$. Then it turns into the following problem.

For $x \in \mathcal{F}_p(T_d)^\perp$, we consider its dense subset $(I - T_d)(L_1(\mathcal{M}) \cap \mathcal{M})$ instead.

In the following, we describe the ordinary situation. Let $y \in L_1(\mathcal{M}) \cap \mathcal{M}$ and $z = y - T(y) \in (I - T)(L_1(\mathcal{M}) \cap \mathcal{M})$, $T \in DS^+$, $\underline{k}$ be a subsequence of $\{0, 1, 2, \cdots\}$.

(i) When $\underline{k}$ has a density of one, by denoting $\{0, 1, 2, \cdots, k_n\}$ simply as $[0, k_n]$, we have

$$\begin{aligned}
\big\|A_n(\underline{k}, T)(z)\big\|_\infty &= \Big\|\frac{1}{n}\sum_{i=1}^{n} T^{k_i}(z)\Big\|_\infty \\
&= \Big\|\frac{1}{n}\Big(\sum_{i=0}^{k_n} T^i(y) - \sum_{i=1}^{n} T^{k_i}(Ty)\Big) - \frac{1}{n}\sum_{[0,k_n]-\{k_i\}_{i=1}^n} T^i(y)\Big\|_\infty \\
&\leq \Big\|\frac{1}{n}\Big(\sum_{i=0}^{k_n} T^i(y) - \sum_{i=1}^{n} T^{k_i}(Ty)\Big)\Big\|_\infty + \Big\|\frac{1}{n}\sum_{[0,k_n]-\{k_i\}_{i=1}^n} T^i(y)\Big\|_\infty \\
&= \Big\|\frac{1}{n}\sum_{[0,k_n]-\{k_i+1\}_{i=1}^{n-1}} T^i(y) - \frac{1}{n}T^{k_n+1}(y)\Big\|_\infty + \Big\|\frac{1}{n}\sum_{[0,k_n]-\{k_i\}_{i=1}^n} T^i(y)\Big\|_\infty \\
&\leq \frac{k_n + 2 - (n-1)}{n}\|y\|_\infty + \frac{k_n + 1 - n}{n}\|y\|_\infty \\
&= \frac{4 + 2(k_n - n)}{n}\|y\|_\infty.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{k_n}{n} = 1$, we have

$$\|A_n(\underline{k}, T)(z)\|_\infty \to 0 \text{ as } n \to \infty.$$

(ii) When $\underline{k}$ is a uniform sequence, we discuss it as follows.

Let $(\Omega, \mathcal{B}, \mu, \varphi)$ and ω_0, Y be the apparatus connected with the sequence $\underline{k}$. Firstly, by the definition of a uniform subsequence, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} T^{k_i}(y - Ty) &= \frac{k_n + 1}{n}\frac{1}{k_n + 1}\sum_{i=1}^{n} T^{k_i}(y - Ty) \\
&= \frac{k_n + 1}{n}\frac{1}{k_n + 1}\sum_{i=0}^{k_n} T^i(y - Ty)\chi_Y(\varphi^i\omega_0).
\end{aligned}$$

Since $\lim_n \frac{k_n}{n} = \frac{1}{\mu(Y)}$, $M_{\underline{k}} = \sup_{n\geq 1} \frac{k_n+1}{n}$ is finite and positive. Next, we just need to estimate $\frac{1}{n}\sum_{i=0}^{n-1} T^i(y - Ty)\chi_Y(\varphi^i\omega_0)$. By Lemma 2, for any $\varepsilon > 0$, there exist open sets Y_1 and W such that for each point $\omega \in W$, there exists a non-negative integer n_0 such that

$$0 \leq \frac{1}{n}\sum_{i=0}^{n-1}[\chi_Y(\varphi^i\omega_0) - \chi_{Y_1}(\varphi^i\omega)] < \varepsilon \text{ for } n \geq n_0. \tag{1}$$

