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Applications of Lucas Balancing Polynomial to Subclasses of Bi-Starlike Functions

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Abstract: The Lucas balancing polynomial is linked to a family of bi-starlike functions denoted as $\mathcal{S}_{sc}^{c}(\vartheta, \Xi(x))$, which we present and examine in this work. These functions are defined with respect to symmetric conjugate points. Coefficient estimates are obtained for functions in this family. The classical Fekete–Szegö inequality of functions in this family is also obtained.

Keywords: analytic functions; univalent and bi-univalent functions; convolution; Taylor–Maclaurin series; starlike functions; convex functions; principle of subordination; Lucas balancing polynomial; coefficient estimates; Fekete–Szegö inequality

MSC: 30C45; 30C50; 33C45

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Academic Editor: Hyun Soo Chung

Received: 19 November 2024
Revised: 2 January 2025
Accepted: 4 January 2025
Published: 10 January 2025

Citation: Murugusundaramoorthy, G.; Cotîrlă, L.-I.; Breaz, D.; El-Deeb, S.M. Applications of Lucas Balancing Polynomial to Subclasses of Bi-Starlike Functions. *Axioms* **2025**, *14*, 50. https://doi.org/10.3390/axioms14010050

1. Introduction and Preliminaries

Complex systems like optical and control systems are difficult to design and optimize in engineering. Engineers employ specialized functions that meet exacting optical requirements in order to precisely represent complex wavefronts. Univalent functions are essential to beam forming in signal processing because they allow electromagnetic waves to be manipulated. Univalent functions are used in control systems engineering to construct filters that produce desired frequency responses with low phase distortion and system stability. Univalent functions are also used in mechanical systems to simulate system dynamics and determine critical parameters for performance optimization. Moreover, bi-univalent functions are instrumental in improving compression ratios in image processing, enhancing image quality during compression and transmission, a persistent engineering challenge. The coefficient problem is a significant component of geometric theory in analytic functions, with a great deal of effort devoted to maximizing initial Taylor coefficient values.

Assume that $\mathcal{A}$ is the family of all analytic functions $\mathcal{F}$ defined on the open unit disk

$$\Delta_{\mathcal{D}} = \{\eta \in \mathbb{C} : |\eta| < 1\}$$

and can be expressed as below:

$$\mathcal{F}(\eta) = \eta + \sum_{n \geq 2} a_n \eta^n, \quad \text{where } \eta \in \Delta_{\mathcal{D}};\ a_n \text{ is a real number.} \tag{1}$$

Also assume that

$$\mathcal{S} = \{\mathcal{F} \in \mathcal{A} : \mathcal{F} \text{ is univalent } \forall \eta \in \Delta_{\mathcal{D}}\}$$

Assuming that $\mathcal{F}$ and $\mathcal{G}$ are analytic in $\Delta_{\mathcal{D}}$, we say that $\mathcal{F}$ is subordinate to $\mathcal{G} \in \Delta_{\mathcal{D}}$, and is represented by $\mathcal{F}(\eta) \prec \mathcal{G}(\eta)$ for all $\eta \in \Delta_{\mathcal{D}}$, provided that there is a Schwarz function ω with $\omega(0) = 0$ and $|\omega(\eta)| < 1$ for all $\eta \in \Delta_{\mathcal{D}}$, such that $\mathcal{F}(\eta) = \mathcal{G}(\omega(\eta))$ for all $\eta \in \Delta_{\mathcal{D}}$. More specifically, $\mathcal{F}(\eta) \prec \mathcal{G}(\eta)$ is equal to $\mathcal{F}(0) = \mathcal{G}(0)$ and $\mathcal{F}(\Delta_{\mathcal{D}}) \subset \mathcal{G}(\Delta_{\mathcal{D}})$ if the function $\mathcal{G}$ is univalent over $\Delta_{\mathcal{D}}$. For more about the Subordination Principle, one may refer to [1–4].

