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A Variational Theory for Biunivalent Holomorphic Functions

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Abstract: Biunivalent holomorphic functions form an interesting class in geometric function theory and are connected with special functions and solutions of complex differential equations. This class has been investigated by many authors, mainly to find the coefficient estimates. The assumption of biunivalence is rigid; this rigidity means that, for example, only the initial Taylor coefficients have been estimated. The aim of this paper is to develop a variational technique for biunivalent functions, which provides a power tool for solving the general extremal problems on the classes of such functions. It involves quasiconformal analysis.

Keywords: biunivalent holomorphic function; quasiconformal homeomorphism; variations; distortion theorem; coefficient estimates

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1. Introductory Remarks

A univalent holomorphic function $w = f(z)$ on a given disk is called **biunivalent** if the inverse $z = f^{-1}(w)$ is also univalent on this disk. One can deal here with the unit disk $\mathbb{D} = \{z \in \mathbb{C} : \|z| < 1\}$.

In this sense, the notion of biunivalence is very broad, because one can take, for example, all functions of the form $f(z) = 4g(z)$, where $g(z) = z + a_2z^2 + \dots$ belongs to the canonical class S of univalent functions on $\mathbb{D}$ with $g(0) = 0$, $g'(0) = 1$. These functions occupy a substantial part of the classical geometric function theory in view of their remarkable features and have been widely investigated.

The normalization $f(0) = 0$, $|f'(0)| \leq 4$ ensures the compactness of this collection in topology generated by uniform convergence on closed (compact) subsets of $\mathbb{D}$, which plays a crucial role.

Another more special class of biunivalent functions originated in the 1960s. It consists of functions $f(z) = z + a_2z^2 + \dots$, which are univalent on a given disk (usually this is the unit disk $\mathbb{D}$) together with their inverse functions $z = f^{-1}(w) = \sum_1^\infty b_n w^n$. We denote this class by $\mathcal{B}$.

(This normalization is customary. It would be interesting to consider the collection of functions subject to another normalization.)

The biunivalent functions are connected with special functions and solutions of complex differential equations, with the so-called q-calculus, etc. From these points of view, these functions have been and remain intensively investigated by many authors, who have considered and defined new special subclasses of biunivalent functions depending on different parameters; see, e.g., [1–9] and the references cited there. These investigations resulted mainly in the estimates of the initial Taylor coefficients a_2 and a_3 and their combinations.

The conditions of normalization are essential, because the assumption of biunivalence is rather rigid. Together with the classical Schwarz lemma and holomorphy of the inverse function f^{-1} on the disk $\mathbb{D}$, it implies that $f(z)$ must have a holomorphic extension into a broader domain D containing $\mathbb{D}$ (this domain depends on f), and the same is valid for the inverse functions. This rigidity causes a scarcity of results obtained for biunivalent functions; in particular, only a few partial distortion results mentioned above are known. Actually, the basic methods of the classical geometric function theory touch on biunivalence only to a small extent.

Among the important open problems here are to develop an extended distortion theory and find new applications of biunivalent functions.

2. A General Distortion Theorem for Biunivalent Functions

Our approach is completely different. It links the biunivalence of holomorphic functions with quasiconformality.

We develop here a variational technique for biunivalent functions, which provides a powerful tool to solve the general extremal maximization problems on the classes of such functions.

We shall denote the class of biunivalent functions $f(z)$ on the disk $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$ by $\mathcal{B}$ and also consider its subclasses $\mathcal{B}_k$ formed by functions $f \in \mathcal{B}$ whose restrictions to the disk $\mathbb{D}$ admit k-quasiconformal extensions across the circle $\mathbb{S}^1 = \{|z| = 1\}$ onto the whole plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (here, $0 < k < 1$). One can assume that these extensions preserve the point at $z = \infty$ fixed. Finally, denote $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$.

Recall that a quasiconformal map of a domain $G \subset \widehat{\mathbb{C}}$ is a generalized homeomorphic solution $w(z)$ of the Beltrami equation $\partial_{\bar{z}} w = \mu(z) \partial_z w$, where μ is a given measurable function on G with $\|\mu\|_\infty < 1$ (called the Beltrami coefficient of w). Quasiconformal maps require the additional third normalization, which insures their compactness, the holomorphic dependence of their Beltrami coefficients μ_f on complex parameters, etc. The maps with $\|\mu\|_\infty \le k < 1$ are called k-quasiconformal. For the properties of quasiconformal maps see, e.g., [10–13].

