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A Framework for I^*-Statistical Convergence of Fuzzy Numbers

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Abstract: In this study, we investigate the concept of I^*-statistical convergence for sequences of fuzzy numbers. We establish several theorems that provide a comprehensive understanding of this notion, including the uniqueness of limits, the relationship between I^*-statistical convergence and classical convergence, and the algebraic properties of I^*-statistically convergent sequences. We also introduce the concept of I^*-statistical pre-Cauchy and I^*-statistical Cauchy sequences and explore its connection to I^*-statistical convergence. Our results show that every I^*-statistically convergent sequence is I^*-statistically pre-Cauchy, but the converse is not necessarily true. Furthermore, we provide a sufficient condition for an I^*-statistically pre-Cauchy sequence to be I^*-statistically convergent, which involves the concept of $I^* - lim\ inf$.

Keywords: I^*-statistical convergence; I^*-statistical Cauchy sequences; I^*-statistical pre-Cauchy; $I^* - lim\ inf$

MSC: 40A05; 40A35; 40D25

1. Introduction

There has been considerable progress in the convergence theory concerning fuzzy number sequence due to seminal works and innovative extensions that have taken place. Matloka [1] introduced the primary definition of convergence of sequences of fuzzy numbers and defined its limit and discussed its algebraic properties, while Nanda [2] studied the spaces of bounded and convergent sequences of fuzzy numbers and showed that they are complete metric spaces which furthered its theoretical background. Variations are manifested by sequences that do not converge under classical convergence conditions. Most mathematical problems involve sequences that are not convergent in the usual sense. There is now a realization of the necessity of considering more classes of sequences for determining or discussing their convergences. One of the approaches is to consider sequences that converge when we restrict our attention to large subsets of natural numbers in some meaningful sense. For example, if we define an important subset as all natural numbers apart from those with finitely many, then we get the traditional concept of convergence. On the other hand, recourse may be made to subsets having zero natural density. The natural density of a subset $\mathcal{A}$ of $\mathbb{N}$ is formally expressed as $\delta(\mathcal{A})$, and it is defined as follows:

$$\delta(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n}|\{k < n : k \in \mathcal{A}\}|,$$

which will lead us to a type of convergence namely, statistical convergence. The concept of statistical convergence for sequences of real numbers was independently introduced by Fast [3] and Schoenberg [4]. This foundational idea was later expanded by Savaş [5], who discussed alternative conditions for sequences of fuzzy numbers to be statistically Cauchy. Subsequent research further explored the nuances of this area, notably by Connor [6], who

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introduced the concept of statistically pre-Cauchy sequences and demonstrated that statistically convergent sequences are inherently statistically pre-Cauchy. The exploration of statistical convergence from a sequence space perspective and its connection to summability theory was advanced by researchers like Fridy [7] and Salát [8]. For a foundational understanding of statistical convergence, we recommend consulting works such as [9–13]. Some of the applications of statistical convergence can be found in [14,15]. Kostyrko et al. [16] extended the concept of statistical convergence by introducing I-convergence and I^*-convergence, which utilize ideals in metric spaces. They discussed several basic properties of these new types of convergence. For a detailed examination of I-convergence, we suggest referring to [17–21].

Kumar and Kumar [22] applied the concepts of I-convergence, I^*-convergence, and I-Cauchy sequences to sequences of fuzzy numbers, with further developments in this area discussed in works such as [23,24].

Savaş and Das [25] later extended I-convergence to I-statistical convergence, aiming to unify λ-statistical and A-statistical convergence using ideals. They introduced the notion of I-statistically pre-Cauchy sequences, which were further investigated by Debnath et al. [26]. Later on, Debnath et al. [27] discussed I-statistical convergence, introducing I-statistical limit points and cluster points, and exploring their basic properties. They extended I-statistical convergence and proved that I^*-statistical convergence implies I-statistical convergence. In recent years, various authors have studied different kinds of convergence by generalising statistical convergence via ideals in different spaces and for different types of sequences, for example, [28–30]. However, the properties and consequences of I^*-statistical convergence have not been thoroughly discussed, which motivated our current research.

This article investigates the concept of I^*-statistical convergence for sequences of fuzzy numbers in metric space. We have proved that under I^*-statistical convergence the limit of the sequence is unique. We established several theorems that comprehensively understand this notion, which include the relationship between I^*-statistical convergence and classical convergence and the algebraic properties of I^*-statistically convergent sequences. We also defined I^*-statistically pre-Cauchy sequences and I^*-statistical Cauchy sequences and explored their connection to I^*-statistical convergence. Our results show that every I^*-statistically convergent sequence is I^*-statistically pre-Cauchy, but the converse is not necessarily true. Furthermore, we provide a sufficient condition for an I^*-statistically pre-Cauchy sequence to be I^*-statistically convergent, which involves the concept of $I^* - lim\, inf$.

2. Preliminaries

In the theory of fuzzy numbers, we start by considering intervals denoted by $\mathcal{A}$ with endpoints $\underline{\mathcal{A}}$ and $\overline{\mathcal{A}}$. The set D comprises all closed, bounded intervals on the real line $\mathbb{R}$, represented as:

$$D = \{\mathcal{A} \subset \mathbb{R} : \mathcal{A} = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]\}.$$

For any $\mathcal{A}, \mathcal{B}$ in D, we define $\mathcal{A} \leqslant \mathcal{B}$ iff $\underline{\mathcal{A}} \leqslant \underline{\mathcal{B}}$ and $\overline{\mathcal{A}} \leqslant \overline{\mathcal{B}}$, with the distance function $d(\mathcal{A}, \mathcal{B})$ being the maximum of $|\underline{\mathcal{A}} - \underline{\mathcal{B}}|$ and $|\overline{\mathcal{A}} - \overline{\mathcal{B}}|$.

The metric d establishes a Hausdorff metric on D, rendering (D, d) a complete metric space. Moreover, $\leqslant$ acts as a partial order on D.