It is known that univalent functions are injective, or one-to-one. Inverse functions are invertible as they may not be defined on $\Delta_{\mathcal{D}}$. In fact, according to Koebe's one-quarter Theorem [1], the disk $\mathcal{D}(0, 1/4)$ with center 0 and radius 1/4 is included in the image of $\Delta_{\mathcal{D}}$ under any function $\mathcal{F} \in \mathcal{S}$. Accordingly, every function $\mathcal{F} \in \mathcal{S}$ has an inverse $\mathcal{F}^{-1} = \mathcal{G}$ which is defined as

$$\mathcal{G}(\mathcal{F}(\eta)) = \eta, \quad \eta \in \Delta_{\mathcal{D}}$$

$$\mathcal{F}(\mathcal{G}(w)) = w, \quad |w| < r(\mathcal{F}); \; r(\mathcal{F}) \geq 1/4.$$

Moreover, the inverse function is given by

$$\mathcal{G}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots. \tag{2}$$

Denote by Σ the class of all bi-univalent functions, defined as below:

$$\Sigma = \{\mathcal{F} \in \mathcal{S} : \mathcal{F}^{-1} \text{ is univalent } \forall \eta \in \Delta_{\mathcal{D}}\}.$$

For more about univalent and bi-univalent functions, see [1,5–8].

The study of a functional composed of combinations of the initial coefficients of the functions $\mathcal{F} \in \mathcal{A}$ is a common subject in the field of geometric function theory research in recent years. It is well knowledge that $|a_n|$ is confined by n for a function in the class $\mathcal{S}$. Additionally, the geometric features of those functions are shown by the coefficient boundaries. For instance, the bound for the second coefficients provides the growth and distortion bounds for the class $\mathcal{S}$. Investigations of functions $\mathcal{F} \in \Sigma$ that are related to coefficients started in about 1970. The bound for $|a_2|$ was derived by Lewin [6] in 1967 while studying the class of bi-univalent functions. The highest value of $|a_2|$ for functions belonging to the class Σ is $\frac{4}{3}$, as demonstrated later in 1969 by Netanyahu [7]. Furthermore, in 1979, Brannan and Clunie [9] demonstrated that $|a_2| \leq \sqrt{2}$ for functions belonging to the class Σ. Numerous academics have examined the coefficient boundaries for different subclasses of Σ. However, little is known about the bounds of the general coefficients $|a_n|$ for $n \geq 4$. As a matter of fact, the general coefficient $|a_n|$ still has an open coefficient estimation problem.

The maximum of $|a_3 - \chi a_2^2|$, as a function of the real parameter $0 \leq \chi \leq 1$ for a univalent function $\mathcal{F}$, was determined by Fekete and Szegö [10] in the year 1933. Since then, the Fekete–Szegö problem has been defined as maximizing the modulus of the functional $\Upsilon_\chi(\mathcal{F}) = a_3 - \chi a_2^2$ for $\mathcal{F} \in \mathcal{A}$ with any complex ϑ. Fekete–Szegö functional and the other coefficient estimates problems, have been discussed extensively in [10–18] and the references therein.

Lucas-Balancing Polynomial (LBP)

Behera and Panda were the first to propose the idea of balancing numbers $(B_n), n \geq 0$ [19]. With initial values set at $B_0 = 0$ and $B_1 = 1$, these numbers are defined by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$. An associated series, known as the Lucas-Balancing numbers (LBNs), which are represented as

$$\mathfrak{C}_n = \sqrt{8B_n^2 + 1},$$

has received a lot of attention. They contain the starting terms $\mathfrak{C}_0 = 1$ and $\mathfrak{C}_1 = 3$, and also fulfill the recurrence relation $\mathfrak{C}_{n+1} = 6\mathfrak{C}_n - \mathfrak{C}_{n-1}$ like B_n for $n \geq 1$. These numbers (LBNs) have since been the focus of many generalizations and investigations using a range of methodologies, including generating functions, hybrid convolutions, research into sum and ratio formulas for balancing numbers, different approaches to summing balancing numbers, the representation of sums using binomial coefficients, reciprocals of sequences related to these numbers, incomplete balancing, and matrix-based methods for studying series. A variety of perspectives and methods are presented in these references, which expand the idea to generalized balancing sequences (see [20–24]). As was first shown in [25], the analysis of LBP is the logical next step in this line of inquiry. Here is a recursive definition of these polynomials:

$$\begin{aligned} \mathfrak{C}_0(x) &= 1 \\ \mathfrak{C}_1(x) &= 3x && (3) \\ \mathfrak{C}_2(x) &= 18x^2 - 1, \textit{and} && (4) \\ \mathfrak{C}_n(x) &= 6x\mathfrak{C}_{n-1}(x) - \mathfrak{C}_{n-2}(x), n \geq 2, && (5) \end{aligned}$$

the generating function of the LBP is denoted as $\Xi(x,\eta)$z, and is expressed by

$$\Xi(x,\eta) = \sum_{n\geq 0} \mathfrak{C}_n(x)\eta^n = \frac{1 - 3x\eta}{1 - 6x\eta + \eta^2}$$

where $x \in [-1,1]$ and $\eta \in \Delta_{\mathcal{D}}$.

