axioms

Article

Fuzzy Fixed Point Theorems in $\mathcal{S}$-Metric Spaces: Applications to Navigation and Control Systems

Maryam Iqbal [1], Afshan Batool [1], Aftab Hussain [2,*] and Hamed Alsulami [2]

[1] Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi 46000, Pakistan; 20-22041-002@mth.fjwu.edu.pk (M.I.); afshan.batool@fjwu.edu.pk (A.B.)
[2] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; hhaalsalmi@kau.edu.sa
* Correspondence: aniassuirathka@kau.edu.sa

Abstract: This manuscript examines fuzzy fixed point results using the concepts of $\mathcal{S}$-metric space. We introduce two contractive maps, γ- and γ-weak contractions, within the context of $\mathcal{S}$-metric spaces. These contractive maps form the cornerstone of our research, offering a novel approach to solving mathematical problems. We explore fixed point results derived from the application of these maps, showcasing their utility in finding solutions in diverse mathematical scenarios. Furthermore, we provide concrete examples that illustrate the practical relevance and versatility of our theorems, emphasizing their potential applications across a wide range of scientific and engineering domains. This manuscript presents the novel concepts of γ- and γ-weak contractions and establishes their importance in mathematical research. By demonstrating their effectiveness in solving real-world problems and offering illustrative examples, our work contributes valuable tools and insights to the broader scientific community, enhancing our understanding of contractive maps and their applications.

Keywords: $\mathcal{S}$-metric space; fixed point; γ-contraction

MSC: 47H09; 47H10

check for updates

Citation: Iqbal, M.; Batool, A.; Hussain, A.; Alsulami, H. Fuzzy Fixed Point Theorems in $\mathcal{S}$-Metric Spaces: Applications to Navigation and Control Systems. *Axioms* **2024**, *13*, 650. https://doi.org/10.3390/axioms13090650

Academic Editors: Sanda Micula and Monica-Felicia Bota

Received: 22 August 2024
Revised: 18 September 2024
Accepted: 19 September 2024
Published: 22 September 2024

1. Introduction

Poincaré, Lefschetz–Hopf, and Leray–Schauder made substantial contributions to the field of fixed point theory, which is referred to as FP theory, during the late 19th and early 20th centuries. Their pioneering work laid the foundation for what would become a rich and expansive domain of study. Over time, FP theory evolved into a diverse and intricate field, encompassing a wide range of concepts and methodologies.

This theory rapidly developed into a significant area of research that integrates various mathematical disciplines, including topology, discrete mathematics, and analysis. It also extends into algebra and geometry, reflecting its broad applicability and relevance. Additionally, FP theory has influenced computational and mathematical methods, showcasing its impact on practical and theoretical aspects of mathematics. The continued exploration and expansion of FP theory underscore its importance and versatility in addressing complex mathematical problems and advancing our understanding of fundamental principles.

The field of metric theory, which has its roots in the foundational work of prominent mathematicians such as Cauchy, Liouville, Lipschitz, Peano, Fredholm, and, notably, Émile Picard, is centered around results and techniques that emphasize isometric properties. These early scholars laid the groundwork for metric theory by developing key concepts related to distances and convergence within mathematical spaces.

Cauchy's contributions, for instance, established fundamental principles concerning the convergence of sequences and series, while Liouville's work extended these ideas into broader contexts. Lipschitz introduced important notions regarding the behavior of functions and their rates of change, and Peano's contributions furthered the understanding of function continuity and differentiability. Fredholm's work on integral equations and Émile Picard's significant advancements in the theory of differential equations added crucial depth to the study of metrics.

Metric theory explores various mathematical phenomena through the lens of isometric characteristics, which are crucial for understanding distances and transformations within different spaces. This theoretical framework underpins many modern mathematical methods and results, demonstrating its importance in both pure and applied mathematics. The rich historical development of metric theory highlights its evolution from these foundational contributions to its current applications in diverse areas of mathematics.

The primary objective of metric theory is to establish the conditions under which unique and existing solutions can be found for first-order nonlinear initial value problems. This pursuit is fundamental to ensuring that such problems, which are common in both theoretical and applied contexts, have well-defined solutions.

Stefan Banach, a distinguished Polish mathematician, played a pivotal role in this field by organizing the core concepts into a comprehensive abstract framework. His contributions significantly advanced the theory, allowing it to extend beyond the traditional scope of differential and integral equations. Banach's work provided a robust theoretical foundation that is applicable to a broad spectrum of mathematical problems, demonstrating its utility in various advanced contexts. This abstract framework has proven instrumental in solving complex mathematical issues, and it has greatly influenced numerous areas of research and application [1].

Metric FP theory has significantly expanded into a specialized field due to its broad range of real-world applications. Researchers have developed and studied various classes of mappings within generalized metric spaces, which are referred to as MSs and have led to the emergence of numerous advanced theoretical concepts. Among these, (ψ, φ)-weakly contractive mappings in ordered G-MSs, weakly α-admissible pairs of F-contractions, and two families of multivalued dominated-contractive maps defined on closed balls within complete multiplicative MSs are particularly noteworthy.

These advancements have proven instrumental in addressing a variety of practical problems. For example, the theoretical results have been effectively applied to solve complex boundary value problems, tackle systems of nonlinear Volterra-type integral equations, and determine common solutions for systems of elliptic boundary value problems [2–9]. The relevance of these findings extends across multiple disciplines, including image processing, engineering, physics, computer science, economics, and telecommunications.

Recent advancements in FP theory have significantly broadened our understanding of MSs and their applications. A notable area of progress involves the generalization of quasi-contractions within b-MSs. This generalization extends traditional FP theorems to accommodate a wider variety of MSs, enhancing the applicability of these theorems to complex scenarios across diverse mathematical and practical contexts. In parallel, the exploration of bipolar b-MSs, particularly within graph settings, has introduced a novel dimension to FP analysis. This approach integrates graph theory with b-MSs, allowing for a nuanced examination of FPs where the metric structure is influenced by the underlying graph. Such studies, which consider both positive and negative weights in the graph, contribute to solving problems in networked and relational contexts, offering new insights into the behavior of FPs in these generalized settings. Both lines of research generalizing quasi-contractions in b-MSs and exploring bipolar b-MSs in graph settings represent significant strides in extending and applying FP theory to a broader range of scenarios and challenges [10].

Around the middle of the 20th century, FPs for set-valued operators were of particular interest because of the well-known extensions of Brown [11]. Later, Nadler [12] added multivalued contractions to the Banach contraction principle which is referred to as BCP. Many authors introduced different contractions such as φ-contractions in complete MSs. They applied their FP results for multivalued maps, including novel existence techniques on the generalized φ-Caputo fractional inclusion boundary problem, differential inclusions, convex and non-smooth optimization, control theory, and economics [13–17].

In recent years, FP theory has been increasingly applied to a variety of engineering problems, demonstrating its versatility and practical utility. Santos et al., (2019) provided a comprehensive review of FP theory, emphasizing its use in addressing complex challenges in control systems, optimization, and signal processing. Zhang et al., (2020) built on this by applying FP theorems to nonlinear control systems, highlighting improvements in stability and performance. Singh et al., (2021) focused on networked systems, illustrating how FP theory aids in solving issues related to network stability and resource allocation. Li et al., (2022) explored the use of FP theorems in image processing, proposing new algorithms that significantly enhance image reconstruction and enhancement. Kumar et al., (2023) discussed the application of FP theory to optimization problems in smart grids, demonstrating how these methods can optimize energy distribution and integrate renewable energy sources. Each of these studies showcases the evolving and impactful use of FP theory across diverse engineering disciplines [18–22].

The concepts of contraction and weak contraction were introduced into the realm of fuzzy logic in (2019) by Sezen [23]. This innovative approach marked a significant expansion of fuzzy logic applications. More recently, Tahair et al. have made notable advancements by developing new FP theorems within complete fuzzy MSs and complete b-multiplicative MSs. Their work is particularly focused on symmetric coupled dominating fuzzy mappings and is supported by a novel contraction criterion applied within a closed-ball framework. These theoretical contributions have proven useful in addressing Fredholm-type integral equations, demonstrating the practical value of their research [9,24–26].

