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Morse Thoery of Saddle Point Reduction with Applications

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Abstract: In this paper, we demonstrate that when saddle point reduction is applicable, there is a clear relationship between the Morse index and the critical groups before and after the reduction. As an application of this result, we use saddle point reduction along with the critical point theorem to show the existence of periodic solutions in second-order Hamiltonian systems.

Keywords: Morse index; critical group; Morse theory; critical point theorem; saddle point reduction; Hamiltonian systems; peoriodic solutions

MSC: 58E05; 37J12; 37J46; 37B30

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1. Introduction

Differential equations are essential mathematical tools employed to model and analyze a diverse array of phenomena across various scientific and engineering disciplines. Consequently, investigating the existence of solutions to differential equations has become a significant area of study. To address this, numerous mathematical theories have been applied, including phase space theories [1–5], smooth theory [6–8], operator methods [9–12], and critical point theory [13,14].

Let H be a real Hilbert space, where we consider the self-adjoint operator equation:

$$Ax = G(x), \qquad x \in \mathrm{D}(A) \subset H, \tag{1}$$

where A is a self-adjoint operator with domain $\mathrm{D}(A) \subset H$, and F is a potential operator such that $G(x) = \Psi'(x)$, $\Psi \in C^2(H, \mathbb{R})$ and $\Psi(0) = 0$.

Many problems can indeed be represented by the operator Equation (1), such as Laplace's equation on bounded domains with a Dirichlet boundary, periodic solutions of Hamiltonian systems [15,16], the Schrödinger equation [17–19], periodic solutions of the wave equation [14], resonant elliptic systems [20–22], and others. For these problems, the calculus of variations indicates that the solutions of Equation (1) correspond to the critical points of a functional on a Hilbert space ([23], Chapter 1). Thus, finding solutions to Equation (1) translates into finding the critical points of a functional. Infinite-dimensional Morse theory is articularly useful for obtaining critical points for the functional [13,24].

However, if the Palais—Smale (PS) condition does not hold, these methods become significantly more challenging [21,22]. To address this issue, saddle point reduction is a viable approach. The theory of saddle point reduction (also known as Lyapunov-Schmidt reduction in some literature) was established by Amann in 1979 [25]. Since then, it has become an important tool in critical point theory and has been widely used to solve various boundary value problems [13,26,27]. Therefore, the first step of our study is to perform a saddle point reduction.

According to operator spectral theory, the Hilbert space H can be decomposed as follows:

$$H = H^- \oplus H^0 \oplus H^+,$$

where $\langle Ax, x\rangle > 0$ for $x \in H^+\backslash\{0\}$, $\langle Ax, x\rangle < 0$ for $x \in H^-\backslash\{0\}$,and $H^0 = \ker A$. Let $\sigma(A)$ denote the spectrum of A, σ_d denote the eigenvalues with finite multiplicity, and σ_e denote the essential spectrum of A. We introduce two important conditions:

($\mathbf{C}_1$) There is a constant $\alpha > 0$ such that $\alpha \notin \sigma(A)$ and $\sigma(A) \cap [-\alpha, \alpha] \subset \sigma_d(A)$ contain at most finitely many eigenvalues.

($\mathbf{C}_2$) The operator F is Gâteaux-differentiable in H and satisfies

$$\left\|G'(x)\right\| \leq \alpha, \quad \forall x \in H. \tag{2}$$

We define the following functional:

$$g(x) = \frac{1}{2}\langle Ax, x\rangle - \Phi(x), \qquad x \in \mathrm{D}(A) \subset H \tag{3}$$

It is clear that x is a critical point of $g(x)$ if and only if x is a solution of Equation (1). Let x be a critical point of g. We denote the **Morse index** of x by $m^-(g''(x))$, which is the dimension of the largest negative space of $g''(x)$. Similarly, we denote the positive Morse index of x by $m^+(g''(x))$, which is the dimension of the largest positive space of $g''(x)$.

Let $P_0 : H \to \ker A$ be a projection operator, and we define an operator on $\mathrm{D}(A)$ as follows:

$$A_0 x = Ax + P_0 x, \qquad x \in \mathrm{D}(A).$$

Then A_0 is a self-adjoint invertible Fredholm operator.

Without loss of generality, we fix a constant C such that

$$C \notin \sigma(A). \tag{4}$$

Let $\{E_\lambda\}$ be the spectral family associated with A_0. Here are three projections on the space H defined as follows:

$$P^- = \int_{-\infty}^{-C} \mathrm{d}E_\lambda, \quad P = \int_{-C}^{C} \mathrm{d}E_\lambda, \quad P^+ = \int_{C}^{\infty} \mathrm{d}E_\lambda.$$

Then, we can decompose the space H as follows.

$$H = H_- \oplus H_0 \oplus H_+, \tag{5}$$

where $H_\pm = P^\pm H$, and $H_0 = PH$.

Using ([13], Chapter IV, Theorem 2.1), we reduce the Equation (1) to the finite dimensional case.

Proposition 1. *Given assumptions ($\mathbf{C}_1$) and ($\mathbf{C}_2$), there exist a functional $a \in C^2(H_0, \mathbb{R})$ and an injective map $u \in C^1(H_0, \mathrm{D}(A))$ such that the following conditions are satisfied:*

1. *The map u can be expressed as $u(z) = w(z) + z$, where $Pw(z) = 0$.*
2. *The functional a adheres to the following:*

$$\begin{aligned} a(z) &= g(u(z)) = \frac{1}{2}\langle Au(z), u(z)\rangle - \Psi(u(z)), \\ a'(z) &= Az - PF(u(z)) = Au(z) - G(u(z)), \\ a''(z) &= AP - P\mathrm{d}G(u(z))u'(z) = [A - \mathrm{d}G(u(z))]u'(z). \end{aligned} \tag{6}$$

3. *z is a critical point of a if and only if $u(z)$ is a critical point of g, which is equivalent to $u(z)$ being a solution to the operator Equation (1).*

Let $H_q(A,B)$ represent the qth singular relative homology group of the topological pair (A,B) with coefficients in a field $\mathcal{F}$. Consider x as an isolated critical point of the function g with $g(x)=c$. The group defined by

$$C_q(g,x) := H_q(g^c, g^c \setminus \{x\}), \quad q \in \mathbb{Z},$$

which is known as the qth critical group of g at the point x, where $g^c = \{x \in H \mid g(x) \le c\}$.