Definition 1 ([22]). *A fuzzy number is a function $\mathcal{X}$ from $\mathbb{R}$ to $[0,1]$, which satisfy the following conditions:*

(i) *$\mathcal{X}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mathcal{X}(x_0) = 1$;*
(ii) *$\mathcal{X}$ is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, $\mathcal{X}(\lambda x + (1-\lambda)y) \geqslant \min\{\mathcal{X}(x), \mathcal{X}(y)\}$;*
(iii) *$\mathcal{X}$ is upper semi-continuous;*
(iv) *The closure of the set $\{x \in \mathbb{R} : \mathcal{X}(x) > 0\}$, denoted by $\mathcal{X}^0$ is compact.*

The properties (i)–(iv) imply that for each $\alpha \in (0,1]$, the α-level set:

$$\mathcal{X}^{\alpha} = \{x \in \mathbb{R} : \mathcal{X}(x) \geqslant \alpha\} = \left[\underline{\mathcal{X}}^{\alpha}, \overline{\mathcal{X}}^{\alpha}\right].$$

where $\mathcal{X}^{\alpha}$ represents a non-empty, compact, and convex subset of the real numbers $\mathbb{R}$.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$, and the set comprising all sequences of fuzzy numbers is represented by $L(\$)$. We define a mapping, denoted as $\overline{d}$, which takes pairs of fuzzy numbers from $L(\mathbb{R}) \times L(\mathbb{R})$ and maps them to the real numbers $\mathbb{R}$. Formally, this mapping $\overline{d}$ can be expressed as follows:

$$\overline{d}(\mathcal{X}, \mathcal{Y}) = \sup_{\alpha \in [0,1]} d(\mathcal{X}^{\alpha}, \mathcal{Y}^{\alpha}).$$

where $\overline{d}(\mathcal{X}, \mathcal{Y})$ computes the supremum of the distance, d, between the α-level sets of fuzzy numbers $\mathcal{X}$ and $\mathcal{Y}$ across all values of α within the interval $[0, 1]$.

Puri and Ralescu [31] demonstrated that the space $(L(\mathbb{R}), \overline{d})$ constitutes a complete metric space: "We define the relation $\mathcal{X} \leqslant \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} \in L(\mathbb{R})$ if $\underline{\mathcal{X}}^{\alpha} \leqslant \underline{\mathcal{Y}}^{\alpha}$ and $\overline{\mathcal{X}}^{\alpha} \leqslant \overline{\mathcal{Y}}^{\alpha}$ for each $\alpha \in [0, 1]$. Furthermore, we define $\mathcal{X} < \mathcal{Y}$ if $\mathcal{X} \leqslant \mathcal{Y}$ and there exists some $\alpha_0 \in [0, 1]$ such that $\underline{\mathcal{X}}^{\alpha_0} < \underline{\mathcal{Y}}^{\alpha_0}$ or $\overline{\mathcal{X}}^{\alpha_0} < \overline{\mathcal{Y}}^{\alpha_0}$. If neither $\mathcal{X} \leqslant \mathcal{Y}$ nor $\mathcal{Y} \leqslant \mathcal{X}$ holds, we say that $\mathcal{X}$ and $\mathcal{Y}$ are incomparable fuzzy numbers". Moreover, they continue that in the metric space $L(\mathbb{R})$, "we can define addition $\mathcal{X} + \mathcal{Y}$ and scalar multiplication $\lambda\mathcal{X}$, where λ is a real number, in terms of α-level sets as follows:

$$[\mathcal{X} + \mathcal{Y}]^{\alpha} = [\mathcal{X}]^{\alpha} + [\mathcal{Y}]^{\alpha}$$

for each $\alpha \in [0, 1]$, and

$$[\lambda\mathcal{X}]^{\alpha} = \lambda[\mathcal{X}]^{\alpha}$$

for each $\alpha \in [0, 1]$, respectively".

Regarding fuzzy integers within a subset S of $L(\mathbb{R})$, if there exists a fuzzy integer denoted by μ such that $\mathcal{X} \leqslant \mu$ holds for every $\mathcal{X}$ in the subset S, we designate S as having an upper bound, with μ serving as the upper bound for the set. Similarly, we define the lower bound.

For each $\alpha \in [0, 1]$, if we define $\overline{\mathcal{Z}}^{\alpha} := \overline{\mathcal{X}}^{\alpha} + \overline{\mathcal{Y}}^{\alpha}$ and $\underline{\mathcal{Z}}^{\alpha} := \underline{\mathcal{X}}^{\alpha} + \underline{\mathcal{Y}}^{\alpha}$, we can express $\mathcal{Z}$ as the sum of $\mathcal{X}$ and $\mathcal{Y}$, denoted as $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. Similarly, following a comparable pattern, we represent $\mathcal{Z}$ as the difference of $\mathcal{X}$ and $\mathcal{Y}$, expressed as $\mathcal{Z} = \mathcal{X} - \mathcal{Y}$, iff $\overline{\mathcal{Z}}^{\alpha} := \overline{\mathcal{X}}^{\alpha} - \underline{\mathcal{Y}}^{\alpha}$ and $\underline{\mathcal{Z}}^{\alpha} := \underline{\mathcal{X}}^{\alpha} - \overline{\mathcal{Y}}^{\alpha}$ for each $\alpha \in [0, 1]$.

Definition 2 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers are said to be convergent to a fuzzy number $\mathcal{X}_0$ if, for every $\varepsilon > 0$, there exists a positive integer m such that $\overline{d}(\mathcal{X}_n, \mathcal{X}_0) < \varepsilon$ for every $n \geqslant m$. The fuzzy number $\mathcal{X}_0$ is referred to as the ordinary limit of the sequence $(\mathcal{X}_n)$, denoted as $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}_0$.*

Definition 3 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers are regarded as a Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\overline{d}(\mathcal{X}_n, \mathcal{X}_m) < \varepsilon$ for all $n, m \geqslant n_0$.*

Definition 4 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers are categorized as a bounded sequence if the set $\{\mathcal{X}_n : n \in \mathbb{N}\}$, comprising all the fuzzy numbers in the sequence is itself a bounded set of fuzzy numbers.*