We consider on these classes $\mathcal{B}$ and $\mathcal{B}_k$ the continuously differentiable real or complex functionals of the form

$$J(f) = J(a_{n_1}, a_{n_2}, \ldots, a_{n_s}, f(z_1), \ldots, f(z_m)), \tag{1}$$

where $z_1, \ldots, z_m$ are the distinguished fixed points in $\mathbb{D} \setminus \{0\}$, and J is a continuously differentiable real or complex function of its arguments, with

$$\operatorname{grad} J(f) \neq 0.$$

We define for any $f \in \mathcal{B}$ the function

$$g(z) = z + \sum_2^\infty b_n z^n, \quad |z| < 1, \tag{2}$$

with the same coefficients as the inverse f^{-1}. In view of the biunivalence of f, this function g is moved to the class $\mathcal{B}_k$ or $\mathcal{B}$ simultaneously with f.

Now, using the Lagrange formula for the coefficients of the inverse function

$$f^{-1}(w) = \sum_1^\infty \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ \left[\frac{z}{f(z)} \right]^n \right\}_{z=0} w^n, \tag{3}$$

which determines the coefficients of g as the polynomials of the initial coefficients a_j of f, and vice versa, after substituting these expressions of a_j into the representation of the initial functional $J(f)$, one obtains a new functional $\widetilde{J}(f)$ on classes $\mathcal{B}$ and $\mathcal{B}_k$ depending on the corresponding coefficients b_n and the values $w_j = f(z_j)$. These functionals satisfy

$$\max_{\mathcal{B}_k} |J(g)| = \max_{\mathcal{B}_k} |\widetilde{J}(f)|,$$

and similarly for the maxima on $\mathcal{B}$. Hence, to find the extremals of $J(f)$ in these classes, one can use the second functional $\widetilde{J}(f)$.

Our approach involves the variational method of quasiconformal analysis created by Belinskii and the author in [13,14]. The variational technique is one of the basic techniques in quasiconformal theory; its different variants were given, for example, in [15–21].

The main result of this paper is given by the following:

Theorem 1. *For any functional $J(f)$ of the form (1) and any $k < 1$, we have the equalities*

$$\max_{\mathcal{B}_k} |J(f)| = \max_{\mathcal{B}_k} |\widetilde{J}(f)| = |\widetilde{J}(f^{\mu_k})|, \tag{4}$$

where

$$\mu_k(z) = k|\varphi_0(z)|/\varphi_0(z), \quad z \in \mathbb{D}^*, \tag{5}$$

and

$$\varphi_0(z) = \sum_{l=1}^{s} \frac{\partial \widetilde{J}}{\partial a_{n_l}}(z) + \sum_{j=1}^{m} \frac{\partial \widetilde{J}}{\partial w_j}.$$

This theorem shows that the extremals of all indicated functionals J on classes $\mathcal{B}_k$ are of the Teichmüller type.

The extremal function f_0 for J on the entire class $\mathcal{B}$ is obtained in the limit as

$$|J(f_0)| = \max_{\mathcal{B}} |J(f)| = \sup_{k} |J(f^{\mu_k})|, \tag{6}$$

and this supremum is attained on some function from $\mathcal{B}$.

3. Proof

Proof. As was mentioned above, the proof is variational. We start with variations given by the local existence theorem from [13]. Its special case for simply connected plain domains states the following:

Lemma 1. *Let D be a simply connected domain on the Riemann sphere $\widehat{\mathbb{C}}$. Assume that there are a set E_0 of positive two-dimensional Lebesgue measures and a finite number of points $z_1, z_2, ..., z_m$ distinguished in D. Let $\alpha_1, \alpha_2, ..., \alpha_m$ be non-negative integers assigned to $z_1, z_2, ..., z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E_0$.*