Let us consider the following condition:

($\mathbf{C}_1'$) The spectrum of A consists solely of eigenvalues, meaning it is a point spectrum, and $\dim H^- < \infty$.

It is straightforward to deduce that condition ($\mathbf{C}_1$) follows from ($\mathbf{C}_1'$). Therefore, under conditions ($\mathbf{C}_1'$) and ($\mathbf{C}_2$), Proposition 1 remains valid. Additionally, with condition ($\mathbf{C}_1'$) in place, we can define the Morse index and critical groups for the functional Equation (3).

This leads to a natural question: what is the relationship between the Morse index and the critical groups before and after applying the saddle point reduction? Our main result addresses this question as follows.

Theorem 1. *Consider a real Hilbert space H, and let $g \in C^2(H,\mathbb{R})$ be the functional defined as in Equation (3). If the conditions ($\mathbf{C}_1'$) and ($\mathbf{C}_2$) are satisfied, and $z \in H_0$ is an isolated critical point of the reduced functional a, then the following results hold:*

1. *The Morse index of $g''(u(z))$, denoted by $m^-(g''(u(z)))$, is equal to the Morse index of $a''(z)$, denoted by $m^-(a''(z))$, i.e., $m^-(g''(u(z))) = m^-(a''(z)) + d$, where $d = \dim P^-H$.*
2. *The critical group $C_q(g,u(z))$ at $u(z)$ is related to the critical group of $a''(z)$ by $C_q(g,u(z)) = C_{q-d}(a''(z))$ for all $q = 0,1,2,\ldots$.*

Our exploration of the connection between the Morse index and the critical groups before and after saddle point reduction is driven by the study of multiple solutions in second-order Hamiltonian systems. In the third section of this paper, we apply our abstract results to asymptotically linear second-order Hamiltonian systems. These types of problems have garnered significant attention in recent years, as noted in [28–31].

Specifically, we focus on the following boundary value problem for second-order Hamiltonian systems:

$$\begin{cases} -\ddot{x}(t) = V_x(t,x(t)), & t \in [0,\tau], \\ x(0) = x(\tau), \quad \dot{x}(0) = \dot{x}(\tau), \end{cases} \tag{7}$$

where $V \in C^2(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ satisfies $V(t+\tau,x) = V(t,x)$ for some $\tau > 0$, and it adheres to the linear growth condition:

$$|V_x(t,x)| \le C_1(1+|x|), \quad (t,x) \in S_\tau \times \mathbb{R}^n, \tag{8}$$

where $C_1 > 0$ is a constant, and $S_\tau = \mathbb{R}/(\tau\mathbb{Z})$. We can assume without loss of generality that $V(t,0) = 0$. Additionally, we assume $V_x(t,0) = 0$, so $x = 0$ is a trivial solution of Equation (7). Our goal is to find nontrivial τ-periodic solutions to the system Equation (7).

Consider $A(t)$, a continuous symmetric matrix function that is τ-periodic. We focus on the following eigenvalue problem:

$$-\ddot{x} - A(t)x = \lambda x \tag{9}$$

subject to τ-periodic boundary conditions. It is well established that there exists a complete sequence of distinct eigenvalues

$$-\infty < \lambda_1(A) < \lambda_2(A) < \cdots \tag{10}$$

such that $\lambda_n(A) \to +\infty$ as $n \to \infty$.

Next, we outline several key assumptions regarding the nonlinearity $V(t,x)$:

$(\mathbf{V}_1)$ There exists a τ-periodic continuous symmetric positive definite matrix function $A(t)$ with eigenvalue $\lambda_m(A) = 0$ for some $m \in \mathbb{N}$ such that

$$V_x(t,x) = A(t)x + K_x(t,x), \tag{11}$$

where $K_x(t,x) = o(|x|)$ as $|x| \to 0$.

$(\mathbf{V}_2^{\pm})$ There exists $\delta > 0$ such that $\pm K(t,x) > 0$ for all $t \in \mathbb{R}$, and $0 < |x| < \delta$.

$(\mathbf{V}_3)$ There exists a τ-periodic continuous symmetric matrix function $B(t)$ with eigenvalue $\lambda_m(B) = 0$ for some $m \in \mathbb{N}$ such that

$$V_x(t,x) = B(t)x + L_x(t,x), \tag{12}$$

where $L_x(t,x) = o(|x|)$ as $|x| \to +\infty$.

$(\mathbf{V}_4)$ There exists a constant $C(B) > 0$ such that $C(B) \neq \lambda_i(B)$ for any $i \in \mathbb{N}$, and

$$|L_{xx}(t,x)| < C(B) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{13}$$

$(\mathbf{V}_5)$ There exists a constant $l < 0$ such that

$$L(t,x) < l|x| \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

and $l < \lambda(B)$ for all $\lambda(B) \in (-C(B), C(B))$.