Definition 5 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers are considered to be statistically convergent to a fuzzy number $\mathcal{X}_0$ if, for any $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : \overline{d}(\mathcal{X}_n, \mathcal{X}_0) \geqslant \varepsilon\}$ exhibits a natural density of zero. In this context, the natural density of a set refers to the proportion of natural numbers within the set concerning the whole set of natural numbers. The fuzzy number $\mathcal{X}_0$ is termed the statistical limit of the sequence $(\mathcal{X}_n)$, denoted as $st - \lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}_0$.*

Definition 6 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers are termed statistically Cauchy if, for any $\varepsilon > 0$, there exists a positive integer $m = m(\varepsilon)$ such that the set $\{n \in \mathbb{N} : \overline{d}(\mathcal{X}_n, \mathcal{X}_m) \geqslant \varepsilon\}$ has a natural density of zero. In this context, the term "natural density" pertains to the proportion of natural numbers within the set concerning the entire set of natural numbers.*

Throughout this paper, we will use $\mathbb{R}$ and $\mathbb{N}$ to represent, respectively, the set of real numbers and positive integers. We will denote the power set of any set $\mathcal{X}$ as $P(\mathcal{X})$, and the complement of the set $\mathcal{A}$ will be denoted as $\mathcal{A}^c$.

Definition 7 ([22])**.** *Let X be a non-empty set, then a collection of subsets I contained in the power set of X denoted as $P(X)$ is said to be ideal iff it satisfies the following conditions:*

(i) The empty set belongs to I, i.e., $\emptyset \in I$;
(ii) For any set $\mathcal{A}$ and $\mathcal{B}$ belonging to I, $\mathcal{A} \cup \mathcal{B}$ also belongs to I;
(iii) If $\mathcal{A} \in I$ and $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{B} \in I$.

Definition 8. *Let X be a non-empty set. A non-empty family of sets F contained within the power set $P(X)$ is denoted as a filter on X iff it adheres to the following criteria:*

(i) The empty set $\emptyset$ is not an element of the filter, meaning $\emptyset \notin F$;
(ii) For any two sets $\mathcal{A}$ and $\mathcal{B}$ that belong to the filter, their intersection denoted as $\mathcal{A} \cap \mathcal{B}$ is also a part of the filter formally expressed as $\mathcal{A} \cap \mathcal{B} \in F$;
(iii) If a set $\mathcal{A}$ is a member of the filter and $\mathcal{B}$ is a super set of $\mathcal{A}$, then $\mathcal{B}$ is also an element of the filter, i.e., $\mathcal{B} \in F$.

Conditions (i), (ii), and (iii) jointly define the properties of a filter on set X.

An ideal I is termed non-trivial if it satisfies two conditions: it is not an empty set ($I \neq \emptyset$), and it does not contain the entire set X ($X \notin I$). Notably, a non-trivial ideal $I \subset P(X)$ corresponds to a filter, denoted as $F(I)$, which is formed by taking the set complement of each element of I with respect to the entire set X. The filter $F(I)$ is referred to as the filter associated with the ideal I.

An ideal I in X is considered admissible iff it includes all singleton sets, i.e.,$\{\{x\} : x \in X\}$.

Definition 9 ([22])**.** *Suppose $I \subset P(\mathbb{N})$ is a non-trivial ideal. We define a sequence $\mathcal{X} = (\mathcal{X}_n)$ of fuzzy numbers as I-convergent to a fuzzy number $\mathcal{X}_0$ if, for any ϵ, the set $A(\epsilon) = \{n \in \mathbb{N} : \overline{d}(\mathcal{X}_n, \mathcal{X}_0) \geqslant \epsilon\} \in I$.*

The fuzzy number $\mathcal{X}_0$ is then referred to as the I-limit of the sequence $(\mathcal{X}_n)$, and this is denoted as $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}_0$.

The set of fuzzy number sequences that are both convergent and I-convergent can be denoted by ℓ_1. These sequences exhibit both conventional convergence and convergence according to the ideal I, providing a rich framework for the study of their convergence properties. Throughout the paper, we consider I as an admissible ideal.

Definition 10 ([22])**.** *A sequence $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$ of fuzzy numbers is said to be I^*-convergent to a fuzzy number $\mathcal{X}_0$ iff there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 < \cdots < m_k < \dots\} \subset \mathbb{N}$ such that $\mathcal{K} \in F(I)$ and $\overline{d}\left(\mathcal{X}_{m_k}, \mathcal{X}_0\right) \to 0$ as $n \to \infty$.*

3. I^*-Statistical Convergence of Sequence of Fuzzy Numbers

Definition 11. *A sequence $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$ is said to be I^*-statistically convergent to a fuzzy number $\mathcal{X}_0$ if and only if there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 \cdots < m_k < \dots\} \subset \mathbb{N}$ and for each $\epsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) < \epsilon\} \in F(I)| = 1$. $\mathcal{X}_0$ is the I^*-statistical limit of $\mathcal{X}_n$ and is denoted by $I^* - st - lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$.*

Example 1. *Consider the sequence $\mathcal{X} = (\mathcal{X}_n)$, which is defined as follows:*

$$\mathcal{X} = (\mathcal{X}_n) = \begin{cases} 0 & \text{for } n = \mathcal{k}^2 \text{ where } \mathcal{k} \in \mathbb{N} \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

which is I^*-statistically convergent to 0. Let $\mathcal{K} = \{m_1 < m_2 < m_3 < \cdots < m_k < \ldots\} \subset \mathbb{N}$, where $m_1, m_2, m_3, \ldots, m_k, \ldots$ are all non-perfect square natural numbers. Then, for each $\epsilon > 0$, we have:

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{ m_k < n : \overline{d}(\mathcal{X}_{m_k}, 0) < \epsilon \right\} \in \mathcal{K}\right| = 1.$$

It is trivial to show that I is an ideal if it is the collection of subsets of the set $X = \{n \in \mathbb{N} : n = k^2\}$. This implies that $\mathcal{K} \in F(I)$. Therefore:

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{ m_k < n : \overline{d}(\mathcal{X}_{m_k}, 0) < \epsilon \right\} \in F(I)\right| = 1.$$