In (2024), several significant advancements in FP theory and its applications were made. Lateef et al., extended traditional FP theory to L-fuzzy sets within F-MSs, offering more nuanced descriptions of distance and convergence under uncertainty. Rasham et al. introduced two families of multivalued dominated mappings and examined their properties under generalized nonlinear contractive inequalities, broadening the scope of FP results to include mappings with multiple outputs. Dwivedi et al., focused on fractional fuzzy differential equations, integrating fractional calculus with fuzzy logic to model systems with uncertainty and imprecision. Meanwhile, Younis et al., introduced the novel "Kannan-graph-fuzzy contraction" concept by combining graph mappings, Kannan mappings, and fuzzy contractions. This innovative framework aims to address nonlinear models in real-world scenarios, demonstrating the benefits of integrating diverse theoretical perspectives to solve complex engineering and technological problems [27–30].

Many mathematical problems require the determination of distances between objects, a task that can often be challenging to measure with precision. In response to this, Sedghi, Shobe, and Aliouche [31,32] introduced the concept of an $\mathcal{S}$-MS. The $\mathcal{S}$-metric is a relatively novel concept that extends the traditional notion of MSs, providing a more generalized framework for measuring distances.

Building on this foundation, Dung, Hieu, and Radojevic [33] developed several FP theorems applicable to partially ordered $\mathcal{S}$-MSs. Their work demonstrated the versatility of the $\mathcal{S}$-MS in the context of FP theory, showcasing its potential in addressing complex problems.

Further advancements were made by Gupta and Deep, who explored FP theorems in $\mathcal{S}$-MSs using a combination of mixed weakly monotone characteristics and shifting distance functions [34]. Their research provided valuable insights and extended the applications of $\mathcal{S}$-MSs, paving the way for further theoretical developments.

Inspired by these contributions, our work builds on the ideas introduced by Sezen [23], integrating and expanding upon these concepts to explore new dimensions in FP theory within $\mathcal{S}$-MSs. This progression underscores the ongoing evolution and application of $\mathcal{S}$-MSs in mathematical research and problem-solving.

Fuzzy MSs and $\mathcal{S}$-MSs represent an advanced integration of fuzzy set theory with traditional MSs. This integration allows for a more flexible representation of distance and accommodates the inherent imprecision found in mathematical structures. Building on this foundation, our paper introduces two novel types of contractions: γ-contractions and γ-weak contractions, within the framework of $\mathcal{S}$-MSs. These concepts were previously explored in the context of fuzzy MSs by Sezen [23].

In this work, we extend and generalize several FP theorems to $\mathcal{S}$-MSs using these new contraction types. Our generalizations are supported by practical examples that illustrate the applicability of these theorems.

To demonstrate the utility of our results, we explore their applications in real-world scenarios, specifically in navigation systems and equilibrium points within control systems. These examples highlight how our theoretical advancements can be applied to solve practical problems, bridging the gap between abstract mathematical concepts and tangible engineering challenges.

2. Preliminaries

This section provides a comprehensive overview of the foundational concepts and essential background required for establishing the key theorems presented in the article. It covers all preliminary information necessary for understanding the theoretical framework and methodologies utilized in the proofs. By detailing the underlying principles and definitions, this section ensures that readers are well equipped to follow the logical progression of the arguments and appreciate the significance of the results. This preparatory material is crucial for building a solid base from which the main theoretical contributions of the paper can be rigorously demonstrated and validated.

Definition 1 ([35])**.** *Let Ω be a nonempty set. For all $\dot{f}, \dot{k}, \dot{l}, \dot{a}$ in Ω, a function, $\mathcal{S} : \Omega^3 \to [0, \infty)$, is said to be an $\mathcal{S}$-metric if the following properties hold:*

(1) $\mathcal{S}(\dot{f}, \dot{k}, \dot{l}) = 0 \iff \dot{f} = \dot{k} = \dot{l}$,

(2) $\mathcal{S}(\dot{f}, \dot{k}, \dot{l}) \leq \mathcal{S}(\dot{f}, \dot{f}, \dot{a}) + \mathcal{S}(\dot{k}, \dot{k}, \dot{a}) + \mathcal{S}(\dot{l}, \dot{l}, \dot{a})$.

The pair $(\Omega, \mathcal{S})$ is known as $\mathcal{S}$-MS.

Example 1 ([35])**.** *Let Ω be a nonempty set, and let $\check{d}_1$ and $\check{d}_2$ represent two non-ordinary metrics on Ω. An $\mathcal{S}$-metric on Ω is thus defined as $\mathcal{S}(\dot{f}, \dot{k}, \dot{l}) = \check{d}_1(\dot{f}, \dot{l}) + \check{d}_2(\dot{k}, \dot{l})$.*

Example 2 ([34])**.** *Let Ω be a nonempty set, and let $\check{d}_1$ and $\check{d}_2$ be two ordinary metrics on Ω. Then,*

$$\mathcal{S}(\dot{f}, \dot{k}, \dot{l}) = \check{d}_1(\dot{f}, \dot{l}) + \check{d}_2(\dot{k}, \dot{l}),$$

is an $\mathcal{S}$-metric on Ω.

Definition 2 ([35])**.** *Let $(\Omega, \mathcal{S})$ be an $\mathcal{S}$-MS.*

(a) ***$\mathcal{S}$-Convergent****: A sequence, $\{\dot{f}_r\}$, in $\mathcal{S}$ is called the $\mathcal{S}$-convergent to $\dot{f} \in \mathcal{S}$ if $\mathcal{S}(\dot{f}_r, \dot{f}_r, \dot{f}) \to 0$ as $r \to \infty$. That is, for all $\epsilon \geq 0$, there exists $r_0 \in N$ such that, for all $r \geq r_0$, we have $\mathcal{S}(\dot{f}_r, \dot{f}_r, \dot{f}) < \epsilon$. We write $\dot{f}_r \to \dot{f}$.*

(b) ***$\mathcal{S}$-Cauchy****: A sequence, $\{\dot{f}_r\}$ in $\mathcal{S}$, is called an $\mathcal{S}$-cauchy sequence if $\mathcal{S}(\dot{f}_r, \dot{f}_r, \dot{f}_m) \to 0$ as $r, m \to \infty$. That is, for all $\epsilon > 0$, there exists $r_0 \in N$ such that, for all $r, m \geq r_0$, we have $\mathcal{S}(\dot{f}_r, \dot{f}_r, \dot{f}_m) < \epsilon$.*

(c) ***Complete $\mathcal{S}$-MS****: The $\mathcal{S}$-MS $(\Omega, \mathcal{S})$ is complete if every $\mathcal{S}$-Cauchy sequence is $\mathcal{S}$-convergent.*

Lemma 1 ([33])**.** *In every $\mathcal{S}$-MS $(\Omega, \mathcal{S})$, we have*

$$\mathcal{S}(\dot{f}, \dot{f}, \dot{k}) = \mathcal{S}(\dot{k}, \dot{k}, \dot{f}),$$

for all $\dot{f}, \dot{k} \in \Omega$.

An MS is a fundamental concept in mathematics used to define and analyze distances between points. Traditionally, this is represented by a distance function that assigns a non-negative real number to every pair of points, which quantifies their separation in a precise manner. However, in many real-world scenarios, distances are not always exact or easily quantifiable due to inherent imprecision or uncertainty. To address these limitations, the concept of an $\mathcal{S}$-MS was introduced, which expands upon the traditional notion of MSs by incorporating elements of fuzzy set theory.

An $\mathcal{S}$-MS integrates the principles of fuzzy set theory into the framework of MSs. This approach allows for a more nuanced and flexible description of distances between points. In a conventional MS, the distance function provides a clear, binary measure of how far apart two points are. In contrast, an $\mathcal{S}$-MS employs a fuzzy set-valued function in place of the traditional distance function. This function assigns a degree of membership to each pair of points, reflecting how "close" or "far" they are in a gradual and less rigid manner. The core idea behind $\mathcal{S}$-MSs is to capture the imprecision or uncertainty associated with distance measurements. By using fuzzy sets, an $\mathcal{S}$-MS allows for a range of values that represent varying degrees of proximity between points. This is particularly useful in situations where precise distances cannot be defined due to factors such as measurement error, subjective interpretation, or the inherent vagueness of the concepts being modeled.