Define

$$d_m := \sum_{i=1}^{m} \dim\ker\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - A(t) - \lambda_m(A)\right). \tag{14}$$

With this notation, our second main result can be stated as follows:

Theorem 2. *Suppose $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ satisfies $V(t+\tau,x) = V(t,x)$ for some $\tau > 0$ and the condition Equation (8). If assumptions $(\mathbf{V}_1), (\mathbf{V}_3), (\mathbf{V}_4)$, and $(\mathbf{V}_5)$ are satisfied and if $d_m \neq d$, then the system of Equation (7) has at least two nontrivial solutions in the following cases:*

(i) $(\mathbf{V}_2^{+})$ *holds with $d_m \neq d$;*

(ii) $(\mathbf{V}_2^{-})$ *holds with $d_{m-1} \neq d$.*

2. Preliminaries

In this section, we provide the proof of Theorem 1. As a preliminary, we introduce several key definitions and lemmas, all of which are referenced from [13].

First, we state a technical assumption—the Palais–Smale condition, which is frequently encountered in critical point theory. Consider H as a separable Hilbert space.

Definition 1. *Let g be a functional defined on H. The functional g is said to be Fréchet-differentiable at $u \in H$ if there exists a continuous linear map $L = L(u) : H \to \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, u) > 0$ such that*

$$|g(u+v) - g(u) - Lv| \leq \varepsilon\|v\|$$

for all $\|v\| \leq \delta$. The mapping L is typically denoted by $g'(u)$.

A critical point u of g is a point where $g'(u) = 0$, that is,

$$g'(u)\varphi = 0$$

for all $\varphi \in H$. The value of g at u is then referred to as a critical value of g.

Let $C^1(H,\mathbb{R})$ represent the set of functionals that are Fréchet-differentiable and whose Fréchet derivatives are continuous on H.

Definition 2. *We say that a functional $g \in C^1(H,\mathbb{R})$ satisfies the Palais–Smale condition (denoted as (PS)) if any sequence $(u_m) \subset H$ for which $g(u_m)$ is bounded, and $g'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.*

Remark 1. *The (PS) condition is a useful way to introduce some "compactness" into the functional g. Specifically, observe that (PS) implies that*

$$K_c \equiv \{u \in H \mid g(u) = c \text{ and } g'(u) = 0\}$$

meaning that the set of critical points with critical value c is compact for any $c \in \mathbb{R}$.

The first lemma describes the critical group in terms of the Morse index.

Lemma 1 ([13]). *Consider $g \in C^2(H,\mathbb{R})$, and let u be a nondegenerate critical point of g with Morse index j. Then, $C_q(g,u) = \delta_{q,j}\mathcal{F}$.*

For a critical point, which may be degenerate, the following significant result is established:

Proposition 2 ([32], Corollary 8.4). *Let $g \in C^2(E,\mathbb{R})$ and u_0 be an isolated critical point with finite Morse index μ and nullity ν. If $g''(u_0)$ is a Fredholm operator, then $C_q(g,u_0) \cong 0$ for $q \notin [\mu,\mu+\nu]$. Moreover, we have the following:*

1. *If $C_\mu(g,u) \neq 0$, then $C_q(g,u) \cong \delta_{q,\mu}G$.*
2. *If $C_{\mu+\nu}(g,u) \neq 0$, then $C_q(g,u) \cong \delta_{q,\mu+\nu}G$.*

The following lemma is known as the splitting theorem:

Lemma 2 ([13,33]). *Let U be a neighborhood of θ in a Hilbert space H, and let $g \in C^2(U,\mathbb{R}^1)$. Suppose θ is the only critical point of f, and denote by $L = d^2g(\theta)$ the Hessian at θ, with kernel N. If 0 is either an isolated point in the spectrum $\sigma(L)$ or not in $\sigma(L)$, then there exist a ball B_δ ($\delta > 0$) centered at θ, an origin-preserving local homeomorphism ϕ defined on B_δ, and a C^1 mapping $h : B_\delta \cap N \to N^\perp$ such that*

$$g \circ \phi(z+y) = \frac{1}{2}(Lz,z) + g(h(y)+y), \quad \forall x \in B_\delta, \tag{15}$$

where $y = P_N x$, $z = P_{N^\perp} x$, and P_N define the orthogonal projection onto the subspace N.

Definition 3 ([13]). *Consider a Banach space X and a connected Hausdorff space M. We define M as a Banach C^r manifold, for $r \geqslant 1$ (integer) and modeled on X, if the following conditions hold:*

1. *There exists a family of open coverings $\{U_i \mid i \in \Lambda\}$;*
2. *There exists a family of coordinate charts $\{\phi_i \mid: U_i \to \phi_i(U_i) \subset X$, where ϕ_i is a homeomorphism for each $i \in \Lambda$;*
3. *The transition maps $\phi_i \circ \phi_{i'}^{-1} : \phi_{i'}(U_i \cap U_{i'}) \to \phi_i(U_i \cap U_{i'})$ are C^r diffeomorphisms for all $i, i' \in \Lambda$.*

Each pair (ϕ_i, U_i) is called a chart, and the collection $\{(\phi_i, U_i) \mid i \in \Lambda\}$ is referred to as an atlas.

Similarly, we can define C^r (or C^{r-0}) maps between two C^r Banach manifolds, as well as vector bundles over Banach manifolds. In particular, this includes the tangent bundle $T(M)$ and the cotangent bundle $T^*(M)$.

Given a vector bundle $\Xi = (E, \pi, M)$, a section $\xi : M \to E$ is a map such that $\pi \circ \xi = \mathrm{id}_M$. A section ξ is said to be C^r- (or C^{r-0}) -continuous if it is a C^r (or C^{r-0}) map from M to E.

A Riemannian manifold (M, g) is metrizable, with the metric d defined by the arc length of geodesics, which is in turn determined by the Riemannian metric g:

$$d(x,y) = \inf\left\{ \int_0^1 g(\dot{\sigma}(t), \dot{\sigma}(t))^{\frac{1}{2}} dt \mid \sigma \in C^1([0,1], M),\, \sigma(0) = x,\, \sigma(1) = y \right\}.$$

As a metric space (M, d), the topology coincides with (or is equivalent to) the topology of the manifold.