Theorem 1. *If I is an admissible ideal, then a sequence $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$ that is I^*-statistically convergent will converge to a unique limit.*

Proof. Let $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$ be an I^*-statistically convergent sequences to two different fuzzy numbers $\mathcal{X}_0$ and $\mathcal{Y}_0$. Without the loss of generality, suppose that $\mathcal{X}_0$ and $\mathcal{Y}_0$ are comparable fuzzy numbers. Consequently, there exists $\alpha_0 \in [0, 1]$ such that:

$$\underline{\mathcal{X}}_0^{\alpha_0} < \underline{\mathcal{Y}}_0^{\alpha_0} \quad \text{and} \quad \overline{\mathcal{X}}_0^{\alpha_0} > \overline{\mathcal{Y}}_0^{\alpha_0} \tag{1}$$

or

$$\underline{\mathcal{X}_0}^{\alpha_0} > \underline{\mathcal{Y}_0}^{\alpha_0} \quad \text{and} \quad \overline{\mathcal{X}_0}^{\alpha_0} < \overline{\mathcal{Y}_0}^{\alpha_0}. \tag{2}$$

We will prove that (1) and (2) can be performed in a similar manner.

Let us assume that (1) is valid. Choose $\xi_1 = \underline{\mathcal{Y}_0}^{\alpha_0} - \underline{\mathcal{X}_0}^{\alpha_0}$ and $\xi_2 = \underline{\mathcal{X}_0}^{\alpha_0} - \underline{\mathcal{Y}_0}^{\alpha_0}$. Clearly $\xi_1 > 0$ and $\xi_2 > 0$. Let $\xi' = min\{\xi_1, \xi_2\}$. Select ϵ such that $0 < \epsilon < \xi'$. Given that $(\mathcal{X}_n)$ is I^*-statistical convergent to both $\mathcal{X}_0$ and $\mathcal{Y}_0$ therefore, we have $\mathcal{M} = \{m_1 < m_2 < m_3 < \cdots < m_k < \ldots\} \subset \mathbb{N}$ and $\mathcal{K} = \{n_1 < n_2 < n_3 < \cdots < n_k < \ldots\} \subset \mathbb{N}$ such that for every $\epsilon > 0$:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1 \text{ and} \\ &\lim_{n\to\infty} \frac{1}{n}|\{n_k < n : \overline{d}(\mathcal{X}_{n_k}, \mathcal{Y}_0) \leqslant \epsilon\} \in F(I)| = 1 \end{aligned} \tag{3}$$

since $F(I)$ is a filter on $\mathbb{N}$ therefore, by the definition of filter $\mathcal{M} \cap \mathcal{N} \neq \phi$.

Let $m \in \mathcal{M} \cap \mathcal{N}$ then by (3) there exists positive integers k_1 and k_2 such that:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1 \text{ for every } m_k \in \mathcal{M} \\ &\text{with} m_k > \mathcal{K}_1 \text{ and} \\ &\lim_{n\to\infty} \frac{1}{n}|\{n_k < n : \overline{d}(\mathcal{X}_{n_k}, \mathcal{Y}_0) \leqslant \epsilon\} \in F(I)| = 1 \text{ for every } n_k \in \mathcal{N} \\ &\text{with } n_k > \mathcal{K}_2. \end{aligned} \tag{4}$$

Let $k = max\{k_1, k_2\}$ the (4) follows for $m \in \mathcal{M} \cap \mathcal{N}$ with $n_k, m_k > k$. For each $\alpha \in [0, 1]$ and $m = max\{m_k, n_k\}$ we have, $\lim_{n\to\infty} \frac{1}{n}|\{m < n : \overline{d}(\mathcal{X}_m^{\alpha_0}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$ and $\lim_{n\to\infty} \frac{1}{n}|\{m < n : \overline{d}\ (\mathcal{X}_m^{\alpha_0}, \mathcal{Y}_0) \leqslant \epsilon\} \in F(I)| = 1$. Now the definition of d implies:

$$\begin{aligned} &\left|\underline{\mathcal{X}}_m^{\alpha_0} - \underline{\mathcal{X}}_0^{\alpha_0}\right| < \varepsilon \text{ and } \left|\underline{\mathcal{X}}_m^{\alpha_0} - \underline{\mathcal{Y}}_0^{\alpha_0}\right| < \varepsilon, \\ &\left|\overline{\mathcal{X}}_m^{\alpha_0} - \overline{\mathcal{X}}_0^{\alpha_0}\right| < \varepsilon \text{ and } \left|\overline{\mathcal{X}}_m^{\alpha_0} - \overline{\mathcal{Y}}_0^{\alpha_0}\right| < \varepsilon. \end{aligned}$$

$\underline{\mathcal{X}}_m^{\alpha_0} \in \left(\underline{\mathcal{X}}_0^{\alpha_0} - \varepsilon, \underline{\mathcal{X}}_0^{\alpha_0} + \varepsilon\right) \cap \left(\underline{\mathcal{Y}}_0^{\alpha_0} - \varepsilon, \underline{\mathcal{Y}}_0^{\alpha_0} + \varepsilon\right) = \Phi$. Thus, a contradiction arises, implying the comparability of fuzzy numbers $\mathcal{X}_0$ and $\mathcal{Y}_0$. Consider $\mathcal{X}_0 \leqslant \mathcal{Y}_0$ and the neighborhoods $\mathcal{A} = \left\{n \in \mathbb{N} : \overline{d}(\mathcal{X}_n, \mathcal{X}_0) < \varepsilon\right\}$ and $\mathcal{B} = \left\{n \in \mathbb{N} : \overline{d}(\mathcal{X}_n, \mathcal{Y}_0) < \varepsilon\right\}$ of $\mathcal{X}_0$ and $\mathcal{Y}_0$, respectively, are disjoint for $\varepsilon = \frac{\overline{d}(\mathcal{X}_0, \mathcal{Y}_0)}{3} > 0$. By Definition (8), both the sets $\mathcal{A}, \mathcal{B} \in F(I)$ so that $\mathcal{A} \cap \mathcal{B} \neq \Phi$. A contradiction has arrived that the neighborhoods of $\mathcal{X}_0$ and $\mathcal{Y}_0$ are disjoint. Hence, $\mathcal{X}_0$ is determined uniquely. □