For example, consider a scenario where distances between objects are difficult to measure accurately due to limitations in measurement tools or environmental conditions. In such cases, a traditional MS may not adequately represent the nuances of these distances. An $\mathcal{S}$-MS, however, can model these distances more effectively by allowing for a spectrum of closeness, thereby providing a more flexible and realistic representation.

The relationship between $\mathcal{S}$-MSs and fuzzy MSs is particularly significant. Fuzzy MSs were among the first to introduce the idea of using fuzzy sets to describe distances. They utilize a membership function that indicates the degree to which a pair of points is "close". An $\mathcal{S}$-MS builds upon this concept by formalizing and generalizing it further. Specifically, an $\mathcal{S}$-MS can be seen as an extension of fuzzy MSs, offering a more refined framework that incorporates various aspects of fuzzy set theory into the MS structure.

The introduction of $\mathcal{S}$-MSs represents a significant advancement in the mathematical treatment of distances, offering a more flexible and nuanced approach than traditional MSs. By leveraging the principles of fuzzy set theory, $\mathcal{S}$-MSs provide a means to model and analyze scenarios where distances are not precisely defined, thereby enhancing the ability to address complex problems involving uncertainty and imprecision.

Sezen made significant contributions to the field of fuzzy MSs by introducing the concepts of γ-contraction and γ-weak contraction. These notions represent advanced types of contractive mappings designed to address the complexities of distance measurement within the framework of fuzzy logic. In her work [23], Sezen explored how these contractions can be applied to fuzzy MSs, providing a generalized approach that extends beyond traditional MS theory. The γ-contraction and γ-weak contraction concepts are particularly valuable because they allow for a more flexible treatment of proximity and distance, accommodating the inherent uncertainties in fuzzy metrics. By defining these new types of contractions, Sezen laid the groundwork for further research and application in areas where precise distance measurement is challenging, thus enriching the theoretical landscape of fuzzy MSs and broadening the scope of FP theory.

Definition 3 ([36])**.** *A binary operation, $* : [0,1] \times [0,1] \to [0,1]$, is called a continuous triangular norm (in short, a continuous t-norm) if it satisfies the following conditions:*

(1) *$*$ is commutative and associative;*
(2) *$*$ is continuous;*
(3) *$*(\dot{f}, 1) = \dot{f} \; \forall \; \dot{f} \in [0,1]$;*
(4) *$*(\dot{f}, \dot{k}) \leq *(\dot{l}, \dot{m})$ whenever $\dot{f} \leq \dot{l}, \dot{k} \leq \dot{m}$ and $\dot{f}, \dot{k}, \dot{l}, \dot{m} \in [0,1]$.*

Definition 4 ([37]). *A fuzzy MS is an ordered triple $(\Omega, \breve{M}, *)$ such that Ω is a nonempty set, $*$ is a continuous t-norm, and $\breve{M}$ is a fuzzy set of $\Omega^2 \times (0, \infty)$, satisfying the following conditions, for all $\dot{f}, \dot{k}, \dot{l} \in \Omega, s, t > 0$:*

(1) $\breve{M}(\dot{f}, \dot{k}, t) > 0$;
(2) $\breve{M}(\dot{f}, \dot{k}, t) = 1 \iff \dot{f} = \dot{k}$;
(3) $\breve{M}(\dot{f}, \dot{k}, t) = \breve{M}(\dot{k}, \dot{f}, t)$;
(4) $\breve{M}(\dot{f}, \dot{l}, t + s) \geq \breve{M}(\dot{f}, \dot{k}, t) * \breve{M}(\dot{k}, \dot{l}, s)$;
(5) $\breve{M}(\dot{f}, \dot{k}, .) : (0, \infty) \to [0, 1]$ *is continuous.*

If, in the above definition, the triangular inequality (4) is replaced with $\breve{M}(\dot{f}, \dot{l}, \max\{t, s\}) \geq \breve{M}(\dot{f}, \dot{k}, t) * \breve{M}(\dot{k}, \dot{l}, s) \ \forall\ \dot{f}, \dot{k}, \dot{l} \in \Omega, s, t > 0$, *or equivalently,* $\breve{M}(\dot{f}, \dot{l}, t) \geq \breve{M}(\dot{f}, \dot{k}, t) * \breve{M}(\dot{k}, \dot{l}, t)$, *then the* $(\Omega, \breve{M}, *)$ *is called a non-Archimedean fuzzy MS.*

Definition 5 ([23]). *Let $\gamma : [0, 1] \to \mathbb{R}$ be a strictly increasing continuous function. For each sequence, $\{\dot{x}_r\}$, of positive numbers and $\lim_{r\to\infty} \dot{x}_r = 1 \iff \lim_{r\to\infty} \gamma(\dot{x}_r) = +\infty$, suppose that Γ is the collection of all γ-functions.*

A mapping, $\psi : \Omega \to \Omega$, is called γ-contraction if there exists a $\delta \in (0, 1)$ such that

$$\breve{M}(\psi_{\dot{f}}, \psi_{\dot{k}}, t) < 1 \implies \gamma(\breve{M}(\psi_{\dot{f}}, \psi_{\dot{k}}, t)) \geq \gamma(\breve{M}(\dot{f}, \dot{k}, t)) + \delta, \tag{1}$$

for all $\dot{f}, \dot{k} \in \Omega$ and $\gamma \in \Gamma$, where $\breve{M}$ is a fuzzy set of non-Archimedean fuzzy MS and t is the norm.

Definition 6 ([23]). *A mapping, $\psi : \Omega \to \Omega$, is called a γ-weak contraction if there is $\delta \in (0, 1)$ such that*

$$\breve{M}(\psi_{\dot{f}}, \psi_{\dot{k}}, t) < 1 \implies \gamma(\breve{M}(\psi_{\dot{f}}, \psi_{\dot{k}}, t) \geq \gamma(\min\{\breve{M}(\dot{f}, \dot{k}, t), \breve{M}(\dot{f}, \psi_{\dot{f}}, t), \breve{M}(\dot{k}, \psi_{\dot{k}}, t)\}) + \delta,$$

for all $\dot{f}, \dot{k} \in \Omega$ and $\gamma \in \Gamma$, where $\breve{M}$ is the fuzzy set of a non-Archimedean fuzzy MS, and t is the norm.

3. Main Results

In this section, we introduce the concept of γ-contractions within the framework of $\mathcal{S}$-MSs. γ-contractions are a specific type of contractive mapping that extends the traditional notion of contraction mappings to the more flexible setting of $\mathcal{S}$-MSs. This extension allows for a broader application of FP theorems in scenarios where distance measurements are represented as fuzzy sets, rather than exact values. γ-contractions are designed to accommodate the imprecision inherent to fuzzy MSs, providing a useful tool for analyzing and solving FP problems under these conditions. To validate the theoretical findings associated with γ-contractions, we present several illustrative examples. These examples are carefully constructed to showcase the practical effectiveness of γ-contractions in demonstrating FP results. By applying these contractive mappings to specific problems, we highlight how they can be utilized to achieve reliable and meaningful solutions. The examples serve to illustrate the robustness of γ-contractions and their capacity to address real-world problems within the context of $\mathcal{S}$-MSs, reinforcing the theoretical framework and confirming the applicability of the proposed results.

Definition 7. *Suppose that $(\Omega, \mathcal{S})$ is $\mathcal{S}$-MS, and let $\gamma : [0, 1] \to \mathbb{R}$ be a strictly increasing continuous function. For each sequence, $\{\dot{x}_r\}$, of positive numbers and $\lim_{r\to\infty} \dot{x}_r = 0 \iff \lim_{r\to\infty} \gamma(\dot{x}_r) = +\infty$, suppose that Γ is the collection of all γ-functions.*

A mapping, $\psi : \Omega \to \Omega$, is called γ-contraction if there exists a $\delta \in (0, 1)$ such that

$$\mathcal{S}(\psi_{\dot{f}}, \psi_{\dot{k}}, \dot{a}) < 1 \implies \gamma(\mathcal{S}(\psi_{\dot{f}}, \psi_{\dot{k}}, \dot{a})) \geq \gamma(\mathcal{S}(\dot{f}, \dot{k}, \dot{a})) + \delta, \tag{2}$$

for all $\dot{f}, \dot{k}, \dot{a} \in \Omega$ and $\gamma \in \Gamma$.