By the definition of a recurring uniform sequence $\underline{k}$, for the neighborhood W of ω_0 as above, there are $\omega_1 \in W$ and $i_0 \in \mathbb{N}$ such that $\varphi^{i_0}\omega_1 \in W$ and $\varphi^{i_0+1}\omega_1 \in W$. It is obvious that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}|\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \chi_{Y_1}(\varphi^i(\varphi^{i_0+1}\omega_1))| \\
=&\frac{1}{n}\sum_{i=0}^{n-1}|\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \chi_Y(\varphi^i\omega_0) + \chi_Y(\varphi^i\omega_0) - \chi_{Y_1}(\varphi^i(\varphi^{i_0+1}\omega_1))| \\
\leq&\frac{1}{n}\sum_{i=0}^{n-1}|\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \chi_Y(\varphi^i\omega_0)| + \frac{1}{n}\sum_{i=0}^{n-1}|\chi_Y(\varphi^i\omega_0) - \chi_{Y_1}(\varphi^i(\varphi^{i_0+1}\omega_1))|.
\end{aligned}$$

Then applying again the result as in Equation (1), there is n_0' such that for $n \geq n_0'$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1}|\chi_{Y_1}(\varphi^{i+i_0}\omega_1) - \chi_{Y_1}(\varphi^{i+i_0+1}\omega_1)| < 2\varepsilon.$$

Then it follows that for any $\varepsilon > 0$, there is a non-negative integer n_0'' such that for $n \geq n_0''$,

$$\begin{aligned}
&\Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y-Ty)\chi_Y(\varphi^i\omega_0)\Big\|_\infty \\
=&\Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y-Ty)[\chi_Y(\varphi^i\omega_0) - \chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) + \chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1))]\Big\|_\infty \\
\leq&\Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y-Ty)[\chi_Y(\varphi^i\omega_0) - \chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1))]\Big\|_\infty \\
&+\Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y-Ty)\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1))\Big\|_\infty \\
<&2\|y\|_\infty\varepsilon + \Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y)\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \frac{1}{n}\sum_{i=0}^{n-1}T^{i+1}(y)\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1))\Big\|_\infty \\
=&2\|y\|_\infty\varepsilon + \Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y)\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \frac{1}{n}\sum_{i=0}^{n-1}T^{i+1}(y)[\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) \\
&-\chi_{Y_1}(\varphi^{i+1}(\varphi^{i_0}\omega_1)) + \chi_{Y_1}(\varphi^{i+1}(\varphi^{i_0}\omega_1))]\Big\|_\infty \\
=&2\|y\|_\infty\varepsilon + \Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^i(y)\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \frac{1}{n}\sum_{i=0}^{n-1}T^{i+1}(y)\chi_{Y_1}(\varphi^{i+1}(\varphi^{i_0}\omega_1)) \\
&-\frac{1}{n}\sum_{i=0}^{n-1}T^{i+1}(y)[\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \chi_{Y_1}(\varphi^{i+1}(\varphi^{i_0}\omega_1))]\Big\|_\infty \\
\leq&2\|y\|_\infty\varepsilon + \frac{2}{n}\|y\|_\infty + \Big\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+1}(y)[\chi_{Y_1}(\varphi^i(\varphi^{i_0}\omega_1)) - \chi_{Y_1}(\varphi^{i+1}(\varphi^{i_0}\omega_1))]\Big\|_\infty \\
\leq&\|y\|_\infty(2\varepsilon + \varepsilon + 2\varepsilon) \\
=&5\|y\|_\infty\varepsilon.
\end{aligned}$$

So finally, we have that for any $\varepsilon > 0$, there is $n_0'' \geq 0$ such that

$$\|\frac{1}{n}\sum_{i=1}^{n}T^{k_i}(y-Ty)\|_\infty \leq 5M_{\underline{k}}\|y\|_\infty\varepsilon \text{ for } n \geq n_0''.$$