Theorem 2. *Let $\mathcal{X} = (\mathcal{X}_n)$ and $\mathcal{Y} = (\mathcal{Y}_n) \in L(\$)$ then:*

(i) $lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$ *implies* $I^* - st - lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$;
(ii) $I^* - st - lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$ *and* $c \in \mathbb{R}$, *then* $I^* - st - lim_{n\to\infty}c\mathcal{X}_n = c\mathcal{X}_0$;
(iii) *If* $I^* - st - lim_{n\to\infty} = \mathcal{X}_0$ *and* $I^* - st - lim_{n\to\infty}\mathcal{Y}_n = \mathcal{Y}_0$ *then* $I^* - st - lim_{n\to\infty}(\mathcal{X}_n + \mathcal{Y}_n) = (\mathcal{X}_0 + \mathcal{Y}_0)$.

Proof.

(i) Let $lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$, then for each $\epsilon > 0$ there exists a positive integer m(say) such that $\overline{d}(\mathcal{X}_n, \mathcal{X}_0) < \epsilon$ for every $n \geqslant m$. Then, for $\epsilon > 0$ let $A(\epsilon) = \{m_{\mathcal{K}} : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) < \epsilon\}$ for set $\mathcal{K} = \{m_1 < m_2 < \ldots < m_{\mathcal{K}} < \ldots\} \subset \mathbb{N}$ is an infinite set then there exists a set $H = \{n_1, n_2, n_3, \ldots, n_k\}$ such that $\mathbb{N} - H = \mathcal{K}$ and H is a finite set, and therefore, $H \in I$ as I is an admissible ideal. This implies that $\mathcal{K} \in F(I)$ and $\delta(\mathcal{K}) = 1$. Thus, $\lim_{n\to\infty} \dfrac{1}{n}|\{m_{\mathcal{K}} < n : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$. Hence, $lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$ implies $I^* - st - lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$.

(ii) Let $\alpha \in [0,1]$ and $c \in \mathbb{R}$. Let $\mathcal{K} = \{m_1 < m_2 < \ldots < m_{\mathcal{K}} < \ldots\} \subset \mathbb{N}$ and $\epsilon > 0$ be given. Since $d(c\mathcal{X}_n{}^{\alpha}, c\mathcal{X}_0{}^{\alpha}) = |c|d(\mathcal{X}_n{}^{\alpha}, \mathcal{X}_0{}^{\alpha})$. Therefore, $\overline{d}(c\mathcal{X}_{m_{\mathcal{K}}}, c\mathcal{X}_0) = |c|\overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0)$. As $I^* - st - lim_{n\to\infty}\mathcal{X}_n = \mathcal{X}_0$. Therefore, the set $A(\epsilon) = \{m_{\mathcal{K}} : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)$ and $\delta(A(\epsilon)) = 1$. Let $B(\epsilon) = \{m_{\mathcal{K}} : \overline{d}(c\mathcal{X}_{m_{\mathcal{K}}}, c\mathcal{X}_0) \leqslant \epsilon\}$. We will show that $B(\epsilon)$ is contained in $A(\epsilon_1)$ for some $0 < \epsilon_1 < \epsilon$. Let $m_p \in B(\epsilon)$, then $\overline{d}(c\mathcal{X}_{m_{\mathcal{K}}}, c\mathcal{X}_0) \leqslant \epsilon$, which implies that $|c|\overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) \leqslant \epsilon$, that is, $\overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) \leqslant \frac{\epsilon}{|c|} = \epsilon_1$(say). Therefore, $m \in A(\epsilon_1)$. Since $\mathcal{X}_n$ is I^*-statistically convergent therefore, $A(\epsilon_1) \in F(I)$ and by this $B(\epsilon) \in F(I)$. Hence, $I^* - st - lim_{n\to\infty}c\mathcal{X}_n = c\mathcal{X}_0$.

(iii) For $\alpha \in [0,1]$, let $\mathcal{X}_n{}^{\alpha}$, $\mathcal{Y}_n{}^{\alpha}$, $\mathcal{X}_0{}^{\alpha}$, and $\mathcal{Y}_0{}^{\alpha}$ be the α level sets of $\mathcal{X}_n$, $\mathcal{Y}_n$, $\mathcal{X}_0$, and $\mathcal{Y}_0$, respectively. Since $\overline{d}(\mathcal{X}_n{}^{\alpha} + \mathcal{Y}_n{}^{\alpha}, \mathcal{X}_0{}^{\alpha} + \mathcal{Y}_0{}^{\alpha}) \leqslant \overline{d}(\mathcal{X}_n{}^{\alpha}, \mathcal{X}_0{}^{\alpha}) + \overline{d}(\mathcal{Y}_n{}^{\alpha}, \mathcal{Y}_0{}^{\alpha})$, therefore, $\overline{d}(\mathcal{X}_n + \mathcal{Y}_n, \mathcal{X}_0 + \mathcal{Y}_0) \leqslant \overline{d}(\mathcal{X}_n, \mathcal{X}_0) + \overline{d}(\mathcal{Y}_n, \mathcal{Y}_0)$. Let $\epsilon > 0$ be given. Since $\mathcal{X}_n$ and $\mathcal{Y}_n$ are I^*-statistically convergent, therefore, there exists $\mathcal{K} = \{m_1 < m_2 < \cdots < m_{\mathcal{K}} < \ldots\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n}|\{m_{\mathcal{K}} < n : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$ and $\lim_{n\to\infty} \frac{1}{n}|\{m_{\mathcal{K}} < n : \overline{d}(\mathcal{Y}_{m_{\mathcal{K}}}, \mathcal{Y}_0) \leqslant \epsilon\} \in F(I)| = 1$. Take $A(\frac{\epsilon}{2}) = \{m_{\mathcal{K}} : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}}, \mathcal{X}_0) < \frac{\epsilon}{2}\}$, $B(\frac{\epsilon}{2}) = \{m_{\mathcal{K}} : \overline{d}(\mathcal{Y}_{m_{\mathcal{K}}}, \mathcal{Y}_0) < \frac{\epsilon}{2}\}$ and $C(\epsilon) = \{m_{\mathcal{K}} : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}} + \mathcal{Y}_{m_{\mathcal{K}}}, \mathcal{X}_0 + \mathcal{Y}_0) < \epsilon\}$. Since, $A(\frac{\epsilon}{2}) \in F(I)$ and $B(\frac{\epsilon}{2}) \in F(I)$, therefore, $A(\frac{\epsilon}{2}) \cap B(\frac{\epsilon}{2}) \neq \phi$ and belongs to the filter; thus, we have for all $n \in A(\frac{\epsilon}{2}) \cap B(\frac{\epsilon}{2}) \subset C(\frac{\epsilon}{2}) \in F(I)$, i.e., $\lim_{n\to\infty} \frac{1}{n}|\{m_{\mathcal{K}} < n : \overline{d}(\mathcal{X}_{m_{\mathcal{K}}} + \mathcal{Y}_{m_{\mathcal{K}}}, \mathcal{X}_0 + \mathcal{Y}_0) \leqslant \epsilon\} \in F(I)| = 1$. Hence, $I^* - st - lim(\mathcal{X}_n + \mathcal{Y}_n) = (\mathcal{X}_0 + \mathcal{Y}_0)$.