Theorem 1. *Let $\psi : \Omega \to \Omega$ be a γ-contraction, and let $(\Omega, \mathcal{S})$ be a complete $\mathcal{S}$-MS. Then, ψ has a unique FP in Ω.*

The proof of the theorem relies fundamentally on the BCP, a cornerstone result in MS theory. To establish the theorem, we start by considering a Cauchy sequence within a complete $\mathcal{S}$-MS. By definition, a Cauchy sequence is one in which the elements become arbitrarily close to each other as the sequence progresses. In the context of a complete MS, every Cauchy sequence converges to a limit within that space, ensuring that the space is sufficiently robust to handle the convergence of such sequences. In our approach, we demonstrate that this convergent point of the Cauchy sequence serves as the FP of the given contraction mapping. By applying the BCP, which guarantees the existence of a unique FP for a contraction mapping in a complete MS, we show that the limit of the Cauchy sequence, which is guaranteed to exist, indeed satisfies the FP condition. This rigorous process confirms that the convergent point of the sequence is an FP of the contraction, thereby validating the theorem within the framework of $\mathcal{S}$-MSs.

Proof. Suppose that $\dot{f}_0 \in \Omega$ is a fixed arbitrary point. Define a sequence, $\{\dot{f}_r\}$, as

$$\psi \dot{f}_r = \dot{f}_{r+1} \; \forall \; r \in N,$$

If $\dot{f}_r = \dot{f}_{r+1}$, then $\dot{f}_{r+1}$ is an FP of ψ, and the proof is, therefore, complete.

Suppose that $\dot{f}_r \neq \dot{f}_{r+1}$ for all $r \in N$. So, according to (2), we have

$$\gamma(\mathcal{S}(\psi_{\dot{f}_{r-1}}, \psi_{\dot{f}_r}, \dot{a})) \geq \gamma(\mathcal{S}(\dot{f}_{r-1}, \dot{f}_r, \dot{a})) + \delta,$$

Reiterating this, we get

$$\begin{aligned} \gamma(\mathcal{S}(\psi_{\dot{f}_{r-1}}, \psi_{\dot{f}_r}, \dot{a})) &\geq \gamma(\mathcal{S}(\dot{f}_{r-1}, \dot{f}_r, \dot{a})) + \delta \\ &= \gamma(\mathcal{S}(\psi_{\dot{f}_{r-2}}, \psi_{\dot{f}_{r-1}}, \dot{a})) + \delta \\ &\geq \gamma(\mathcal{S}(\dot{f}_{r-2}, \dot{f}_{r-1}, \dot{a})) + 2\delta ... \\ &\geq \gamma(\mathcal{S}(\dot{f}_0, \dot{f}_1, \dot{a})) + r\delta, \end{aligned} \tag{3}$$

Suppose that $r \to \infty$ in (3); we have

$$\lim_{r \to \infty} \gamma(\mathcal{S}(\psi_{\dot{f}_{r-1}}, \psi_{\dot{f}_r}, \dot{a})) = +\infty.$$

Then, we get

$$\mathcal{S}(\psi_{\dot{f}_{r-1}}, \psi_{\dot{f}_r}, \dot{a}) = 0. \tag{4}$$

We must now prove through contradiction that $\{\dot{f}_r\}$ is a Cauchy sequence. To do this, we suppose that a sequence, $\{\dot{f}_r\}$, is not a Cauchy sequence. Then, there exist $\epsilon \in (0, 1)$ and $\dot{a}_0 \geq 0$ such that, for all $s \in N$, there exist $m(s), r(s) \in N$ with $r(s) > m(s) > s$ and

$$\mathcal{S}(\dot{f}_{r(s)}, \dot{f}_{m(s)}, \dot{a}_0) \geq \epsilon, \tag{5}$$

Suppose that $m(s)$ is lowest integer exceeding $r(s)$, satisfying Equation (5). Then, we have

$$\mathcal{S}(\dot{f}_{m(s)-1}, \dot{f}_{r(s)}, a_0) > \epsilon,$$

and so, for all $s \in N$, we have

$$\begin{aligned} (\epsilon) &\geq \mathcal{S}(\dot{f}_{r(s)}, \dot{f}_{m(s)}, \dot{a}_0) \\ &\geq \mathcal{S}(\dot{f}_{m(s)-1}, \dot{f}_{m(s)}, \dot{a}_0)\mathcal{S}(\dot{f}_{m(s)-1}, \dot{f}_{r(s)}, \dot{a}_0) \\ &\geq \mathcal{S}(\dot{f}_{m(s)-1}, \dot{f}_{r(s)}, \dot{a}_0)(\epsilon). \end{aligned} \tag{6}$$

By applying $s \to \infty$ (6) and using (4), we get

$$\lim_{s\to\infty} \mathcal{S}(\dot{f}_{r(s)}, \dot{f}_{m(s)}, \dot{a}_0) = \epsilon. \tag{7}$$

By applying (2) with $\dot{f} = \dot{f}_{m(s)}$ and $\dot{k} = \dot{f}_{r(s)}$, we have

$$\gamma(\mathcal{S}(\dot{f}_{r(s)}, \dot{f}_{m(s)}, \dot{a})) \geq \gamma(\mathcal{S}(\dot{f}_{r(s)}, \dot{f}_{m(s)}, \dot{a})) + \delta. \tag{8}$$

Taking the limit as $\lim_{s\to\infty}$ in Equation (8), applying Equation (2) to Equation (3), and applying the continuity of γ, we get

$$\gamma(\epsilon) \geq \gamma(\epsilon) + \delta,$$

This is a contradiction. So, $\{\dot{f}_r\}$ is a Cauchy sequence in Ω. As Ω is taken to be a complete S-MS, by using the completeness of Ω, there exists $\dot{z} \in \Omega$, such that

$$\lim_{r\to\infty} \dot{f}_r = \dot{z}.$$

Using the continuity of ψ yields $\mathcal{S}(\psi_{\dot{z}}, \dot{z}, \dot{a}) = \lim_{r\to\infty} \mathcal{S}(\psi_{\dot{f}_r}, \dot{f}_r, \dot{a}) = \lim_{r\to\infty} \mathcal{S}(\dot{f}_{r+1}, \dot{f}_r, \dot{a}) = 0$. Now, we have to show the uniqueness of FP of ψ. We show it through contradiction. Let $\dot{z}_1$ and $\dot{z}_2$ be two different FPs of ψ. This means that $\dot{z}_1, \dot{z}_2 \in \Omega, \psi_{\dot{z}_1} = \dot{z}_1 \neq \dot{z}_2 = \psi_{\dot{z}_2}$, so we get

$$\gamma(\mathcal{S}(\dot{z}_1, \dot{z}_2, \dot{a})) \geq \gamma(\mathcal{S}(\dot{z}_1, \dot{z}_2, \dot{a})) + \delta,$$

which is a contradiction. Therefore, ψ has a unique FP. Hence, the proof is completed. □

Example 3. *Consider the space $(\Omega, \mathcal{S})$, where $\Omega = [0,1]$ with $\mathcal{S}$ being the standard metric, and $\mathcal{S}(\dot{f}, \dot{k}) = |\dot{f} - \dot{k}|$ for $\dot{f}, \dot{k} \in [0,1]$.*

Define the function $\gamma : [0,1] \to \mathbb{R}$ as

$$\gamma(\dot{f}) = \begin{cases} \frac{1}{\dot{f}} & \text{for } \dot{f} > 0, \\ +\infty & \text{for } \dot{f} = 0. \end{cases}$$

Note that γ is strictly increasing and continuous on $(0,1]$ and $\gamma(\dot{f}) \to +\infty$ as $\dot{f} \to 0^+$.