This also implies that

$$\|A_n(\underline{k}, T)(z)\|_\infty \to 0 \text{ as } n \to \infty.$$

(iii) When $\underline{k}$ is a block sequence with a positive lower density such that $\lim_{n\to\infty} \frac{N_{\mathcal{I}}(n)}{n} \to 0$,

$$\begin{aligned}
A_n(\underline{k}, T)(z) &= \frac{1}{n}\sum_{j=1}^{n} T^{k_j}(y - T(y)) = \frac{1}{n}\sum_{j=1}^{n}\Big(T^{k_j}(y) - T^{k_j+1}(y)\Big) \\
&= \frac{1}{n}\Big((T^{a_{N_{\mathcal{I}}(n)}}(y) - T^{k_n+1}(y)) + \sum_{i=1}^{N_{\mathcal{I}}(n)-1}\sum_{j=a_i}^{b_i}\Big(T^{j}(y) - T^{j+1}(y)\Big)\Big) \\
&= \frac{1}{n}\Big((T^{a_{N_{\mathcal{I}}(n)}}(y) - T^{k_n+1}(y)) + \sum_{i=1}^{N_{\mathcal{I}}(n)-1}\Big(T^{a_i}(y) - T^{b_i+1}(y)\Big)\Big)
\end{aligned}$$

and since T is a contraction on $\mathcal{M}$, we obtain

$$\|A_n(\underline{k}, T)(z)\|_\infty \leq 2\|y\|_\infty \frac{N_{\mathcal{I}}(n)+1}{n}.$$

By $\lim_{n\to\infty} \frac{N_{\mathcal{I}}(n)}{n} \to 0$, we have

$$\|A_n(\underline{k}, T)(z)\|_\infty \to 0 \text{ as } n \to \infty.$$

Since Dunford–Schwarz operators are contractions on $\mathcal{M}$, we come to a conclusion summing up the above results: Let $z \in (I - T_d)(L_1(\mathcal{M}) \cap \mathcal{M})$ and $\underline{\mathbf{k}} \in \Delta$; we have

$$\big\|A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(z)\big\|_\infty \leq \big\|A_{n_d}(\underline{k}^{(d)}, T_d)(z)\big\|_\infty \to 0 \text{ as } \mathbf{n} \to \infty.$$

This implies that we have the unit operator $I \in P(\mathcal{M})$ such that

$$\tau(I^\perp) \leq \varepsilon \text{ and } \|A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(z)I\|_\infty \to 0 \text{ for all } \underline{\mathbf{k}} \in \Delta.$$

Then, as $\mathcal{F}_p(T_d) \oplus (I - T_d)(L_1(\mathcal{M}) \cap \mathcal{M})$ is dense in $L_p(\mathcal{M})$, from Theorem 6, we know Δ is of L_p-bsWW type.

For the $2 < p < \infty$ and τ finite Orlicz spaces case, these are the corresponding consequences of applying Theorem 6. □

Corollary 1. *Let $\mathbf{T} = (T_1, \cdots, T_d)$, and let F_j be the projection onto the fixed-point subspace for $T_j \in DS^+$, $j = 1, \cdots, d$. Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$; then, for any $\varepsilon > 0$, there exists $e \in P(\mathcal{M})$ such that $\tau(e^\perp) \leq \varepsilon$ and*

$$\lim_{\mathbf{n}} \big\|e\big(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1 \cdots F_d(x)\big)e\big\|_\infty = 0 \textit{ for all } \underline{\mathbf{k}} \in \Delta,$$

and if $p > 2$, there exists $\tilde{e} \in P(\mathcal{M})$ such that $\tau(\tilde{e}^\perp) \leq \varepsilon$ and

$$\lim_{\mathbf{n}} \big\|\big(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1 \cdots F_d(x)\big)\tilde{e}\big\|_\infty = 0 \textit{ for all } \underline{\mathbf{k}} \in \Delta.$$