Motivated by the study on the class of functions that are starlike with respect to their symmetric points by Sakaguchi [26] in the year 1987, El-Ashwah and Thomas [27] introduced and investigated the class of starlike functions with respect to symmetric conjugate points, denoted by $\mathcal{S}^*_{sc}$ and given by

$$\mathcal{S}^*_{sc} = \left\{ \mathcal{F}(\eta) \in \mathcal{S} : \Re\left\{ \frac{\eta \mathcal{F}'(\eta)}{\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}} \right\} > 0; \quad \eta \in \Delta_{\mathcal{D}} \right\}.$$

The class can be expanded to include convex functions with respect to symmetric conjugate points [28], which is another class in $\mathcal{S}$, if for all $\eta \in \Delta_D$, the following condition holds:

$$\mathcal{C}_{sc} = \left\{ \mathcal{F}(\eta) \in \mathcal{S} : \Re\left\{ \frac{(\eta \mathcal{F}'(\eta))'}{\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'} \right\} > 0; \quad \eta \in \Delta_{\mathcal{D}} \right\}.$$

For the first time, in this article, we define a new subclass $\mathcal{S}^c_{sc}(\vartheta, \Xi(x))$ where $0 \leq \vartheta \leq 1$ and $x \in \mathbb{R}$ of Σ associating with Lucas-Balancing polynomials (LBP), as given in Definition 1.

Definition 1. *A function $\mathcal{F} \in \Sigma$ is in $\mathcal{S}^c_{sc}(\vartheta, \Xi(x))$ if it satisfies the following subordinations:*

$$\frac{2\vartheta\eta^3\mathcal{F}'''(\eta) + 2(1+\vartheta)\eta^2\mathcal{F}''(\eta) + 2\eta\mathcal{F}'(\eta)}{\vartheta\left\{\eta^2\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'' + \left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)\right\} + (1-\vartheta)\eta\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'} \quad (6)$$

$$\prec \Xi(x,\eta),$$

and

$$\frac{2\vartheta w^3\mathcal{G}'''(w)+2(1+\vartheta)w^2\mathcal{G}''(w)+2w\mathcal{G}'(w)}{\vartheta\left\{w^2\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)''+\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)\right\}+(1-\vartheta)w\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)'} \prec \Xi(x,w), \tag{7}$$

where $\mathcal{G}\,(w) = \mathcal{F}^{-1}(w)$ *as given in (2).*

By fixing the parameter $\vartheta = 0$ and $\vartheta = 1$, we derive the new subclasses that have not been discussed yet by connecting LBNs.

Definition 2. *A function* $\mathcal{F} \in \Sigma$ *is in* $\mathcal{CV}^c_{sc}(\Xi(x))$ *if it satisfies the following subordinations:*

$$\frac{2\eta\mathcal{F}''(\eta)+2\mathcal{F}'(\eta)}{\left(\mathcal{F}(\eta)-\overline{\mathcal{F}(-\overline{\eta})}\right)'} \prec \Xi(x,\eta),$$

and

$$\frac{2w\mathcal{G}''(w)+2\mathcal{G}'(w)}{\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)'} \prec \Xi(x,w),$$

where $\mathcal{G}(w) = \mathcal{F}^{-1}(w)$ *as given in (2).*

Definition 3. *A function* $\mathcal{F} \in \Sigma$ *is said to be in the class* $\mathcal{K}^c_{sc}(\vartheta, \Xi(x))$ *if it satisfies the following subordinations:*

$$\frac{2\eta^3\mathcal{F}'''(\eta)+4\eta^2\mathcal{F}''(\eta)+2\eta\mathcal{F}'(\eta)}{\eta^2\left(\mathcal{F}(\eta)-\overline{\mathcal{F}(-\overline{\eta})}\right)''+\left(\mathcal{F}(\eta)-\overline{\mathcal{F}(-\overline{\eta})}\right)} \prec \Xi(x,\eta),$$

and

$$\frac{2w^3\mathcal{G}'''(w)+4w^2\mathcal{G}''(w)+2w\mathcal{G}'(w)}{w^2\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)''+\left(\mathcal{G}(w)-\overline{\mathcal{G}(-\overline{w})}\right)} \prec \Xi(x,w),$$

where $\mathcal{G}\,(w) = \mathcal{F}^{-1}(w)$ *as given in (2).*

This paper's research is inspired by the studies conducted in [29–31] and on Lucas-Balancing polynomials [32–36]. This study's main objective is to estimate the initial Taylor–Maclarin coefficients $|a_2|$ and $|a_3|$ for $\mathcal{F} \in \mathcal{S}^c_{sc}(\vartheta, \Xi(x))$ that are subordinate to Lucas-Balancing polynomials and some of their special cases. Additionally, we look at the Fekete–Szegö functional problem that corresponds to $\mathcal{F} \in \mathcal{S}^c_{sc}(\vartheta, \Xi(x))$.