Let $\psi : [0,1] \to [0,1]$ be defined by

$$\psi(\dot{f}) = \frac{\dot{f}}{2}.$$

To check whether ψ is a γ-contraction, we need to verify the condition from the definition:

For $\mathcal{S}(\psi(\dot{f}), \psi(\dot{k}), \dot{a}) < 1$, we have

$$\mathcal{S}(\psi(\dot{f}), \psi(\dot{k}), \dot{a}) = \left|\frac{\dot{f}}{2} - \frac{\dot{k}}{2}\right| = \frac{1}{2}|\dot{f} - \dot{k}| = \frac{1}{2}\mathcal{S}(\dot{f}, \dot{k}, \dot{a}).$$

Thus,

$$\gamma(\mathcal{S}(\psi(\dot{f}), \psi(\dot{k}), \dot{a})) = \gamma\left(\frac{1}{2}\mathcal{S}(\dot{f}, \dot{k}, \dot{a})\right) = \frac{2}{\mathcal{S}(\dot{f}, \dot{k}, \dot{a})}.$$

Also,

$$\gamma(\mathcal{S}(\dot{f}, \dot{k}, \dot{a})) = \frac{1}{\mathcal{S}(\dot{f}, \dot{k}, \dot{a})}.$$

We need

$$\gamma(\mathcal{S}(\psi(\dot{f}), \psi(\dot{k}), \dot{a})) \geq \gamma(\mathcal{S}(\dot{f}, \dot{k}, \dot{a})) + \delta.$$

Substituting, we get

$$\frac{2}{\mathcal{S}(\dot{f},\dot{k},\dot{a})} \geq \frac{1}{\mathcal{S}(\dot{f},\dot{k},\dot{a})} + \delta.$$

The solution is

$$\frac{2-1}{\mathcal{S}(\dot{f},\dot{k},\dot{a})} \geq \delta,$$

$$\frac{1}{\mathcal{S}(\dot{f},\dot{k},\dot{a})} \geq \delta,$$

$$\mathcal{S}(\dot{f},\dot{k},\dot{a}) \leq \frac{1}{\delta}.$$

Since $\delta \in (0,1)$, $\frac{1}{\delta} > 1$, *the condition is satisfied if* $\mathcal{S}(\dot{f},\dot{k},\dot{a}) < 2$.
The unique FP of ψ *is found by solving*

$$\psi(\dot{f}) = \dot{f} \implies \frac{\dot{f}}{2} = \dot{f} \implies \dot{f} = 0.$$

Thus, ψ *has a unique FP, which is* 0.

Application of Theorem 1—navigation in robotics:

Setting the context:

In the context of navigation robotics, Theorem 1 addresses the crucial issue of ensuring the existence and uniqueness of an FP for a contraction mapping within a complete MS. This theorem provides a theoretical foundation for algorithms used in robotics navigation, particularly in ensuring that such algorithms converge to a precise solution. In practical terms, Theorem 1 can be applied to design navigation systems that rely on iterative methods in order to determine optimal paths or locations. By utilizing a contraction mapping that satisfies the conditions of the theorem, we can guarantee that the iterative process will converge to a unique FP, which represents the desired navigation outcome. This application is essential for developing reliable and accurate navigation systems, as it ensures that the algorithm will consistently yield a well-defined and reproducible result, thereby enhancing the robustness and effectiveness of the robotic navigation strategies employed.

MS $(\Omega, \mathcal{S})$: In robotics and navigation systems, Ω could represent a set of states or configurations that the robot can occupy. The metric $\mathcal{S}$ could represent a distance or similarity measure between these states, which is typically defined to quantify the "closeness" of two states.

γ**-Contraction(ψ):** A mapping, $\psi : \Omega \to \Omega$, is said to be a γ-contraction if there exists a constant $0 \leq \gamma < 1$ such that, for all $\dot{f}, \dot{k} \in \Omega$,

$$S(\psi(\dot{f}), \psi(\dot{k})) \leq \gamma.\mathcal{S}(\dot{f},\dot{k}).$$

This condition ensures that ψ compresses the distances between points in Ω, indicating a tendency to bring points closer together.

Complete MS: The space $(\Omega, \mathcal{S})$ is complete if every Cauchy sequence in Ω converges to a point in Ω. This completeness ensures that the space has no "gaps" or missing points where sequences might fail to converge.

Existence of FP: Theorem 1 guarantees that there exists at least one point, $\dot{z} \in \Omega$, such that $\psi(\dot{z}) = \dot{z}$. In navigation robotics, this could correspond to a stable state or configuration where the robot's position or state remains unchanged under some mapping, ψ. For instance, it might represent a stable configuration where the robot has reached its target position or has stabilized its orientation.

Uniqueness of FP: Moreover, Theorem 1 asserts that this FP $\dot{z}$ is unique within the space Ω. This uniqueness property is crucial in navigation because it ensures that there is only one stable state or configuration that the robot will converge to under the mapping

ψ. This can be interpreted as ensuring deterministic behavior or convergence to a single desired state in robotic navigation tasks.

Practical implementation:

Navigation control: Understanding and leveraging Theorem 1 can help in designing navigation algorithms that guarantee convergence to a desired state or configuration. For example, in path planning, ψ can represent the robot's iterative adjustment towards a target position or orientation, and Theorem 1 assures that such adjustments will eventually lead to a unique stable state.

Error correction: In practical robotic systems, noise, disturbances, or measurement errors can affect navigation. The contraction mapping property can be used to design controllers or algorithms that are robust against such uncertainties, ensuring eventual convergence to a correct state despite initial errors. In summary, the application of Theorem 1 in the field of navigation robotics plays a crucial role in ensuring the robustness, stability, and convergence of robotic systems towards their desired states or configurations. This theorem provides a solid theoretical foundation for developing and implementing algorithms that guide autonomous robots in complex environments. Guaranteeing that the contraction mappings used in these algorithms will converge to a unique FP Theorem 1 ensures that the navigation process is reliable and consistent. This theoretical assurance is essential for creating practical navigation solutions that can effectively handle the uncertainties and challenges inherent to real-world scenarios.

Moreover, Theorem 1 contributes significantly to the design and implementation of algorithms by offering a framework that can be systematically applied to various navigation tasks. The robustness ensured via the theorem means that robots can maintain stability and achieve accurate navigation results even in the presence of disturbances or changing conditions. This is particularly important for autonomous systems operating in dynamic or unknown environments, where precision and adaptability are critical. The theoretical guarantees provided via Theorem 1 thus support the development of advanced navigation systems that are both dependable and efficient, reinforcing their practical utility and effectiveness in diverse robotic applications [1,38].

Definition 8. *Suppose that $(\Omega, \mathcal{S})$ is an $\mathcal{S}$-MS. A mapping $\psi : \Omega \to \Omega$ is called a γ-weak contraction if there is $\delta \in (0,1)$, such that*

$$\mathcal{S}(\psi_{\dot{f}}, \psi_{\dot{k}}, \dot{a})) < 1 \implies \gamma(\mathcal{S}(\psi_{\dot{f}}, \psi_{\dot{k}}, \dot{a}) \geq \gamma(\min\{\mathcal{S}(\dot{f}, \dot{k}, \dot{a}), \mathcal{S}(\dot{f}, \psi_{\dot{f}}, \dot{a}), \mathcal{S}(\dot{k}, \psi_{\dot{k}}, \dot{a})\}) + \delta, \tag{9}$$

for all $\dot{f}, \dot{k} \in \Omega$.

FP result of γ-weak contraction defined on $\mathcal{S}$-MS is constructed in Theorem 2. The BCP is the foundation of this theorem.

Theorem 2. *Let $(\Omega, \mathcal{S})$ be a complete $\mathcal{S}$-MS, and let $\psi : \Omega \to \Omega$ be a γ-weak contraction. Then, ψ has a unique FP.*

Proof. The proof of the theorem is based on BCP. A Cauchy sequence is taken from a complete S-MS. Every Cauchy sequence is convergent in a complete MS. The converging point of the sequence is proved to be the FP of contraction.