Consequently, every $\underline{\mathbf{k}} \in \Delta$ is bilaterally good universal in $L_p(\mathcal{M})$ if $1 < p < \infty$ and good universal if $p > 2$.
Moreover, if the trace τ is finite, let $x \in L_1 \log^{2(d-1)} L(\mathcal{M})$; then for any $\varepsilon > 0$, there exists $e \in P(\mathcal{M})$ such that $\tau(e^\perp) \leq \varepsilon$ and

$$\lim_{\mathbf{n}} \big\|e\big(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1 \cdots F_d(x)\big)e\big\|_\infty = 0 \textit{ for all } \underline{\mathbf{k}} \in \Delta,$$

and for $x \in L_2 \log^{2(d-1)} L(\mathcal{M})$, there exists $\tilde{e} \in P(\mathcal{M})$ such that $\tau(\tilde{e}^\perp) \leq \varepsilon$ and

$$\lim_{\mathbf{n}} \big\|\big(A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1 \cdots F_d(x)\big)\tilde{e}\big\|_\infty = 0 \textit{ for all } \underline{\mathbf{k}} \in \Delta;$$

thus, every $\underline{\mathbf{k}} \in \Delta$ *is bilaterally good universal in* $L_1 \log^{2(d-1)}(\mathcal{M})$ *and good universal in* $L_2 \log^{2(d-1)}(\mathcal{M})$.

Proof. We consider only the typical case $d = 2$. Note that

$$A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T}) = A_{n_1}(\underline{k}^{(1)}, T_1) A_{n_2}(\underline{k}^{(2)}, T_2).$$

For $1 < p < \infty$, fix $x \in L_p(\mathcal{M})$ and decompose x as $x = F_2(x) + y_k + u_k$ with

$$y_k \in (I - T_2)(L_1(\mathcal{M}) \cap \mathcal{M}), \quad u_k \in L_p(\mathcal{M}), \quad \|u_k\|_p \le \frac{1}{k}, \ k \in \mathbb{N}.$$

Similarly, we decompose $F_2(x)$ with respect to $T_1 : F_2(x) = F_1(F_2(x)) + z_k + v_k$, with

$$z_k \in (I - T_1)(L_1(\mathcal{M}) \cap \mathcal{M}), \quad v_k \in L_p(\mathcal{M}), \quad \|v_k\|_p \le \frac{1}{k}, \ k \in \mathbb{N}.$$

Applying $A_{n_2}(\underline{k}^{(2)}, T_2)$ to x and $A_{n_1}(\underline{k}^{(1)}, T_1)$ to $F_2(x)$, we obtain

$$A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1F_2(x) = A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y_k) + A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(u_k) + A_{n_1}(\underline{k}^{(1)}, T_1)(z_k) + A_{n_1}(\underline{k}^{(1)}, T_1)(v_k).$$

By the multi-parameter maximal inequality (Theorem 2),

$$\|{\sup_{\mathbf{n}}}^+ A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(u_k)\|_p \le C_p^2 M_{\underline{k}^{(1)}} M_{\underline{k}^{(2)}} \|u_k\|_p \le \frac{C_p^2 M_{\underline{k}^{(1)}} M_{\underline{k}^{(2)}}}{k} \to 0 \text{ as } k \to \infty.$$

Similarly,

$$\lim_{k\to\infty} \|{\sup_{n_1}}^+ A_{n_1}(\underline{k}^{(1)}, T_1)(v_k)\|_p = 0.$$

Thus,

$$\lim_{k\to\infty} (A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y_k) + A_{n_1}(\underline{k}^{(1)}, T_1)(z_k)) = A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(x) - F_1(x)F_2(x) \quad \text{in } L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^2)).$$

Since $L_p(\mathcal{M}; c_0(\mathbb{N}^2))$ is closed in $L_p(\mathcal{M}; \ell_\infty(\mathbb{N}^2))$, it remains to be shown that

$$\{A_{\mathbf{n}}(\underline{\mathbf{k}}, \mathbf{T})(y_k)\}_{\mathbf{n}} \in L_p(\mathcal{M}; c_0(\mathbb{N}^2) \text{ and } \{A_{n_1}(\underline{k}^{(1)}, T_1)(z_k)\}_{n_1} \in L_p(\mathcal{M}; c_0).$$