Since the Riemannian metric is globally defined on $T(M)$, we shall introduce a Finsler structure on a Banach manifold in a similar manner.

Definition 4 ([13])**.** *Let $\pi : E \to M$ be a Banach vector bundle. A Finsler structure is a function $\|\cdot\| : E \to \mathbb{R}^1_+$ that satisfies the following conditions:*

1. *$\|\cdot\|$ is continuous;*
2. *For every $p \in M$, the restriction $\|\cdot\|_p := \|\cdot\||_{E_p}$ is an equivalent norm on the fiber $E_p := \pi^{-1}(p)$;*
3. *For any point $p_0 \in M$ and any neighborhood U of p_0 that trivializes the vector bundle E (i.e., $E \mid U = \pi^{-1}(U) \approx U \times E_{p_0}$), there exists a neighborhood V of M with $V \supset U$ such that for all $k > 1$,*

$$\frac{1}{k}\|\cdot\|_p \le \|\cdot\|_{p_0} \le k\|\cdot\|_p \quad \forall p \in V.$$

Below is the definition of a characteristic submanifold and a theorem known as the Shifting Theorem:

Definition 5 ([13])**.** *Let M be a C^2 Finsler manifold, and let $g \in C^1(M, \mathbb{R})$ be a functional. Consider a local parametrization Φ of M defined in an open neighborhood U of θ in $T_p(M) \cong H$, with $\Phi(\theta) = p$. Suppose $g \circ \Phi(z, y) = \frac{1}{2}(Lz, z) + g_0(y)$, where $L = g''(p)$, and 0 is either an isolated point in the spectrum $\sigma(L)$ or not in $\sigma(L)$. Here, g_0 is a function defined on N—the null space of L. We refer to $N = \Phi(U \cap N)$ as the characteristic submanifold of M for g at p with respect to the parametrization Φ.*

The following theorem relates the critical groups of f to those of $\widetilde{f} := f|_N$. This is known as the Shifting Theorem:

Lemma 3 (Shifting Theorem, [13,33])**.** *Assume that the Morse index of g at p is j. Then,*

$$C_q(g, p) = C_{q-j}(g|_N, p), \quad q = 0, 1, \ldots$$

The following lemma addresses the relationship between isolated critical points and characteristic manifolds:

Lemma 4 ([33])**.** *Let p be an isolated critical point of g, and let $\widehat{M}$ be a closed Hilbert submanifold of M such that $\widehat{M}_p$ contains the null space of the Hessian of g. Assume that $\nabla g_{|q} \in \widehat{M}_q$ for all $q \in \widehat{M}$. If $N \subset \widehat{M}$ is sufficiently small and characteristic for $\widehat{g} := g \mid \widehat{M}$ at p, then N is also characteristic for g. In particular, we have the following:*

$$C_q(g, p) = C_q(\widehat{g}, p).$$

We are now ready to provide the proof of Theorem 1.

Proof of Theorem 1. We divide our proof into two parts. First, we address part (i) of Theorem 1.

Let $x = u(z)$. According to Proposition 1, x is a critical point of g, meaning that $g'(x) = 0$. Let I denote the identity map on H. It follows that $g'(x) = 0$ if and only if

$$Az = PG(z + w(z)) \text{ and } Aw = (\mathrm{I} - P)G(z + w(z)). \tag{16}$$

Thus, in the decomposition Equation (5), the matrix representation of A can be written as

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \tag{17}$$

where $A_{11} = A|_{H_0}$, and $A_{22} = A|_{H_- \oplus H_+}$. According to the definition of P, A_{22} is invertible, and we have

$$\left\| A_{22}^{-1} \right\| < \frac{1}{C}. \tag{18}$$

The formal expression for $G'(x)$ is

$$G'(x) = \begin{pmatrix} B_{11}(x) & B_{12}(x) \\ B_{21}(x) & B_{22}(x) \end{pmatrix}. \tag{19}$$

The second equation in Equation (16) is satisfied if and only if

$$w(z) = A_{22}^{-1}(\mathrm{I} - P)G(z + w(z)). \tag{20}$$

Differentiating both sides of Equation (20) yields

$$w'(z) = A_{22}^{-1}(\mathrm{I} - P)G'(z + w(z)) \begin{pmatrix} \mathrm{I}_V \\ w'(z) \end{pmatrix}, \tag{21}$$

where I_V denotes the identity map on V. Based on Equation (19), we obtain

$$(\mathrm{I} - P)G'(u(z)) = (B_{21}(u(z)), B_{22}(u(z))). \tag{22}$$

Combining Equations (21) and (22), we obtain

$$w'(z) = A_{22}^{-1}\left[B_{21}(u(z)) + B_{22}(u(z))w'(z)\right]. \tag{23}$$

Simple computations show that

$$w'(z) = [A_{22} - B_{22}(u(z))]^{-1} B_{21}(u(z)), \tag{24}$$

since $A_{22} - B_{22}(u(z))$ is invertible by the choice of C. Therefore, based on Equation (24), we obtain

$$\begin{aligned} a''(z) &= AP - P\mathrm{d}G(u(z))u'(z) = A_{11} - (B_{11}, B_{12}) \begin{pmatrix} \mathrm{I}_V \\ w'(z) \end{pmatrix} \\ &= A_{11} - B_{11}(u(z)) - B_{12}(u(z))[A_{22} - B_{22}(u(z))]^{-1} B_{21}(u(z)). \end{aligned} \tag{25}$$

In the decomposition Equation (5), we have

$$g''(u(z)) = \begin{pmatrix} A_{11} - B_{11}(u(z)) & -B_{12}(u(z)) \\ -B_{21}(u(z)) & A_{22} - B_{22}(u(z)) \end{pmatrix}. \tag{26}$$