Assume that $\dot{f}_0 \in \Omega$ is random and fixed. Define a sequence, $\{\dot{f}_r\}$, as

$$\psi \dot{f}_r = \dot{f}_{r+1}, \forall\, r \in N,$$

if $\dot{f}_r = \dot{f}_{r+1}$ (in this case, $\dot{f}_{r+1}$ is an FP of ψ), then the proof is done. Let $\dot{f}_r \neq \dot{f}_{r+1} \forall\, r \in N$. So, according to inequality (9), we have

$$\begin{aligned}\gamma(\mathcal{S}(\psi\dot{f}_{r-1},\psi\dot{f}_r,\dot{a}) &\geq \gamma(\min\{\mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a})\mathcal{S}(\dot{f}_{r-1},\psi\dot{f}_{r-1},\dot{a}),S(f_r,\psi\dot{f}_r,\dot{a})\})+\delta \\ &= \gamma(\min\{\mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a}),\mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a})\mathcal{S}(\dot{f}_r,f_{r+1},\dot{a})\})+\delta \\ &= \gamma(\min\{S(\dot{f}_{r-1},\dot{f}_r,\dot{a}),\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a})\})+\delta, \end{aligned} \quad (10)$$

If $\exists\, r \in N$, i.e.,

$$\min\{\mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a}),\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a})\} = S(\dot{f}_r,\dot{f}_{r+1},\dot{a}).$$

So, now, Equation (10) becomes

$$\begin{aligned}\gamma(\mathcal{S}(\psi\dot{f}_{r-1},\psi\dot{f}_r,\dot{a}) &= \mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a}) \\ &\geq (\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a}))+\delta \\ &> (\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a})),\end{aligned}$$

which is a contradiction; therefore,

$$\min\{\mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a}),\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a})\} = S(\dot{f}_r,\dot{f}_{r+1},\dot{a}),$$

for all $r \in N$, and when γ is given to be strictly increasing, we have

$$\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a}) > \mathcal{S}(\dot{f}_{r-1},\dot{f}_r,\dot{a}),$$

So, we can write it as

$$\gamma(\dot{f}_r,\dot{f}_{r+1},\dot{a}) \geq \gamma(S(\dot{f}_{r-1},\dot{f}_r,\dot{a}))+\delta.$$

$\forall\, r \in N$, we have

$$\gamma(\dot{f}_r,\dot{f}_{r+1},\dot{a}) \geq \gamma(S(\dot{f}_{r-1},\dot{f}_r,\dot{a}))+r\delta, \quad (11)$$

Applying $\lim_{r\to\infty}$ in Equation (11) gives us

$$\lim_{r\to\infty}\gamma(\mathcal{S}(\dot{f}_r,\dot{f}_{r+1},\dot{a})) = \lim_{r\to}\gamma(\mathcal{S}(\psi\dot{f}_{r-1},\psi\dot{f}_r,\dot{a})) = +\infty.$$

Then, we have

$$\lim_{r\to\infty}\mathcal{S}(\psi\dot{f}_{r-1},\psi\dot{f}_r,\dot{a}) = 0.$$

The proof that $\{\dot{f}_r\}$ is a Cauchy sequence can be shown in Theorem 1. As Ω is given to be complete S-MS, by using the completeness of Ω, there exists $\dot{z} \in \Omega$, such that

$$\lim_{r\to\infty}\dot{f}_r = \dot{z},$$

Now, we have to show that $\dot{z}$ is an FP of ψ. As γ is given to be continuous, there are two cases:

Case 1: For every $r_0 \in N$, there exists $i_r \in N$, such that $\dot{f}_{i_n+1} = \psi_{\dot{z}}$, and $i_r > i_{r-1}$, where $i_o = 0$; then, we get

$$\dot{z} = \lim_{r\to\infty}\dot{f}_{i_n+1} = \lim_{r\to\infty}\psi_{\dot{z}} = \psi_{\dot{z}}, \quad (12)$$

Hence, $\dot{z}$, is an FP of ψ.

Case 2: There exists $r_o \in N$, such that $\dot{f}_{r+1} \neq \psi\dot{z} \,\forall\, r > r_o$, which is $S(\psi_{\dot{f}_r}, \psi\dot{z}, \dot{a} < 0 \,\forall\, r > r_o$. From (9), and by using the property of continuity of ψ, we have

$$\begin{aligned}\gamma(\mathcal{S}(\dot{f}_{r+1}, \psi\dot{z}, \dot{a}) &= \gamma(\min\{\mathcal{S}(\psi\dot{f}_r, \psi\dot{z}, \dot{a})\}) + \delta \\ &\geq \gamma(\min\{\mathcal{S}(\dot{f}_r, \dot{z}, \dot{a}), \mathcal{S}(\dot{f}_r, \psi\dot{f}_r, \dot{a})\mathcal{S}(\dot{f}_r, f_{r+1}, \dot{a})\}) + \delta \\ &= \gamma(\min\{S(\dot{f}_r, \dot{z}, \dot{a}), \mathcal{S}(\dot{f}_r, \dot{f}_{r+1}, \dot{a}), \mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a})\}) + \delta. \end{aligned} \quad (13)$$

If $\mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a}) < 0$, then we have

$$\lim_{r\to\infty} \mathcal{S}(\dot{f}_r, \dot{f}_{r+1}, \dot{a}) = 1.$$

There exists $r_1 \in N$, such that, $\forall r > r_1$, we have

$$\min\{\mathcal{S}(\dot{f}_r, \dot{z}, \dot{a}), \mathcal{S}(\dot{f}_r, \dot{f}_{r+1}, \dot{a})\mathcal{S}(\dot{f}_r, f_{r+1}, \dot{a})\} = \mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a}),$$

So, from (13), we have

$$\gamma(\mathcal{S}(\dot{f}_{r+1}, \psi\dot{z}, \dot{a})) \geq \gamma(\mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a})) + \delta, \quad (14)$$

$\forall\, r \geq \min\{r_o, r_1\}$, and since we know that γ is given to be continuous, by applying the limit $r \to \infty$ to (14), we have

$$\gamma(\mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a})) \geq \gamma(\mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a})) + \delta,$$

which is a contradiction. Therefore, $\mathcal{S}(\dot{z}, \psi\dot{z}, \dot{a})\} = 0$. So, $\dot{z}$ is an FP of ψ. Now, we have to prove the uniqueness of FPs. We prove this through contradiction. Let $\dot{z_1}, \dot{z_2}$ be two FPs of ψ. We suppose that $\dot{z_1} \neq \dot{z_2}$; then, we have $\psi\dot{z_1} \neq \psi\dot{z_2}$. From Equation (9), we have

$$\begin{aligned}\gamma(\mathcal{S}(\dot{z_1}, \dot{z_2}, \dot{a})) &= \gamma(\mathcal{S}(\psi\dot{z_1}, \psi\dot{z_2}, \dot{a})) \\ &\geq \gamma(\min\{\mathcal{S}(\dot{z_1}, \dot{z_2}, \dot{a}), \mathcal{S}(\dot{z_1}, \psi\dot{z_1}, \dot{a}), \mathcal{S}(\dot{z_2}, \psi\dot{z_2}, \dot{a})\} + \delta \\ &= \gamma(\min\{\mathcal{S}(\dot{z_1}, \dot{z_2}, \dot{a}), \mathcal{S}(\dot{z_1}, \dot{z_1}, \dot{a}), \mathcal{S}(\dot{z_2}, dotz_2, \dot{a})\} + \delta \\ &= \gamma(\mathcal{S}(\dot{z_1}, \dot{z_2}, \dot{a}) + \delta. \end{aligned}$$

which is a contradiction. That is, $\dot{z_1} = \dot{z_2}$.
□

Example 4. *Consider the $\mathcal{S}$-MS $(\Omega, \mathcal{S})$, where $\Omega = \mathbb{R}$, and the $\mathcal{S}$-metric $\mathcal{S}$ is defined as follows: For any $\dot{f}, \dot{k}, \dot{a} \in \mathbb{R}$,*

$$\mathcal{S}(\dot{f}, \dot{k}, \dot{a}) = |\dot{f} - \dot{k}| + |\dot{k} - \dot{a}| + |\dot{a} - \dot{f}|.$$

Define a mapping, $\psi : \mathbb{R} \to \mathbb{R}$, as follows:

$$\psi(\dot{f}) = \frac{\dot{f}}{2}.$$

Verification of γ-weak contraction:

Let γ be a function such that, for any $\dot{f}, \dot{k}, \dot{a} \in \mathbb{R}$,

$$\gamma(\mathcal{S}(\dot{f}, \dot{k}, \dot{a})) = \frac{1}{2}\mathcal{S}(\dot{f}, \dot{k}, \dot{a}).$$

We need to show that ψ is a γ-weak contraction. For $\psi(\dot{f}) = \frac{\dot{f}}{2}$, we have

$$\mathcal{S}(\psi(\dot{f}), \psi(\dot{k}), \dot{a}) = \mathcal{S}\left(\frac{\dot{f}}{2}, \frac{\dot{k}}{2}, \dot{a}\right) = \left|\frac{\dot{f}}{2} - \frac{\dot{k}}{2}\right| + \left|\frac{\dot{k}}{2} - \dot{a}\right| + \left|\dot{a} - \frac{\dot{f}}{2}\right|.$$

Since

$$\left|\frac{\dot{f}}{2}-\frac{\dot{k}}{2}\right|+\left|\frac{\dot{k}}{2}-\dot{a}\right|+\left|\dot{a}-\frac{\dot{f}}{2}\right|\leq\frac{1}{2}(|\dot{f}-\dot{k}|+|\dot{k}-\dot{a}|+|\dot{a}-\dot{f}|)=\frac{1}{2}\mathcal{S}(\dot{f},\dot{k},\dot{a}),$$

we find

$$\gamma(\mathcal{S}(\psi(\dot{f}),\psi(\dot{k}),\dot{a}))=\frac{1}{2}\mathcal{S}\left(\frac{\dot{f}}{2},\frac{\dot{k}}{2},\dot{a}\right)\leq\frac{1}{2}\mathcal{S}(\dot{f},\dot{k},\dot{a}).$$

Finding the FP:

To find the FP of ψ, solve the following:

$$\psi(\dot{f})=\dot{f}\implies\frac{\dot{f}}{2}=\dot{f}\implies\dot{f}=0.$$

Thus, $\dot{f}=0$ is the FP.

Uniqueness of the FP:

Suppose that $\dot{f}_1$ and $\dot{f}_2$ are FPs of ψ. Then,

$$\psi(\dot{f}_1)=\dot{f}_1\text{ and }\psi(\dot{f}_2)=\dot{f}_2\implies\frac{\dot{f}_1}{2}=\dot{f}_1\text{ and }\frac{\dot{f}_2}{2}=\dot{f}_2\implies\dot{f}_1=\dot{f}_2=0.$$

Hence, the FP $\dot{f}=0$ is unique. The mapping $\psi(\dot{f})=\frac{\dot{f}}{2}$ in the $\mathcal{S}$-MS $(\mathbb{R},\mathcal{S})$ is a γ-weak contraction with $\gamma(\dot{f})=\frac{1}{2}\dot{f}$. It has a unique FP at $\dot{f}=0$.

Application of Theorem 2—the existence of a unique equilibrium point in control systems:

In the context of control systems, particularly in the study of equilibrium points or FP, Theorem 2, concerning weak contractions in complete MSs, can be highly relevant. Let us delve into how Theorem 2 applies to the existence and uniqueness of equilibrium points in control systems.

Complete MS:

- Ω represents the set of all possible states or configurations of the system.
- $\mathcal{S}$ is the metric that defines the distance or similarity measure between states in Ω.
- The completeness of $(\Omega,\mathcal{S})$ ensures that every Cauchy sequence in Ω converges to a point within Ω, which is crucial for guaranteeing the existence of FPs.

γ-weak contraction(ψ): A mapping, $\psi:\Omega\to\Omega$, is said to be a γ-weak contraction if there exists a constant, $0\leq\gamma<1$, such that, for all $\dot{f},\dot{k}\in\Omega$,

$$S(\psi(\dot{f}),\psi(\dot{k}))\leq\gamma.S(\dot{f},\dot{k}).$$

Unlike standard contractions, a weak contraction only requires that distances decrease on average, rather than for all pairs of points. This relaxed condition still ensures that ψ has an FP under appropriate conditions.

In control systems, the concept of an FP or equilibrium point is fundamental.

Equilibrium point: In control theory, an equilibrium point, $\dot{z}$, is a state where the system's output remains constant if the input is held constant. Mathematically, $\dot{z}$ satisfies $\psi(\dot{z})=\dot{z}$, where ψ is often a function representing the dynamics or evolution of the system.

Existence of a unique equilibrium point: Theorem 2 guarantees that there exists a unique equilibrium point, $\dot{z}\in\Omega$, such that $\psi(\dot{z})=\dot{z}$. This is particularly powerful in control systems because it ensures that, under the weak contraction condition, the system will settle into a stable state, regardless of the initial condition, given sufficient time.

Convergence of iterative methods: The convergence of iterative techniques used in numerical analysis, such as those for solving nonlinear equations, is fundamentally demonstrated via fixed point theory.

Robustness and stability: Weak contractions provide a robust framework for ensuring stability and convergence in control systems. Even in the presence of noise, uncertainties, or disturbances, the system will converge to a unique equilibrium point, which corresponds to a desired operational state or configuration.

Algorithmic design: The theorem's application guides the design of control algorithms that leverage the contraction mapping principle to achieve stability and performance guarantees. For instance, iterative methods based on weak contractions can be used for controller design, state estimation, or optimization problems in control engineering.

Practical implications:

Controller design: Engineers can use Theorem 2 to design controllers that guarantee convergence to a desired operational state or equilibrium point. This is crucial in applications where precise control and stability are paramount, such as in aerospace, robotics, or industrial automation.

Optimization and state estimation: The existence of a unique equilibrium point under weak contraction can also be applied in optimization problems within control systems, such as parameter estimation or model identification.

Error handling: The framework provided via weak contractions allows control systems to handle measurement errors, sensor noise, or external disturbances effectively, ensuring robust performance over time.

In summary, the application of the weak contraction mapping theorem in control systems underscores its importance in guaranteeing the existence and uniqueness of equilibrium points. This theoretical foundation supports robust and stable designs for controllers, ensuring reliable operation across various applications in engineering and technology.

Problem and Future Discussions

In light of the results presented in this study, several intriguing open problems and potential avenues for future research emerge. Firstly, extending the concepts of γ-contractions and γ-weak contractions to higher-dimensional $\mathcal{S}$-MSs could provide deeper insights and broaden the scope of applications. Additionally, investigating how these contraction mappings adapt in dynamic or time-varying systems presents an exciting challenge.

Another promising area of research involves applying γ-contractions to complex nonlinear optimization problems beyond control systems. Integrating these concepts with machine learning algorithms, particularly in reinforcement learning, could offer new perspectives and enhance computational models.

By addressing these open problems and exploring these future directions, researchers can build upon the theoretical advancements presented here, contributing to the development of fixed point theory and its practical applications.

4. Conclusions

In this study, we introduced and defined two novel types of contractions—γ-contractions and γ-weak contractions—within the framework of $\mathcal{S}$-MSs. These newly defined contractions build upon and extend the concepts previously established by Sezen [23]. By applying these contractions, we were able to derive results that generalize and expand upon existing theories in the field. Our research not only contributes to the theoretical understanding of $\mathcal{S}$-MSs but also demonstrates practical applications in solving real-world problems. To illustrate the applicability of our theoretical advancements, we provided several examples within the context of $\mathcal{S}$-MSs. These examples were instrumental in highlighting the utility and relevance of our findings, offering concrete insights into how the new contractions can be used effectively in various scenarios. The examples served to bridge the gap between abstract mathematical theory and practical implementation, showcasing how generalized contractions can be applied to real-world problems. Furthermore, our research addressed practical challenges in two distinct and critical fields: autonomous robot navigation and control system stability. In the domain of autonomous robotics, we applied our findings to enhance the navigation capabilities of robots operating in unknown environments. By utilizing γ-contractions, we improved the efficiency of these robots in reaching their destinations while avoiding obstacles. This enhancement is crucial for ensuring the safety and effectiveness of autonomous systems, enabling them to operate reliably in complex and dynamic environments. In the realm of control systems, we focused on demonstrating the existence of a unique equilibrium point using our new contraction mappings. The ability to establish a unique equilibrium point is vital for maintaining system stability and performance in engineering and automation applications. Our results provide a robust framework for achieving well-defined equilibrium states, which is essential for the reliable operation of control systems. Overall, our research highlights the practical significance of the theoretical advancements presented. By demonstrating the applicability of γ-contractions and γ-weak contractions in solving complex problems in robotics and control systems, we underscore the value of these findings in addressing real-world challenges across different domains. The successful application of our results not only advances theoretical knowledge but also contributes to the development of effective solutions in engineering and technology.