2. Initial Coefficient Bounds for the Class $\mathcal{S}^c_{sc}(\vartheta, \Xi(x))$

In this section, we provide estimates for the initial Taylor–Maclaurin coefficients for $\mathcal{F} \in \mathcal{S}^c_{sc}(\vartheta, \Xi(x))$ and are of the form (1).

For deriving our main results, we need the following lemma.

Lemma 1 ([37]). *If* $h \in \mathfrak{P}$, *then* $|c_n| \leq 2$ *for each* n, *where* $\mathfrak{P}$ *is the family of all functions* h *analytic in* Δ *for which* $\Re(h(\eta)) > 0$ *and*

$$h(\eta) = 1 + c_1\eta + c_2\eta^2 + \ldots \quad \text{for } \eta \in \Delta_{\mathcal{D}}$$

Define the functions $p(\eta)$ and $q(\eta)$ by

$$p(\eta) := \frac{1+u(\eta)}{1-u(\eta)} = 1 + p_1\eta + p_2\eta^2 + \dots$$

and

$$q(\eta) := \frac{1+v(\eta)}{1-v(\eta)} = 1 + q_1\eta + q_2\eta^2 + \dots.$$

It follows that

$$u(\eta) := \frac{p(\eta)-1}{p(\eta)+1} = \frac{1}{2}\left[p_1\eta + \left(p_2 - \frac{p_1^2}{2}\right)\eta^2 + \dots\right]$$

and

$$v(\eta) := \frac{q(\eta)-1}{q(\eta)+1} = \frac{1}{2}\left[q_1\eta + \left(q_2 - \frac{q_1^2}{2}\right)\eta^2 + \dots\right].$$

Then, $p(\eta)$ and $q(\eta)$ are analytic in Δ_D with $p(0) = 1 = q(0)$.

As $u, v : \Delta_D \to \Delta_D$, the functions $p(\eta)$ and $q(\eta)$ have a positive real part in Δ_D, and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i. Now, we have

$$\Xi(x, u(\eta)) \quad = \quad 1 + \frac{\mathfrak{C}_1(x)}{2}p_1\eta + \left[\frac{\mathfrak{C}_1(x)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\mathfrak{C}_2(x)}{4}p_1^2\right]\eta^2 + \dots \tag{8}$$

And, similarly, we obtain

$$\Xi(x, v(w)) \quad = \quad 1 + \frac{\mathfrak{C}_1(x)}{2}q_1 w + \left[\frac{\mathfrak{C}_1(x)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\mathfrak{C}_2(x)}{4}q_1^2\right]w^2 + \dots \tag{9}$$

Theorem 1. *Let the function $\mathcal{F}$ given by (1) be in the class $\mathcal{S}_{sc}^c(\vartheta, \Xi(x))$. Then*

$$|a_2| \leq \frac{2|3x|\sqrt{|3x|}}{\sqrt{|2(4\vartheta+3)(3x)^2 - 4(\vartheta+2)^2|18x^2-3x-1||}}, \tag{10}$$

and

$$|a_3| \leq \frac{3|x|}{2(4\vartheta+3)} + \frac{9x^2}{4(\vartheta+2)^2}. \tag{11}$$