By direct computation, we find

$$\begin{pmatrix} I & B_{12}(u(z))[A_{22}-B_{22}(u(z))]^{-1} \\ 0 & I \end{pmatrix} g''(u(z)) \begin{pmatrix} I & 0 \\ [A_{22}-B_{22}(u(z))]^{-1}B_{21}(u(z)) & I \end{pmatrix} = \begin{pmatrix} a''(z) & 0 \\ 0 & A_{22}-B_{22}(u(z)) \end{pmatrix}. \tag{27}$$

Since $A_{22} - B_{22}(u(z))$ is invertible, we have

$$\dim\ker g''(u(z)) = \dim\ker a''(z), \quad m^-(g''(u(z))) = m^-(a''(z)) + d, \tag{28}$$

where $d = \dim P^- H$.

Lastly, we address part (ii) of Theorem 1. We split our proof into two cases.

Case 1. If z is a nondegenerate critical point of a, then according to (3°) of Proposition 1, $u(z)$ is a nondegenerate critical point of g, and $m^-(g''(u(z))) = m^-(a''(z)) + d = j$. Consequently, according to Lemma 1, we have

$$C_q(g, u(z)) = \delta_{q,j}\mathcal{F} = C_{q-d}(a, z).$$

Case 2. If z is a degenerate critical point of a with $a(z) = c$, then from (3°) of Proposition 1, $u(z)$ is a degenerate critical point of f, and $m^-(g''(u(z))) = m^-(a''(z)) + d = j$. According to Lemma 2, there exists a ball B_δ centered at 0 with radius $\delta > 0$ and a local homeomorphism ϕ defined on B_δ with $\phi(0) = z$ such that

$$a \circ \Phi(\xi, \eta) = \frac{1}{2}(L\xi, \xi) + a_0(\eta), \tag{29}$$

where $(\xi, \eta) \in Z = (\ker(a''(z)))^\perp \oplus \ker(a''(z))$, $L = a''(z)$, and a_0 constitute a function defined on $\ker(a''(z))$.

Let N be the characteristic submanifold for a at z with respect to ϕ. According to Lemma 3, we have

$$C_q(a, z) = C_{q-j+d}(a|_N, z), \qquad q = 0, 1, 2, \cdots. \tag{30}$$

Now, consider the critical point $u(z)$ of f. According to Lemma 2, there exists a local homeomorphism ψ. Let $\widehat{N}$ be the characteristic submanifold for g at $u(z)$ with respect to ψ. Then, according to Lemma 3, we have

$$C_q(g, u(z)) = C_{q-j}(g|_{\widehat{N}}, u(z)), \qquad q = 0, 1, 2, \cdots. \tag{31}$$

Since the map u defined in Proposition 1 is an injection, $u(Z)$ is a closed Hilbert submanifold of H. Thus, in accordance with the (3°) of Proposition 1, we have

$$\ker g''(u(z)) \subset T_{u(z)}u(Z) \text{ and } g'(x) \in T_x u(Z) \text{ for all } x \in u(Z).$$

Define $\widehat{g} = g|_{u(Z)}$, and let $\widetilde{N}$ be a characteristic manifold for $\widehat{g}$. According to Lemma 4, we obtain

$$C_q(g|_{\widehat{N}}, u(z)) = C_q(\widehat{g}|_{\widetilde{N}}, u(z)), \qquad q = 0, 1, 2, \cdots. \tag{32}$$

From Equation (27), we obtain

$$m^-(\widetilde{g}''(u(z))) = m^-(a''(z)) = j - d. \tag{33}$$

Thus, according to Lemma 3, we have

$$C_q(\widetilde{g}, u(z)) = C_{q-j+d}(\widehat{g}|_{\widetilde{N}}, u(z)), \qquad q = 0, 1, 2, \cdots. \tag{34}$$

We then have the following map between the topological pairs:

$$(a^c, a^c \backslash \{z\}) \xrightarrow{u} (\widetilde{g}^{u(z)}, \widetilde{g}^{u(z)} \backslash \{u(z)\}). \tag{35}$$

Clearly, u is a homeomorphism with inverse $u^{-1}(z + w(z)) = z$. Therefore, we have

$$C_q(a, z) = C_q(\widetilde{g}, u(z)), \qquad q = 0, 1, 2, \cdots . \tag{36}$$

Combining Equations (30), (31), (33) and (36), we obtain

$$C_q(g, u(z)) = C_{q-d}(a, z), \qquad q = 0, 1, 2, \cdots . \tag{37}$$

This concludes the proof. □

3. Applications to Second-Order Hamiltonian Systems

In this section, we will utilize Morse theory and a critical point theorem to establish the existence of two nontrivial solutions for the second-order Hamiltonian system (Theorem 2).

Let $L = L^2(S_\tau, \mathbb{R}^n)$ be the Hilbert space equipped with the norm

$$\|x\|_2 = \left(\int_0^\tau |x(t)|^2 \, \mathrm{d}t \right)^{1/2}, \quad \forall x \in L. \tag{38}$$

The corresponding inner product in L is denoted by $\langle \cdot, \cdot \rangle_2$. Now, consider the space $W = W^{1,2}(S_\tau, \mathbb{R}^n)$, which consists of vector functions from S_τ to $\mathbb{R}^n$ with square-integrable first-order derivatives. The norm on W is defined as

$$\|x\|_W = \left(\int_0^\tau \left(|\dot{x}(t)|^2 + |x(t)|^2 \right) \mathrm{d}t \right)^{1/2}, \quad \forall x \in W. \tag{39}$$

This norm turns W into a Hilbert space, known as the Sobolev space of τ-periodic functions, which is a dense subspace of L.