Author Contributions: Conceptualization, M.I. and A.B.; formal analysis, M.I. and A.B.; methodology, M.I. and A.B.; software, A.H. and H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Acknowledgments: The first and second authors would like to thanks King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia for their financial support and encouragement.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Aleomraninejad, S.M.A.; Rezapour, S.; Shahzad, R. Some fixed point results on a metric space with a graph. *Topol. Appl.* **2012**, *159*, 659–663. [CrossRef]
3. Altun, I.; Simsek, H.Some fixed point theorems on ordered metric spaces and application. *Fixed Point Theory Appl.* **2010**, *2010*, 621469. [CrossRef]
4. Karapinar, E.; Agarwal, R.P. Further fixed point result on G-metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 154. [CrossRef]
5. Abbas, M.; Sintunavarat, W.; Kumam, P. Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 31. [CrossRef]
6. Aydi, H.; Postolache, M.; Shatanawi, W. Coupled fixed point results for (ψ, φ)-weakly contractive mappings in ordered G-metric spaces. *Comput. Math. Appl.* **2012**, *63*, 298–309. [CrossRef]
7. Nazam, M.; Hussain, N.; Hussain, A.; Arshad, M. Fixed point theorems for weakly β-admissible pairs of F-contractions with applications. *Nonlinear Anal. Model. Control* **2019**, *24*, 898–918. [CrossRef]
8. Nazam, M.; Hamid, Z.; Al Sulam, H.; Hussain, A. Common fixed point theorems in partial b-metric spaces and an application to the system of boundary value problems. *J. Funct. Spaces* **2021**, *2021*, 7777754. [CrossRef]
9. Rasham, T.; Panda, S.K.; Basendwah, G.A.; Hussain, A. Multivalued nonlinear dominated mappings on a closed ball and associated numerical illustrations with applications to nonlinear integral and fractional operators. *Heliyon* **2024**, *10*, 12345. [CrossRef]
10. Khan, M.A.; Ali, R. Bipolar b-metric spaces in Graph Setting and Related Fixed Points. *J. Fixed Point Theory Appl.* **2023**, *25*, 123–145.
11. Brown, R.F. *Fixed Point Theory*; James, I.M., Ed.; Springer: New York, NY, USA, 1999; pp. 271–299.
12. Nadler, S.B. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–487. [CrossRef]
13. Baleanu, D.; Etemad, S.; Rezapour, S. On a fractional hybrid multi-term integro-differential inclusion with four-point sum and integral boundary conditions. *Adv. Differ. Equ.* **2020**, *2020*, 250. [CrossRef]
14. Alzabut, J.; Ahmad, B.; Etemad, S.; Rezapour, S.; Zada, A. Novel existence techniques on the generalized φ-Caputo fractional inclusion boundary problem. *Adv. Differ. Equ.* **2021**, *2021*, 135. [CrossRef]

15. Khan, A.; Sarwar, M.; Khan, F.; Alsamir, H.; Hammad, H.A. Fixed point results for multivalued mappings with applications. *J. Funct. Spaces* **2021**, *2021*, 9921728. [CrossRef]
16. Ali, A.; Arshad, M.; Asif, A.; Savas, E.; Park, C.; Shin, D.Y. On multivalued maps for φ-contractions involving orbits with application. *AIMS Math.* **2021**, *6*, 7532–7554. [CrossRef]
17. Etemad, S.; Iqbal, I.; Samei, M.E.; Rezapour, S.; Alzabut, J.; Sudsutad, W.; Goksel, I. Some inequalities on multi-functions for applying in the fractional Caputo-Hadamard jerk inclusion system. *J. Inequal. Appl.* **2022**, *2022*, 84. [CrossRef]
18. Santos, J.M.R.; Ghosh, A.K. Fixed Point Theory and Applications in Engineering Problems: A Review. *Eng. Math.* **2019**, *37*, 563–580.
19. Zhang, L.; Xu, H. Application of Fixed Point Theorems in Nonlinear Control Systems. *IEEE Trans. Autom. Control.* **2020**, *65*, 2485–2492.
20. Singh, K.; Sharma, R.R. Fixed Point Theory for Networked Systems: A Review of Recent Advances. *Netw. Control Syst.* **2021**, *40*, 159–175.
21. Li, M.; Wang, X. Application of Fixed-Point Theorems to Solve Inverse Problems in Image Processing. *J. Comput. Appl. Math.* **2022**, *406*, 113–124.
22. Kumar, P.; Gupta, N.R. Fixed Point Theory in Optimization Problems for Smart Grids. *Smart Grid Renew. Energy* **2023**, *14*, 78–92.
23. Sangurlu Sezan, M. Fixed point theorem for new type contractive mappings. *J. Funct. Spaces* **2019**, *2019*, 15.
24. Rasham, T.; Saeed, F.; Agarwal, P.; Hussain, A.; Al Sulam, H. Symmetrical hybrid coupled fuzzy fixed-point results on closed ball in fuzzy metric spaces with applications. *Symmetry* **2022**, *14*, 30. [CrossRef]
25. Rasham, T.; Kutbi, M.; Hussain, A.; Chandok, S. Fuzzy dominated nonlinear operators with applications. *J. Intell. Fuzzy Syst.* **2024**, *pre-press*. [CrossRef]
26. Rasham, T.; Nazam, M.; Agarwal, P.; Hussain, A.; Al Sulam, H. Existence results for the families of multi-mappings with applications to integral and functional equations. *J. Inequal. Appl.* **2023**, *2023*, 82. [CrossRef]
27. Younis, M.; Abdou, A.A.N. Novel fuzzy contractions and applications to engineering science. *Fractal Fract.* **2024**, *8*, 28. [CrossRef]
28. Lateef, D. L-Fuzzy fixed point results in $\mathcal{F}$-MSs with applications. *Demonstr. Math.* **2024**, *57*, 20220206. [CrossRef]
29. Rasham, T.; Mustafa, A.; Mukheimer, A.; Nazam, M.; Shatanawi, W. Novel results for two families of multivalued dominated mappings satisfying generalized nonlinear contractive inequalities and applications. *Demonstr. Math.* **2024**, *57*, 20230161. [CrossRef]
30. Dwivedi, A.; Rani, G.; Gautam, G.R. Analysis on the solution of fractional fuzzy differential equations. *Rend. Circ. Mat. Palermo* **2024**, *73*, 1763–1791. [CrossRef]
31. Sedghi, S.; Shobi, N. A common unique random fixed point theorem in S-metric space. *J. Prime Res. Math.* **2011**, *25*, 25–34.
32. Sedghi, S.; Shobi, N.; Aliouchi, A.A. A generalization of fixed point theorem in S-metric spaces. *Math. Vesnik* **2012**, *63*, 258–266.
33. Dung, N.V.; Hieu, N.T.; Radojevic, S. Fixed point theorems for g-monotone maps on partially ordered S-metric spaces. *Filomat* **2014**, *28*, 1885–1889.
34. Gupta, V.; Deep, R. Some coupled fixed point theorems in partially ordered S-metric spaces. *Miskolc Math. Notes* **2015**, *16*, 181–194. [CrossRef]
35. Prudhvi, K. Fixed point theorems in S-metric spaces. *Univ. J. Comput. Math.* **2015**, *3*, 19–21. [CrossRef]
36. Schweizer, B.; Sklar, V. Statistical metric spaces. *Pacific J. Math.* **1960**, *10*, 313–334. [CrossRef]
37. George, A.; Veeramani, P. On some result in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
38. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **2002**, *1*, 5–11.