□

Theorem 3. *For any sequence $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$ if there exists two sequences $\mathcal{Y} = (\mathcal{Y}_n)$, $\mathcal{Z} = (\mathcal{Z}_n) \in L(\$)$ of fuzzy numbers such that $\mathcal{X} = \mathcal{Y} + \mathcal{Z}$, $\overline{d}(\mathcal{Y}_n, \mathcal{X}_0) \to 0$ as $n \to 0$ and $Supp\mathcal{Z} = \{n \in \mathbb{N} : \mathcal{Z}_n \neq 0\} \in I$ and $\delta(Supp\mathcal{Z}) = 0$, then $\mathcal{X}$ is I^*-statistically convergent.*

Proof. Let $\mathcal{Y} = (\mathcal{Y}_n), \mathcal{Z} = (\mathcal{Z}_n) \in L(\$)$ such that $\mathcal{X} = \mathcal{Y} + \mathcal{Z}$, $\overline{d}(\mathcal{Y}_n, \mathcal{X}_0) \to 0$ as $n \to 0$ and $\lim_{n\to\infty} \dfrac{1}{n}|\{n \in \mathbb{N} : \mathcal{Z}_n \neq 0\} \in I| = 0$.

Let $\mathcal{K} = \{n \in \mathbb{N} : \mathcal{Z}_n = 0\}$. Since $Supp\mathcal{Z}$ belongs to I then $\mathcal{K} \in F(I)$ with $\delta(\mathcal{K}) = 1$ and also $\mathcal{K}$ is an infinite set as otherwise $\mathcal{K} \in I$. Let $\mathcal{K} = \{m_1 < m_2 < m_3 \cdots < m_{\mathcal{K}} < \ldots\} \subset \mathbb{N}$ such that $\delta(k) = 1$ then $\mathcal{X}_{m_{\mathcal{K}}} = \mathcal{Y}_{m_{\mathcal{K}}}$ for each $n \in \mathbb{N}$. Since $\mathcal{Z}_n = 0$ for all $n \in K$. It is given that $\overline{d}(\mathcal{Y}_n, \mathcal{X}_0) \to 0$. Therefore, $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \to \infty$. Thus, $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$. This proves that $\mathcal{X}$ is I^*-statistically convergent. □

4. I^*-Statistically Cauchy and I^*-Statistically Pre-Cauchy Sequences

Definition 12. *A sequence $\mathcal{X} = (\mathcal{X}_n)$ is said to be I^*-statistically Cauchy if there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ and for each $\epsilon > 0$, there exists $m_p \in \mathbb{N}(\epsilon)$ such that $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon\} \in F(I)| = 1$. I^*_{ca} denotes the collection of all I^*-statistically Cauchy sequences.*

Definition 13. *A sequence $\mathcal{X} = (\mathcal{X}_n)$ is said to be I^*-statistically pre-Cauchy if there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ and for each $\epsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n^2}|\{(m_k, m_p) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n\} \in F(I)| = 1$.*

Theorem 4. *Every I^*-statistically convergent sequence is I^*-statistically Cauchy.*

Proof. Let $\mathcal{X} = (\mathcal{X}_n)$ be I^*-statistically convergent to $\mathcal{X}_0$. Then, there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 \cdots < m_k < \ldots\} \subset \mathbb{N}$ and for each $\epsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$. Let $C = \{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \frac{\epsilon}{2}\} \in F(I)$ and $\delta(C) = 1$. Since I is an admissible ideal, therefore, we can choose $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \frac{\epsilon}{2}$. Define $B = \{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\}$. We need to show that $C \subset B$. Let $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0)$ be any arbitrary element of C, then $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) < \frac{\epsilon}{2}$, $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) + \overline{d}(\mathcal{X}_{m_p}, \mathcal{X}_0) \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2}$, and $\overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon$, which shows that every element of C is as element of B. Therefore, $C \subset B$. According to the Definition (8) $B \in F(I)$ and since $\delta(C) = 1$, this implies that $\delta(B) = 1$. Hence, we have $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon\} \in F(I)| = 1$. □

Theorem 5. *Every I^*-statistically Cauchy sequence is I^*-statistically pre-Cauchy.*

Proof. Let $\mathcal{X} = (\mathcal{X}_n)$ be any arbitrary sequence of I^*_{ca}. Then, there exists a set $\mathcal{K} = \{m_1 < m_2 < m_3 \cdots < m_k < \ldots\} \subset \mathbb{N}$ and for each $\epsilon > 0$ we have $\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$. Let $C = \{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \frac{\epsilon}{2}\} \in F(I)$ and $\delta(C) = 1$. Now without any loss of generality define T such that $\mathcal{X}_{m_p}$ be any term of the sequence $\mathcal{X}_n$ and $T = \{(m_k, m_p) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n\}$ and by Definition (8) $T \in F(I)$ and $\delta(K) = 1$. That is, $\lim_{n\to\infty} \frac{1}{n^2}|\{(m_k, m_p) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n\} \in F(I)| = 1$, which shows that every I^*-statistically Cauchy sequence is I^*-statistically pre-Cauchy. □

Remark 1. *Every I^*-statistically pre-Cauchy sequence need not be I^*-statistically Cauchy.*

To understand this we will consider the following example.