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_{sj}, s = 0, 1, ..., \alpha_j,\ j = 1, 2, ..., m$, which satisfy the conditions $w_{0j} \in D$,

$$|w_{0j} - z_j| \le \varepsilon, \ \ |w_{1j} - 1| \le \varepsilon, \ \ |w_{sj}| \le \varepsilon \ (s = 0, 1, \dots a_j, \ j = 1, ..., m),$$

there exists a quasiconformal self-map h of D which is conformal on $D \setminus E_0$ and satisfies

$$h^{(s)}(z_j) = w_{sj} \quad \text{for all } s = 0, 1, ..., \alpha_j, \ j = 1, ..., m.$$

Moreover, the Beltrami coefficient $\mu_h(z) = \partial_{\bar{z}}h/\partial_z h$ of h on E_0 satisfies $\|\mu_h\|_\infty \le M\varepsilon$. The constants ε_0 and M depend only upon the sets D, E_0 and the vectors $(z_1, ..., z_m)$ and $(\alpha_1, ..., \alpha_m)$.

If the boundary ∂D is Jordan or is $C^{l+\alpha}$-smooth, where $0 < \alpha < 1$ and $l \ge 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \le l$, respectively.

Applying the variations $\omega(w)$ given by this lemma to the sets $E_0 \subset f(\mathbb{D}^*)$ of positive measure immediately implies that the dilatation $k(f_0$ of the extremal map $f_0(z)$ in $\mathcal{B}_k$ equals k almost everywhere. In view of the general properties of quasiconformal maps, one can set $|\mu_0(z)| = k$ at all points z, where $f_0(z)$ is not conformal.

To establish the explicit form of $\arg \mu_0(z)$, we apply another quasiconformal variation borrowed from [13,14].

First, observe that letting $f_2 = f_1 \circ f$, one has from the chain rule for Beltrami coefficients the equalities
$$\mu_{f_1} \circ f = \mu_{f_2 \circ f^{-1}} \circ f = \frac{\mu_{f_2} - \mu_f}{1 - \overline{\mu}_f \mu_{f_2}} \frac{\partial_z f}{\overline{\partial_z f}}.$$
We can vary the functions $f \in \mathcal{B}_k$ using the variations
$$\omega = H(w, \varepsilon) = w - \frac{w^2}{\pi} \iint\limits_E \frac{\mu_H(\zeta)}{\zeta^2(\zeta - w)} d\xi d\eta + O(\|\mu_H\|_\infty^2), \tag{7}$$
whose Beltrami coefficients μ_H are supported on sufficiently small sets $E \subset f_0(\mathbb{D}^*)$, and $\|\mu_H\|_\infty < \varepsilon$ is small. Then, $H \circ f(z) = f(z) + O(\varepsilon)$, and such composition preserves biunivalence.

In particular, for the extremal map f_0, letting
$$\widetilde{\mu}(w) = \mu_{f_0^{-1} \circ H^{-1}}(\omega(w)) \overline{\partial_w \omega} / \partial_w \omega,$$
one obtains
$$\widetilde{\mu}(w) = \mu_{f_0^{-1}} - \varepsilon \mu_H(w) + \varepsilon \overline{\mu_H(w)} \mu_{f_0^{-1}}(w)^2 + O(\varepsilon^2).$$
This implies
$$|\widetilde{\mu}(w)| = |\mu_{f_0^{-1}}(w)| - |\varepsilon \mu_H(w)|(1 - |\mu_{f_0^{-1}}(w)|^2) \cos[\arg \mu_{f_0^{-1}}(w) - \arg(\varepsilon \mu_H(w))] + O(\varepsilon^2). \tag{8}$$

Now, we specify the choice of E. Since the extremal Beltrami coefficient $\mu_0 = \mu_{f_0}$ is measurable on $\mathbb{D}^*$, one can choose the closed subsets E_δ of $f_0(\mathbb{D}^*)$ so that the measure of $E \setminus E_\delta$ is arbitrarily small and μ_0 is continuously differentiable on these subsets. Further, choose the set E in (7) to be the intersection of E_δ with a small disk centered at a density point of E_δ. In addition, one can assume for simplicity (distorting $f_0^{-1} \circ H^{-1}$ up to quantity $O(\varepsilon^2)$) that $\mu_H \equiv \text{const}$ on E.