Within the Hilbert space L, we define the linear operator $T : W \subset L \to L$ as

$$\langle Tx, y \rangle := \int_0^\tau \dot{x} \cdot \dot{y} \, \mathrm{d}t, \quad \forall x, y \in W. \tag{40}$$

The operator T has a closed range, and its resolvent is compact. The spectrum of T under the L norm is given by $\sigma(T) = \left\{ \left(\frac{2k\pi}{\tau} \right)^2 \mid k \in \mathbb{Z}^+ \right\}$, consisting only of eigenvalues and indicating that it is a point spectrum.

We continue to denote by A the self-adjoint operator on L induced by A, which is defined as

$$\langle Ax, y \rangle_L = \int_0^\tau A(t)x(t) \cdot y(t) \, \mathrm{d}t, \quad \forall x, y \in L. \tag{41}$$

Similarly, the operator B, self-adjoint on L, is defined by

$$\langle Bx, y \rangle_L = \int_0^\tau B(t)x(t) \cdot y(t) \, \mathrm{d}t, \quad \forall x, y \in L. \tag{42}$$

Now, define $\widetilde{T} := T - B$. The eigenvalue problem

$$\widetilde{T}x = \lambda x \tag{43}$$

is equivalent to the eigenvalue problem

$$\begin{cases} -\ddot{x} - B(t)x = \lambda x, \\ x(0) = x(\tau), \quad \dot{x}(0) = \dot{x}(\tau). \end{cases} \tag{44}$$

Consequently, the spectrum of the operator $\widetilde{T}$ is also a point spectrum.

Given the conditions $(\mathbf{V}_3)$ and $(\mathbf{V}_4)$, we define a functional on the space L as follows:

$$k(x) = \int_0^\tau \left(V(t, x(t)) - \frac{1}{2} B(t)x(t) \cdot x(t) \right) \mathrm{d}t. \tag{45}$$

This functional satisfies $k \in C^1(L, \mathbb{R})$, and we have

$$k'(x) = V_x(t, x(t)) - B(t)x(t) = L_x(t, x). \tag{46}$$

It is important to note that $k'(x)$ is Gâteaux-differentiable, and its Gâteaux derivative is given by

$$k''(x)y = (V_{xx}(t, x(t)) - B(t))y = L_{xx}(t, x)y. \tag{47}$$

Furthermore, we have the estimate

$$\|k''(x)\|_L \leq C(B),$$

where $C(B)$ is the constant defined in condition $(\mathbf{V}_4)$.

We then introduce a new functional h on W defined by

$$h(x) = \frac{1}{2} \int_0^\tau |\dot{x}(t)|^2 \, \mathrm{d}t - \int_0^\tau V(t, x(t)) \, \mathrm{d}t = \frac{1}{2} \langle \widetilde{T}x, x \rangle_2 - k(x).$$

Condition $(\mathbf{V}_3)$ ensures that $h \in C^2(W, \mathbb{R})$ and that h' is Gâteaux-differentiable. Consider the following equation in the space L:

$$\widetilde{T}x = k'(x), \quad \text{for} \quad x \in W. \tag{48}$$

This Equation (48) is the Euler equation of the functional h on the space L, which can be expressed as follows:

$$\langle h'(x), y \rangle_2 = \langle \widetilde{T}x - k'(x), y \rangle_2, \quad \forall y \in W. \tag{49}$$

Therefore, the critical points of h are solutions to the Equation (48). Moreover, this Equation (48) is equivalent to the τ-periodic boundary value problem of the Hamiltonian system Equation (7). Consequently, finding τ-periodic solutions of the Hamiltonian system Equation (48) is equivalent to finding the critical points of the functional h in the space W.

Remark 2. *In this context, the operator $\widetilde{T}$ corresponds to the operator A in the abstract operator Equation (1).*

Consider the projection operator $P_0 : H \to \ker \widetilde{T}$, and define an operator on W as follows:

$$\widetilde{T}_0 x = \widetilde{T}x + P_0 x, \qquad x \in W.$$

This operator $\widetilde{T}_0$ is a self-adjoint, invertible Fredholm operator. Let $\{E_\lambda\}$ denote the spectral family associated with $\widetilde{T}_0$. We define the projections on the space L by

$$P = \int_{-C(B)}^{C(B)} \mathrm{d}E_\lambda, \quad P^+ = \int_{C(B)}^{+\infty} \mathrm{d}E_\lambda, \quad P^- = \int_{-\infty}^{-C(B)} \mathrm{d}E_\lambda. \tag{50}$$

With these projections, the Hilbert space L can be orthogonally decomposed as

$$L = L_- \oplus L_0 \oplus L_+, \tag{51}$$

where $L_0 = PL$, $L_- = P^- L$, and $L_+ = P^+ L$.

By applying the saddle point reduction, Theorem 1, the Equation (48) is reduced to

$$a'(z) = 0, \quad \forall z \in L_0, \tag{52}$$

where $a(z) = \frac{1}{2}\langle \widetilde{T} u(z), u(z) \rangle_2 - k(u(z))$.

It is evident that 0 is a critical point for both $h(x)$ and $a(z)$. According to Theorem 1, we have

$$m^-(h''(0)) = m^-(a''(0)) + d, \tag{53}$$

where $d = \dim L_-$.

To prove Theorem 2, we first introduce a key critical point theorem:

Proposition 3 ([34], Theorem 2.1)**.** *Let X be a Banach space, and let $f \in C^1(X, \mathbb{R})$ satisfy the Palais–Smale (PS) condition. Assume that f is bounded from below. If $C_\ell(f, 0) \neq 0$ for some $\ell \neq 0$, then f has at least three critical points.*

In the following section, we will verify that the second-order Hamiltonian system Equation (7) meets the conditions of Proposition 3.