Firstly, we consider the one-parameter case. In general, let $z = y - T_1(y) \in (I - T_1)(L_1(\mathcal{M}) \cap \mathcal{M})$. From the arguments related to the three classes of sequences in the proof of Theorem 5, we obtain the following results, respectively:

(i) $\sup_{m_1 \le j_1 \le n_1} \|A_{j_1}(\underline{k}^{(1)}, T_1)(z)\|_\infty \le \sup_{m_1 \le j_1 \le n_1} \frac{4+2(k_{j_1} - j_1)}{j_1} \|y\|_\infty$;
(ii) $\sup_{n_0'' \le m_1 \le j_1 \le n_1} \|A_{j_1}(\underline{k}^{(1)}, T_1)(z)\|_\infty \le \sup_{n_0'' \le m_1 \le j_1 \le n_1} 5M_{\underline{k}} \|y\|_\infty \varepsilon$;
(iii) $\sup_{m_1 \le j_1 \le n_1} \|A_{j_1}(\underline{k}^{(1)}, T_1)(z)\|_\infty \le 2\|y\|_\infty \sup_{m_1 \le j_1 \le n_1} \frac{N_{\mathcal{I}}(j_1)+1}{j_1}$.

In the following, we focus on class (i); the other classes are similar.

Since $z \in L_q(\mathcal{M})$ for any $1 < q < \infty$, we deduce from Theorem 1 that $\{A_{n_1}(\underline{k}^{(1)}, T_1)(z)\}_{n_1}$ belongs to $L_q(\mathcal{M}; \ell_\infty)$. Choose a $q \in (1, p)$. Then by Proposition 1, for any $m_1 < n_1$,

$$\begin{aligned} \Big\| {\sup_{m_1 \le j_1 \le n_1}}^+ A_{j_1}(\underline{k}^{(1)}, T_1)(z) \Big\|_p &\le \sup_{m_1 \le j_1 \le n_1} \|A_{j_1}(\underline{k}^{(1)}, T_1)(z)\|_\infty^{1-\frac{q}{p}} \Big\| {\sup_{m_1 \le j_1 \le n_1}}^+ A_{j_1}(\underline{k}^{(1)}, T_1)(z) \Big\|_q^{\frac{q}{p}} \\ &\le \Big[\sup_{m_1 \le j_1 \le n_1} \frac{4 + 2(k_{j_1} - j_1)}{j_1} \|y\|_\infty \Big]^{1-\frac{q}{p}} \Big\| {\sup_{j_1 \ge 1}}^+ A_{j_1}(\underline{k}^{(1)}, T_1)(z) \Big\|_q^{\frac{q}{p}}. \end{aligned}$$

As $\lim_{j_1\to\infty}\frac{k_{j_1}}{j_1}=1$, the finite sequence $(A_1(\underline{k}^{(1)},T_1)(z),\cdots,A_l(\underline{k}^{(1)},T_1)(z),0,\cdots)$ converges to $\{A_{n_1}(\underline{k}^{(1)},T_1)(z)\}_{n_1}$ in $L_p(\mathcal{M};\ell_\infty)$ as $l\to\infty$. Combining with $L_p(\mathcal{M};c_0)$ and being closed in $L_p(\mathcal{M};\ell_\infty)$, we have $\{A_{n_1}(\underline{k}^{(1)},T_1)(z)\}_{n_1}\in L_p(\mathcal{M};c_0)$. That is to say, $\{A_{n_1}(\underline{k}^{(1)},T_1)(z_k)\}_{n_1}\in L_p(\mathcal{M};c_0)$, $k\in\mathbb{N}$.