Any such variation shows that the linear term of the increment of $\widetilde{J}(f_0^{-1})$ is equal to
$$\begin{aligned} \widetilde{J}(h \circ f_0^{-1}) - \widetilde{J}(f_0^{-1}) &= \sum_{s=1}^{m} \frac{\partial J}{\partial b_{n_s}} (f_0^{-1}) \\ &= -2|\varphi_0(w)||\varepsilon \mu_H(w)| \cos 2 \arg(\varepsilon \mu_H(w)) - \arg \varphi_0(w). \end{aligned} \tag{9}$$

Comparing (8) and (9), one finds that, in the case $|\mu(w)| \equiv k$, any constructed above variation $H(w)$ with sufficiently small $\|\mu_H\|_\infty$ is admissible, and the differential of $\widetilde{J}(H \circ f_0) - \widetilde{J}(f_0)$ can have any sign. This is impossible for extremal f_0^{-1}, and therefore, the extremal Beltrami coefficient of this function must be of the form
$$\mu(w) = k|\varphi_0(w)|/\varphi_0(w) \quad \text{for all } w \in f_0(\mathbb{D}^*). \tag{10}$$
Now, passing from the inverse functions f_0^{-1} to the corresponding functions (2) (with the same Taylor coefficients), one obtains the desired equalities (4), (5) for the extremals of $J(f)$ in all classes $\mathcal{B}_k$. Then, the limit case $k \to 1$ for $\mathcal{B}$ follows trivially, which completes the proof of the theorem. □

4. Some Applications of Theorem 1

As a consequence of Theorem 1 and of (7), one obtains an explicit approximatively sharp estimate for functions from classes $\mathcal{B}_k$ with small k and the non-explicit bound for arbitrary $k < 1$ (and thereby for all $f \in \mathcal{B}$).

To give an intrinsic formulation, we use the Schwarzian derivatives S_f of these functions defined by
$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \quad z \in \mathbb{D}.$$

These derivatives satisfy

$$S_{\gamma\circ f}(z) = S_f(z), \quad S_{f\circ\gamma}(z) = S_f(\gamma z)(\gamma'(z))^2$$

for any Moebius automorphism γ of $\widehat{\mathbb{C}}$ and range over a bounded domain in the complex Banach space $\mathbf{B}(\mathbb{D})$ of hyperbolically bounded holomorphic functions (quadratic differentials) φ on $\mathbb{D}$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}}(1-|z|^2)^2|\varphi(z)|.$$

This domain plays a crucial role in geometric complex analysis and in Techmüller space theory; it models the universal Techmüller space $\mathbf{T}$, in other words, the space of complex structures on the disk in holomorphic Bers' embedding of $\mathbf{T}$, but we shall use these derivatives in another aspect.

The well-known estimate (obtained, for example, in [13]), yields

Lemma 2. *For all univalent functions $f(z)$ in $\mathbb{D}$ admitting k-quasiconformal extension, their Schwarzians are sharply estimated (for any $k < 1$) by*

$$|S_f(z) \le 6k(1-|z|^2)^{-2}.$$

By the Ahlfors–Weill theorem (see [11,12]), every function φ in the space $\mathbf{B}$ with $\|\varphi\|_{\mathbf{B}} < 2$ is the Schwarzian derivative of a univalent function $f(z)$ on the unit disk $\mathbb{D}$, and this function f has quasiconformal extension onto the disk $\mathbb{D}^*$ with the Beltrami coefficient

$$\mu_\varphi(z) = -\frac{1}{2}(|z|^2-1)^2\varphi(1/\overline{z})(1/\overline{z}^4), \quad z \in \mathbb{D}^*;$$

such Beltrami coefficients are called harmonic.