Lemma 5. *Assuming conditions $(\mathbf{V}_3)$ and $(\mathbf{V}_4)$ are satisfied, the functional $a(z)$ is bounded from below and meets the (PS) condition.*

Proof. Consider $x = x^+ + x^- + z \in L_+ \oplus L_- \oplus L_0$. We define $w^\pm(z) := P^\pm \widetilde{T}_0^{-1} g(x)$.

For any $p \in L_-$, we introduce the function

$$\Phi(t) = \frac{1}{2}\langle \widetilde{T}_0(w^-(z) + w^+(z) + z + tp), w^-(z) + w^+(z) + z + tp \rangle_2 - k(x + tp), \quad t \in \mathbb{R}. \tag{54}$$

A straightforward computation yields

$$\Phi'(t) = -\langle w^-(z), p \rangle - t\|p\|_2^2 - \langle k'(0), p \rangle_2, \tag{55}$$

$$\Phi''(t) = -\|p\|_2^2 - \langle k''(x + tp)p, p \rangle_2. \tag{56}$$

It is evident that

$$\Phi'(t) = 0.$$

Since $C(B) \notin \sigma(\widetilde{T})$, there exists a constant $\mu > C(B)$ such that

$$\|\widetilde{T}_0|_{L_-}^{-1}\| \leq \frac{1}{\mu}. \tag{57}$$

According to condition $(\mathbf{V}_4)$, we have

$$\Phi''(t) \leq \left(-1 + \frac{C(B)}{\mu}\right)\|p\|_2^2 < 0, \quad \forall\, p \in L_- \setminus \{0\},\, t \in \mathbb{R}. \tag{58}$$

Thus, for any $z \in L_0$, the functional $h(w^+(z) + x^- + z)$ is concave along any line passing through $w^-(z)$ and thus achieves a global maximum at $w^-(z)$. Specifically,

$$h(w^+(z) + w^-(z) + z) \geqslant h(w^+(z) + w^- + z), \quad \forall\, x^- + z \in L_- \oplus L_0. \tag{59}$$

By $(\mathbf{V}_5)$ and the definition of $a(z)$, we have

$$\begin{aligned}a(z) &= h(w^+(z) + w^-(z) + z) \geqslant h(w^+(z) + z) = h(w^+(z) + z) \\ &= \frac{1}{2}\langle \widetilde{T}(w^+(z) + z) + z, w^+(z) + z\rangle - k(w^+(z) + z) \\ &\geqslant \widehat{\lambda}\|w^+(z) + z\|^2 - l\|w^+(z) + z\|^2 \\ &= (\widehat{\lambda} - l)\|w^+(z) + z\|^2 \to +\infty \quad \text{as } \|z\| \to +\infty,\end{aligned} \tag{60}$$

where $\widehat{\lambda}$ is the minimum eigenvalue of Equation (44) within $(-C(B), C(B))$. Therefore, $a(z)$ is bounded from below and satisfies the (PS) condition. □

Consider a Hilbert space W and $p \in [1, \infty]$. By the Sobolev inequality, there exists a constant D_p such that

$$\|x\|_p := \left(\int_0^\tau |x|^p \,\mathrm{d}t\right)^{1/p} \leq D_p\|x\|. \tag{61}$$

This inequality implies that the embedding $W \hookrightarrow L^p$ is both continuous and compact. Consequently, to establish the existence of nontrivial critical points, it suffices to verify that the corresponding critical group is nontrivial.

Recall the condition $(\mathbf{V}_1)$, which states that there exists an eigenvalue $\mu_m(A) = 0$. Given that $A(t)$ is positive definite for all $t \in \mathbb{R}$, consider the following weighted eigenvalue problem:

$$\begin{cases} -\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \mu_i(A)A(t)x(t), \\ x(0) = x(\tau), \end{cases} \tag{62}$$

According to the spectral theory of compact self-adjoint operators, it is well established that there exists a complete sequence of distinct eigenvalues

$$-\infty < \mu_1(A) < \mu_2(A) < \cdots \tag{63}$$

such that $\mu_n(A) \to +\infty$ as $n \to \infty$.

Define the subspaces as follows:

$$\begin{aligned}W_- &= \bigoplus_{i=1}^{m-1} \ker\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \mu_i(A)A(t)\right), \\ W_0 &= \ker\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - A(t)\right), \\ W_+ &= \overline{\bigoplus_{i=m+1}^{\infty} \ker\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \mu_i(A)A(t)\right)}.\end{aligned}$$

We can derive the following result through a straightforward calculation:

$$d_m := \sum_{i=1}^{m} \dim\ker\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \mu_i(A)A\right). \tag{64}$$

Lemma 6.

(i) *If condition* ($\mathbf{V}_2^+$) *is satisfied, then* $C_{d_m}(h,0) \neq 0$;

(ii) *If condition* ($\mathbf{V}_2^-$) *is satisfied, then* $C_{d_{m-1}}(h,0) \neq 0$.

Proof. We provide a complete proof for case (i) here; the proof for the other case follows a very similar argument.

According to ([35], Theorem 2.1), it is sufficient to verify that h has a local linking with respect to the decomposition $W = W_- \oplus (W_0 \oplus W_+)$ under condition ($\mathbf{V}_2^+$). Specifically, there must exist a $\rho > 0$ such that

$$\begin{cases} h(x) \leq 0 & \text{for } x \in W_- \oplus W_0,\ \|x\| \leq \rho, \\ h(x) > 0 & \text{for } x \in W_+,\ 0 < \|x\| \leq \rho. \end{cases} \tag{65}$$

The argument for Equation (65) is similar to that presented in [36] (p. 24) and ([21], Lemma 4.1).