For the two-parameter case, let $z=y-T_2(y)\in(I-T_2)(L_1(\mathcal{M})\cap\mathcal{M})$. Since $z\in L_q(\mathcal{M})$ for any $1<q<\infty$, we deduce from Theorem 2 that $\{A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(z)\}_{\mathbf{n}}\in L_q(\mathcal{M};\ell_\infty(\mathbb{N}^2))$. Then by the interpolation theorem and with T_1 being a contraction, for any $\mathbf{m}<\mathbf{n}$,

$$\begin{aligned}\Big\|{\sup_{\mathbf{m}\le\mathbf{j}\le\mathbf{n}}}^{+}A_{\mathbf{j}}(\underline{\mathbf{k}},\mathbf{T})(z)\Big\|_p &\le \sup_{\mathbf{m}\le\mathbf{j}\le\mathbf{n}}\|A_{\mathbf{j}}(\underline{\mathbf{k}},\mathbf{T})(z)\|_\infty^{1-\frac{q}{p}}\Big\|{\sup_{\mathbf{m}\le\mathbf{j}\le\mathbf{n}}}^{+}A_{\mathbf{j}}(\underline{\mathbf{k}},\mathbf{T})(z)\Big\|_q^{\frac{q}{p}}\\ &\le \sup_{m_2\le j_2\le n_2}\|A_{j_2}(\underline{k}^{(2)},T_2)(z)\|_\infty^{1-\frac{q}{p}}\Big\|{\sup_{\mathbf{m}\le\mathbf{j}\le\mathbf{n}}}^{+}A_{\mathbf{j}}(\underline{\mathbf{k}},\mathbf{T})(z)\Big\|_q^{\frac{q}{p}}\\ &\le\left[\sup_{m_2\le j_2\le n_2}\frac{4+2(k_{j_2}-j_2)}{j_2}\|y\|_\infty\right]^{1-\frac{q}{p}}\Big\|{\sup_{\mathbf{j}\ge\mathbf{1}}}^{+}A_{\mathbf{j}}(\underline{\mathbf{k}},\mathbf{T})(z)\Big\|_q^{\frac{q}{p}}.\end{aligned}$$

By a similar argument as above, we have that $\{A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(z)\}_{\mathbf{n}}\in L_p(\mathcal{M};c_0(\mathbb{N}^2))$. That is to say, $\{A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(y_k)\}_{\mathbf{n}}\in L_p(\mathcal{M};c_0(\mathbb{N}^2))$.

Thus, by Lemma 6.2 [13], let $x\in L_p(\mathcal{M})$; we have that for any $\varepsilon>0$ and any $\underline{\mathbf{k}}\in\Delta$, there exists $e\in P(\mathcal{M})$ such that $\tau(e^\perp)\le\varepsilon$ and

$$\lim_{\mathbf{n}}\big\|e\big(A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x)-F_1\cdots F_d(x)\big)e\big\|_\infty=0.$$

Combining with Theorem 5, i.e., Δ is of L_p-bsWW type, we have that for any $\varepsilon>0$, there exists a projection $e\in P(\mathcal{M})$ such that $\tau(e^\perp)\le\varepsilon$ and

$$\lim_{\mathbf{n}}\big\|e\big(A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x)-F_1\cdots F_d(x)\big)e\big\|_\infty=0\text{ for all }\underline{\mathbf{k}}\in\Delta.$$

For $2<p<\infty$, we also need only to show

$$\{A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(y_k)\}_{\mathbf{n}}\in L_p(\mathcal{M};c_0^c(\mathbb{N}^2))\text{ and }\{A_{n_1}(\underline{k}^{(1)},T_1)(z_k)\}_{n_1}\in L_p(\mathcal{M};c_0^c).$$

Applying the second part of Theorem 2 and the interpolation theorem for $L_p(\mathcal{M};c_0^c)$, we can prove this with a similar argument. Combined with the fact that Δ is of L_p-sWW type, there exists a projection $\tilde{e}\in P(\mathcal{M})$ such that $\tau(\tilde{e}^\perp)\le\varepsilon$ and

$$\lim_{\mathbf{n}}\big\|\big(A_{\mathbf{n}}(\underline{\mathbf{k}},\mathbf{T})(x)-F_1\cdots F_d(x)\big)\tilde{e}\big\|_\infty=0\text{ for all }\underline{\mathbf{k}}\in\Delta.$$

In the end, the Orlicz space case can be reasoned analogously. □

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