As a consequence of the above results, we have

Theorem 2. *For any functional (1) with $\max_{\mathcal{B}}|J(f)| \le 1$, there is a number $\varepsilon_0 = \varepsilon_0(J) > 0$ ($\varepsilon_0 > 1/3$) such that for any $k \le \varepsilon_0$ and all $f \in \mathcal{B}_k$, whose Schwarzian derivatives S_f satisfy $\|S_f\|_{\mathbf{B}} \le 6k$, we have the sharp asymptotic estimate*

$$\max_{\mathcal{B}_k}|J(f)| = k + O(k^2) \tag{11}$$

with uniform bound for the reminder. The equality is attained on the map f^{μ_k} with

$$\mu_k(z) = k|\varphi_0(z)|/\varphi_0(z), \quad z \in \mathbb{D}^*,$$

where

$$\varphi_0(z) = \sum_1^s \frac{\partial J}{\partial a_{n_j}}(z) + \sum_1^m \frac{\partial J}{\partial z_j}(z).$$

Proof. Similar to (7),

$$f(z) = z - \frac{z^2}{\pi}\iint_{\mathbb{D}^*}\frac{\mu_{S_f}(\zeta)}{\zeta^2(\zeta-z)}d\xi d\eta + O(\|\mu_{S_f}\|_\infty^2).$$

On the other hand, the extremal Beltrami coefficient μ_k and the corresponding harmonic coefficient μ_{S_f} are related by

$$\mu_k(z) = \mu_{S_f}(z) + \nu(z), \quad \nu \in A_1(\mathbb{D}^*)^\perp,$$

where

$$A_1(\mathbb{D}^*) = \{\psi \in L_1(\mathbb{D}^*) : \ \psi \text{ holomorphic on } \ \mathbb{D}^*\}$$

and $A_1(\mathbb{D}^*)^\perp$ is the collection of Beltrami coefficients on $\mathbb{D}^*$ orthogonal to $A_1(\mathbb{D}^*)$. This implies (11). □

In the case of arbitrary $k < 1$, the extremal map f^{μ_k} in the class $\mathcal{B}_k$ is represented by

$$f^{\mu_k}(z) = z - \frac{z^2}{\pi} \iint\limits_{\mathbb{D}^*} \frac{\rho(\zeta)}{\zeta^2(\zeta - z)} d\xi d\eta,$$

where ρ is the solution of the singular integral equation

$$\rho - \mu_k \Pi \rho = \mu_k,$$

where

$$\Pi\rho = -\frac{1}{\pi} \iint\limits_{\mathbb{D}^*} \frac{\rho(\zeta)}{(\zeta - z)^2} d\xi d\eta$$

(this integral exists as a principal Cauchy value). Hence,

$$\rho(z) = \frac{\partial f^{\mu_k}}{\partial \bar{z}} = \mu_k + \mu_k \Pi \mu_k + \dots$$

Generically, this integral can be calculated only approximatively.

5. Example

We illustrate the above distortion theorems on the coefficient problem for univalent functions with k-quasiconformal extension represented by the functional

$$J(f) = a_n, \quad n \geq 2.$$

It is solved by the author only for small k; the result is given by

Theorem 3 ([17])**.** *For all univalent functions $f(z) = z + a_2 z^2 + \dots$ in $\mathbb{D}$ with k-quasiconformal extension to $\widehat{\mathbb{C}}$ and all*

$$k \leq k_n = 1/(n^2 + 1), \tag{12}$$

we have the sharp estimate

$$|a_n| \leq \frac{2k}{n-1}, \tag{13}$$

with equality only for the function

$$f_{n-1}(z) = \frac{z^{n-1}}{(1 - tkz^{n-1})^2} = z + \frac{2kt}{n-1} z^n + \dots, \quad |t| = 1, \ \ n = 2, 3, \dots$$

This solves the well-known Kühnau–Niske problem. Note that, in contrast to (7) and Theorem 2, the estimate (13) does not contain a reminder term $O(k^2)$.

However, the extremal function f_{n-1} does not belong to $\mathcal{B}_k$. Its perturbation by stretching

$$f_{n-1,r}(z) = \frac{1}{r} f_{n-1}(rz) \tag{14}$$

provides by appropriate $r < 1$ the needed function from $\mathcal{B}_k$ maximizing a_n in this class. A similar estimate is valid also for the coefficients b_n of the inverse functions.

Moreover, the assertions of Theorem 3 are valid for more general functionals of the form (1) on classes S_k of univalent functions in the disk with k-quasiconformal extension (see, e.g., [17] and the references cited there). These classes are slightly connected with classes $\mathcal{B}_k$ by stretching (14), and some distortion results obtained for the classes S_k (with sufficiently small k) can be reformulated for functions from $\mathcal{B}_k$.