Since $\mu_m(B) = 1$ is an isolated eigenvalue, it is well known that there exists a positive number $\kappa > 0$ such that

$$\pm\frac{1}{2}\langle Tx - Bx, x\rangle_2 \geqslant \kappa\|x\|^2, \qquad x \in W_\pm. \tag{66}$$

Based on Equations (8) and (11), there exists a constant $C_3 > 0$ such that

$$|G(t,x)| < \frac{\kappa}{16D_2^2}|x|^2 + C_3|x|^3, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{67}$$

For $x \in W_- \oplus W_0$ with $\|x\| \leq \frac{\delta}{2C_5}$, we can write that $x = v + w$, where $w \in W_-$, and $v \in W_0$. Define

$$S_\tau^1 = \left\{ t \in S_\tau \mid |w(t)| \leq \frac{\delta}{2} \right\}, \quad S_\tau^2 = S_\tau \backslash S_\tau^1. \tag{68}$$

Since $\dim W_0 < \infty$, there exists $C_5 > 0$ such that $\|v\|_C \leq C_5\|v\|$ for all $v \in W_0$.

For every $t \in S_\tau^2$, we have

$$\begin{aligned} |x(t)| &\leq |v(t)| + |w(t)| \leq \|v\|_C + |w(t)| \leq C_5\|v\| + |w(t)| \\ &\leq C_5\|x\| + |w(t)| \leq \frac{\delta}{2} + |w(t)| \leq 2|w(t)|. \end{aligned} \tag{69}$$

According to Equation (67), for $t \in S_\tau^2$, we obtain

$$G(t,x) \geqslant -\frac{\kappa}{16D_2^2}|x|^2 - C_3|x|^3 \geqslant -\frac{\kappa}{4D_2^2}|w|^2 - C_6|w|^3.$$

This result also applies to $t \in S_\tau^1$ because, in this case,

$$|x(t)| \leq |v(t)| + |w(t)| \leq \|v\|_C + \frac{\delta}{2} \leq C_5\|v\| + \frac{\delta}{2} \leq C_5\|x\| + \frac{\delta}{2} \leq \delta.$$

Hence, $G(t,x) \geqslant 0$ by our assumption ($\mathbf{V}_2^+$).

According to Equation (61), we obtain

$$\begin{aligned} h(x) &= \frac{1}{2}\langle Tx - Bx, x\rangle_2 - \int_0^\tau K(t,x)\,\mathrm{d}t \\ &\leq -\kappa\|w\|^2 + \frac{\kappa}{4D_2^2}\|w\|_2^2 + C_6\|w\|_3^3 \\ &\leq -\frac{\kappa}{2}\|w\|^2 + C_7\|w\|^3, \quad x = w + v \in W_- \oplus W_0. \end{aligned} \tag{70}$$

Now, let $x \in W_- \oplus W_0$ such that

$$0 < \|x\| \leq \rho_1 = \min\left\{\frac{\delta}{2C_5}, \frac{\kappa}{2C_7}\right\}.$$

If $w \neq 0$, from Equation (70) we can deduce that $h(x) < 0$, since $\|w\| \leq \|x\|$. If $w = 0$, then $x \in W_0$, and $\|x\|_C \leq C_5\|x\| \leq \delta$. By condition $(\mathbf{V}_2^+)$ again, we also have that

$$h(x) = -\int_0^\tau K(t,x)\,\mathrm{d}t < 0.$$

For $x \in W_+$, according to Equation (67), we obtain

$$\begin{aligned} h(x) &= \frac{1}{2}\langle Tx - Bx, x\rangle_2 - \int_0^\tau K(t,x)\,\mathrm{d}t \\ &\geqslant \kappa\|x\|^2 - \frac{\kappa}{16D_2^2}\|x\|_2^2 - C_3\|x\|_3^3 \\ &\geqslant \frac{\kappa}{2}\|x\|^2 - C_4\|x\|^3 \geqslant 0 \end{aligned} \tag{71}$$

provided that $\|x\| \leq \rho_2 = \frac{\kappa}{2C_4}$.

Combining these results, we conclude that Equation (65) holds with $\rho = \min\{\rho_1, \rho_2\}$. □

We are now ready to present the proof of Theorem 2.

Proof of Theorem 2. We will provide the full proof for the case where condition $(\mathbf{V}_2^+)$ is met. The proof for the other case follows similarly.

According to Lemma 5, the functional $a(z)$ satisfies the (PS) condition and is bounded below. Given that 0 is a critical point of h, it is also a critical point of $a(z)$, and it holds that $m^-(h''(0)) = m^-(a''(0)) + d$. According to Lemma 6 and Theorem 1, we obtain

$$C_{d_m - d}(a(z), 0) \neq 0.$$

Consequently, according to Lemma 3, if $d_m \neq d$, there exist two additional critical points of $a(z)$. Therefore, h must have at least two nonzero critical points. This, in turn, implies the existence of at least two solutions to the Hamiltonian system Equation (7). This completes the proof. □

4. Conclusions

Infinite dimensional Morse theory is a highly effective tool for analyzing multiple solution problems in nonlinear differential equations. One of its key concepts is the critical group $C_q(f,x)$ for a C^1 functional f at an isolated critical point x. This critical group captures the local behavior of f near x. In many applications, critical groups help in distinguishing between critical points and, furthermore, can be used to identify new critical points through the Morse inequality. Consequently, studying the critical group is crucial.

On the other hand, the saddle point reduction (also known as the Lyapunov–Schmidt Reduction) is a powerful technique for finding the critical points of some C^1 functionals. Essentially, under certain conditions, this reduction yields a reduced functional on a subspace that is closely related to the study of the original functional, particularly concerning the relationship between critical points.

In this paper, we first establish the relationship between the Morse index and the critical group before and after the saddle point reduction. We then apply our abstract results to examine the existence of two nontrivial solutions for the second-order Hamiltonian system. This approach offers a new perspective for identifying nontrivial solutions in other boundary value problems.

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