Example 2. *Let $\mathcal{X} = (\mathcal{X}_n)$ be a sequence defined as:*

$$\mathcal{X}_n = \begin{cases} (0, 1, 2) & \text{if } n \text{ is a odd}, \\ (0, 0.5, 1) & \text{if } n \text{ is a even}. \end{cases}$$

where (a, b, c) denotes a triangular fuzzy number [32] with peak at b and support $[a, c]$. Let $\epsilon > 0$ be arbitrary. Without the loss of generality, we can choose $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$, we have:

$$\frac{1}{n^2}|\{(m_k, m_p) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n\}|$$

$$\geqslant \frac{1}{n^2}|\{(m_k, m_p) : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n_0\}|$$

Let K be the collection of all odd natural numbers, $K = \{m_1 < m_2 < m_3 < \dots\} \subset \mathbb{N}$*(say). This implies* $K \in F(I)$*. Since* $m_k, m_p \leqslant n_0$ *and belongs to K implies that* m_k, m_p *are both odd, and therefore:*

$$\overline{d}(x_{m_k}, x_{m_p}) = \overline{d}((0,1,2),(0,1,2)) = 0 \leqslant \epsilon$$

Let $C = \{(m_k, m_p) : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon; x_{m_k}, x_{m_p} \leqslant n_0\}$ *and* C^c *denotes the compliment of C. We will show that* $\lim_{n\to\infty} \frac{1}{n^2}|C^c| = 0$*. Since* C^c *contains all even numbers less than or equal to* n_0*. Thus, we have:*

$$\frac{1}{n^2}|C^c| \leqslant \frac{n_0/2}{n^2} \leqslant \frac{1}{2n}.$$

Since n_0 *is fixed, the right-hand side approaches 0 as* $n \to \infty$*. Therefore, we have* $\lim_{n\to\infty} \dfrac{1}{n^2}|\{(m_k, m_p) : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon; m_k, m_p \leqslant n\} \in F(I)| = 1$*, which shows that* x *is* I^**-statistically pre-Cauchy.*

However, x *is not* I^**-statistically Cauchy. Suppose for the sake of contradiction that* x *is* I^**-statistically Cauchy. Then, there exists a set* $\mathcal{K} = \{m_1 < m_2 < m_3 < \dots\} \subset \mathbb{N}$ *and for each* $\epsilon > 0$*, there exists* $m_p \in \mathbb{N}(\epsilon)$ *such that:*

$$\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon\} \in F(I)| = 1$$

Without the loss of generality we can choose $n_0 \in \mathbb{N}$ *such that* $n \geqslant n_0$*, we have:*

$$\lim_{n\to\infty} \frac{1}{n}|\{m_k < n : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon\}| \geqslant \lim_{n\to\infty} \frac{1}{n}|\{m_k < n_0 : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon\}|$$

Let $D = \{m_k \leqslant n_0 : \overline{d}(x_{m_k}, x_{m_p}) \leqslant \epsilon\}$ *and* D^c *denotes the compliment of D. We will show that* $\lim_{n\to\infty} \frac{1}{n}|D^c| = 0$*. Since* D^c *contains all even numbers less than or equal to* n_0*. Thus, we have:*

$$\frac{1}{n}|D^c| \leqslant \frac{n_0/2}{n} \leqslant \frac{1}{2}$$

which is a contradiction, so x *is not* I^**-statistically Cauchy.*

Theorem 6. *Every* I^**- statistically convergent sequence is* I^**-statistically pre-Cauchy.*

Proof. The proof is trivial from Theorem 4 and 5. See the Appendix A. □

To illustrate the concept of a sequence that is I^*-statistically pre-Cauchy but not I^*-statistically convergent, we can consider the the following example. Understanding that any I^*-statistically convergent sequence must contain a subsequence that converges in the usual sense is crucial. Let us look at the example below.

Example 3. *Let* $x = (x_k)$ *be a sequence. Consider the sequence* $x = (x_k)$ *defined such that for* $(m_k - 1)! < k < m_k!$*, we have* $x_k = \sum_{n=1}^{m_k} \frac{1}{n}$*. This sequence* $x = (x_k)$ *does not possess any convergent subsequences, implying that* x *is not* I^**-statistically convergent. However, despite the lack of convergent subsequences, the sequence is* I^**-statistically pre-Cauchy. This means that while the entire sequence does not converge in the* I^**-statistical sense, it still satisfies the pre-Cauchy criterion under* I^**-statistical conditions.*

Let $\epsilon > 0$ *be given and let* $\mathcal{K} = \{m_1 < m_2 < m_3 \dots < m_k < \dots\} \subset \mathbb{N} \in F(I)$*,* $m_k \in \mathbb{N}$ *satisfy* $\frac{1}{m_k} < \epsilon$*. Now, consider the case where* $m_k! < n < (m_k + 1)!$ *and* $(m_k - 1)! < j, k < n$*, then* $d(x_{m_j}, x_{m_k}) \leqslant \frac{1}{m_k} < \epsilon$*. It follows that, for* $m_k! < n < (m_k + 1)!$*,* $\frac{1}{n^2}|\{(m_j, m_k) : d(x_{m_j}, x_{m_k}) < \epsilon, m_j, m_k \leqslant n)\}|$*, we have* $\geqslant \frac{1}{n^2}[n - (m_k - 1)!]^2$*,* $\geqslant \left[1 - \frac{(m_k-1)!}{m_k!}\right]^2$*, and* $= [1 - \frac{1}{m_k}]^2$*.*

Since $lim_{k\to\infty}(1 - \frac{1}{m_k}) = 1$. *As a result,* $\mathcal{X}$ *is* I^**-statistically pre-Cauchy.*

Before we present the next theorem, we need to introduce the definition of the $I^* - lim\ inf$. Let us first outline this concept.