6. Additional Remarks

1. One can associate with any biunivalent function f also its second quasiconformal dilatation, namely, the maximal dilatation among its quasiconformal extensions across the boundary of the entire domain D_f, where the map f is conformal. This provides a weaker result, because then the representation of type (5) of the extremal Beltrami coefficient is valid only on the complementary domain D_f^*, and there occurs an additional complication to find the extremal domain D_{f_0} explicitly.
2. Any univalent function $f(z) = z + a_2z^2 + \dots$ on the disk $\mathbb{D}$ naturally generates a $\widehat{\mathbb{C}}$-holomorphic univalent zero-free function

$$F_f(z) = 1/f(1/z) = z - c_0 + c_1z^{-1} + c_2z^{-2} + \dots \quad (c_0 = -a - 2)$$

on the complementary disk $\mathbb{D}^*$. This canonical class also plays a crucial role in geometric complex analysis.

The Lagrange formula (3) shows that the coefficients b_n of the inverse function $f^{-1}(w)$ are rather simply connected with coefficients of F_f.

3. We conclude that in fact, the class $\mathcal{B}$ of biunivalent functions is rich enough. For example, the following assertion is valid.

Lemma 3. *All univalent functions $f \in S$ with the second coefficient a_2 satisfying $|a_2| \le 1/2$ belong to $\mathcal{B}$.*

This statement is a consequence of the following covering lemma of Koebe's type proven in [22].

Let χ be a holomorphic map from a domain G in a complex Banach space $X = \{\mathbf{x}\}$ into the universal Teichmüller space $\mathbf{T}$ modeled as a bounded subdomain of $\mathbf{B}$ (indicated in Section 3) and suppose that the image set $\chi(G)$ admits the circular symmetry, which means that for every point $\varphi \in \chi(G)$, the circle $e^{i\theta}\varphi$ belongs entirely to this set. Consider in the unit disk the corresponding Schwarzian differential equations

$$S_w(z) = \chi(\mathbf{x}) \tag{15}$$

and pick their holomorphic univalent solutions $w(z)$ in $\mathbb{D}$ satisfying $w(0) = 0$, $w'(0) = 1$ (hence, $w(z) = z + \sum_2^\infty a_nz^n$). Put

$$|a_2^0| = \sup\{|a_2| : S_w \in \chi(G)\},$$

and let $w_0(z) = z + a_2^0z^2 + \dots$ be one of the maximizing functions (its existence follows from the compactness of the family of these $w(z)$ in the topology of locally uniform convergence in $\mathbb{D}$). Then, we have

Lemma 4 ([22])**.** *For every indicated solution $w(z) = z + a_2z^2 + \dots$ of the differential equation (15), the image domain $w(\mathbb{D})$ covers entirely the disk $\{|w| < 1/(2|a_2^0|)\}$.*

The radius value $1/(2|a_2^0|)$ is sharp for this collection of functions, and the circle $\{|w| = 1/(2|a_2^0|)\}$ contains points not belonging to $w(\mathbb{D})$ if and only if $|a_2| = |a_2^0|$ (i.e., when w is one of the maximizing functions).

In particular, all functions $w(z)$ cover the unit disk $\{|w| < 1\}$, which shows that their inverse functions $z^{-1}(w)$ are also univalent in this disk.

Another corollary of Lemma 4 is that the inverted functions

$$W_w(\zeta) = 1/w(1/\zeta) = \zeta - a - 2 + b_1\zeta^{-1} + b_2\zeta^{-2} + \dots, \quad |\zeta| > 1,$$

map the complementary disk $\mathbb{D}^*$ onto the domains whose boundaries are entirely contained in the disk $\{|W + a_2| \le \alpha_2^0|\}$.

Combining this with the well-known result of Elisha Netanyahu [6] that

$$\max_{\mathcal{B}} |a_2| = \frac{4}{3},$$

one finds that the boundaries of all domains $F_f(\mathbb{D}^*)$ determined by univalent functions $f \in \mathcal{B}$ are placed in the disk $\{|w + a_2| \ \le 4/3\}$.

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