Definition 14. *Let* I *be an admissible ideal of* $\mathbb{N}$ *and let* $\mathcal{X} = (\mathcal{X}_n) \in L(\$)$. *Let* $\mathcal{A}_x = \{\mu \in L(\mathbb{R}) : \{k : x_k < \mu\} \in F(I)\}$ *then the* $I^* - lim\ inf$ *is given by:*

$$I^* - lim\ inf \mathcal{X} = \begin{cases} inf\ \mathcal{A}_x & if\ \mathcal{A}_x \neq \phi \\ \infty & if\ \mathcal{A}_x = \phi \end{cases}$$

It is known that "$I^* - lim\ inf \mathcal{X} = \eta$*(finite) if and only if for arbitrary* $\epsilon > 0$ $\{k : x_k < \eta + \epsilon\} \in F(I)$ *and* $\{k : x_k < \eta - \epsilon\} \notin F(I)$*".*

Theorem 7. *Suppose* $\mathcal{X} = (\mathcal{X}_k) \in L(\$)$ *is* I^**-statistically pre-Cauchy. If* $\mathcal{X}$ *has a subsequence* $(\mathcal{X}_{p_k})$ *that converges to* $\mathcal{X}_0$ *and* $0 < I^* - lim\ inf \frac{1}{n}|\{p_k \leqslant n : k \in \mathbb{N}\}| < \infty$ *then* $\mathcal{X}$ *is* I^**-statistically convergent to* $\mathcal{X}_0$*.*

Proof. Let $\epsilon > 0$ be given. Since $lim \mathcal{X}_{p_k} = \mathcal{X}_0$ choose $r \in \mathbb{N}$ such that $\overline{d}(\mathcal{X}_j, \mathcal{X}_0) < \frac{\epsilon}{2}$ whenever $j > r$ and $j = p_k$ for some k. Let $\mathcal{A} = \{p_k : p_k > r, k \in \mathbb{N}\}$ and $\mathcal{A}(\epsilon) = \{k : \overline{d}(\mathcal{X}_k, \mathcal{X}_0) \geqslant \epsilon\}$. Now note that $\dfrac{1}{n^2}|\{n \in \mathbb{N} : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_p}) \leqslant \frac{\epsilon}{2}, m_k, m_p < n\}|$
$\leqslant \frac{1}{n^2} \sum_{\mathcal{A}(\epsilon)\times\mathcal{A}} (m_j, m_k)$
$= \frac{1}{n}|\{p_k \leqslant n : p_k \in \mathcal{A}\}|.\frac{1}{n}|\{k \leqslant n : \overline{d}(\mathcal{X}_k, \mathcal{X}_0) \geqslant \epsilon\}|$.

Since $\mathcal{X}$ is I^*-statistically pre-Cauchy, then there exists a set $\mathcal{K} = \{m_1 < m_2 < \cdots < m_k < \dots\} \subset \mathbb{N}$ and for each $\epsilon > 0$ we have $\lim_{n\to\infty} \dfrac{1}{n^2}|\{(m_k, m_j) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_j}) \leqslant \epsilon; m_k, m_p < n\} \in F(I)| = 1$. Let $C = \{(m_k, m_j) : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_{m_j}) \leqslant \frac{\epsilon}{2}; m_k, m_p\} \in F(I)$ and $\delta(C) = 1$. Again, since $I^* - lim\ inf \frac{1}{n}|\{p_k \leqslant : k \in \mathbb{N}\}| = b > 0$(say). So, $\{n \in \mathbb{N} : \frac{1}{n}|\{p_k \leqslant n : k \in \mathbb{N}\}| > \frac{b}{2}\} = D$(say) $\in F(I)$. As C and D belongs to $F(I)$ so, $C \cap D \in F(I)$, i.e., consequently $\lim_{n\to\infty} \dfrac{1}{n}|\{m_k < n : \overline{d}(\mathcal{X}_{m_k}, \mathcal{X}_0) \leqslant \epsilon\} \in F(I)| = 1$. This shows that $\mathcal{X}$ is I^*-statistically convergent. □

5. Conclusions

Our study has thoroughly examined the concept of I^*-statistical convergence for sequences of fuzzy numbers within a metric space. Our investigation confirms the uniqueness of the limit under I^*-statistical convergence, establishing a firm foundation for understanding this advanced mathematical concept. Through the development of several key theorems, we have elucidated the relationship between I^*-statistical convergence and classical convergence, alongside the algebraic properties intrinsic to I^*-statistically convergent sequences.

Additionally, our work has introduced and analyzed I^*-statistically pre-Cauchy and I^*-statistically Cauchy sequences, highlighting their intricate connection to I^*-statistical convergence. Notably, we demonstrated that while every I^*-statistically convergent sequence is necessarily I^*-statistically pre-Cauchy, the reverse does not universally apply. To further enrich the theoretical framework, we provided a sufficient condition for an I^*-statistically pre-Cauchy sequence to achieve I^*-statistical convergence, utilizing the concept of I^*-lim inf. These findings contribute significantly to the broader understanding of convergence in the context of fuzzy number sequences and open avenues for future research in this area.

The future scope of this study includes examining the monotonicity and boundedness of sequences of fuzzy numbers within the framework of I^*-statistical convergence. Additionally, this concept can be extended to explore convergence in the context of double and triple sequences, broadening the applicability of I^*-statistical convergence. Further research could also investigate these convergence properties in various other mathematical spaces, potentially unveiling new theoretical insights and applications.

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Appendix A. Proof of Theorem 6.

Proof. From Theorem 4, we know that every I^*-statistically convergent sequence is I^*-statistically Cauchy. Additionally, Theorem 5 establishes that every I^*-statistically Cauchy sequence is I^*-statistically pre-Cauchy. Therefore, it follows that every I^*-statistically convergent sequence is also I^*-statistically pre-Cauchy. □

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