Proof. Let $\mathcal{F}$ be in the class $\mathcal{S}_{sc}^c(\vartheta, \Xi(x))$. Then, using Definition 1, there are two analytic functions, u and v, on the unit disk $\Delta_{\mathcal{D}}$, such that

$$\frac{2\vartheta\eta^3\mathcal{F}'''(\eta) + 2(1+\vartheta)\eta^2\mathcal{F}''(\eta) + 2\eta\mathcal{F}'(\eta)}{\vartheta\left\{\eta^2\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'' + \left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)\right\} + (1-\vartheta)\eta\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'} \prec \Xi(x, u(\eta)), \tag{12}$$

and

$$\frac{2\vartheta w^3\mathcal{G}'''(w) + 2(1+\vartheta)w^2\mathcal{G}''(w) + 2w\mathcal{G}'(w)}{\vartheta\left\{w^2\left(\mathcal{G}(w) - \overline{\mathcal{G}(-\overline{w})}\right)'' + \left(\mathcal{G}(w) - \overline{\mathcal{G}(-\overline{w})}\right)\right\} + (1-\vartheta)w\left(\mathcal{G}(w) - \overline{\mathcal{G}(-\overline{w})}\right)'} \prec \Xi(x, v(w)), \tag{13}$$

where for all $\eta, w \in \Delta_{\mathcal{D}}$. Thus, Let

$$\begin{aligned} M(\eta) &= 2\vartheta\eta^3\mathcal{F}'''(\eta) + 2(1+\vartheta)\eta^2\mathcal{F}''(\eta) + 2\eta\mathcal{F}'(\eta) \\ &= 2\eta + 4(2+\vartheta)a_2\eta^2 + 6(3+4\vartheta)a_3\eta^3 + \ldots \end{aligned}$$

and

$$\begin{aligned} N(\eta) &= \vartheta\left\{\eta^2\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)'' + \left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)\right\} + (1-\vartheta)\eta\left(\mathcal{F}(\eta) - \overline{\mathcal{F}(-\overline{\eta})}\right)' \\ &= 2\eta + 2(3+4\vartheta)a_3\eta^3 + \ldots. \end{aligned}$$

Thus (12) will be

$$\begin{aligned} \frac{M(\eta)}{N(\eta)} &= \frac{2\eta + 4(2+\vartheta)a_2\eta^2 + 6(3+4\vartheta)a_3\eta^3 + \ldots}{2\eta + 2(3+4\vartheta)a_3\eta^3 + \ldots} \\ &= 1 + 2(2+\vartheta)a_2\eta + 2(3+4\vartheta)a_3\eta^2 + \ldots. \end{aligned} \tag{14}$$

Similarly by taking we get left hand side of (13) as

$$\frac{M(w)}{N(w)} = 1 - 2(2+\vartheta)a_2 w + 2(3+4\vartheta)(2a_2^2 - a_3)w^2 + \ldots. \tag{15}$$

Thus by (12) and (13), (8) and (9) and comparing coefficients we get the following equations:

$$2(\vartheta+2)a_2 = \frac{\mathfrak{C}_1(x)p_1}{2}, \tag{16}$$

$$2(4\vartheta+3)a_3 = \left[\frac{\mathfrak{C}_1(x)}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\mathfrak{C}_2(x)}{4}p_1^2\right] \tag{17}$$

$$-2(\vartheta+2)a_2 = \frac{\mathfrak{C}_1(x)q_1}{2}, \tag{18}$$

and

$$2(4\vartheta+3)(2a_2^2 - a_3) = \left[\frac{\mathfrak{C}_1(x)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{\mathfrak{C}_2(x)}{4}q_1^2\right]. \tag{19}$$

From Equations (16) and (18), we find that

$$p_1 = -q_1, \tag{20}$$

and

$$8(\vartheta+2)^2a_2^2 = \frac{\mathfrak{C}_1^2(x)}{4}(p_1^2 + q_1^2) \tag{21}$$

Equation (21) gives us the following

$$a_2^2 = \frac{\mathfrak{C}_1^2(x)(p_1^2 + q_1^2)}{32(\vartheta+2)^2}, \tag{22}$$

and

$$p_1^2 + q_1^2 = \frac{32(\vartheta+2)^2a_2^2}{\mathfrak{C}_1^2(x)}. \tag{23}$$

If we add Equations (17) and (19), then make use of Equations (20) and (23), we obtain

$$4(4\vartheta+3)a_2^2 = \frac{\mathfrak{C}_1(x)}{2}(p_2 + q_2) + \frac{(\mathfrak{C}_2(x) - \mathfrak{C}_1(x))}{4}(p_1^2 + q_1^2),$$

which gives

$$4(4\vartheta+3)a_2^2 = \frac{\mathfrak{C}_1(x)}{2}(p_2+q_2) + \frac{8(\mathfrak{C}_2(x)-\mathfrak{C}_1(x))(\vartheta+2)^2 a_2^2}{\mathfrak{C}_1^2(x)}.$$

Therefore, we obtain the following

$$a_2^2 = \frac{\mathfrak{C}_1^3(x)(p_2+q_2)}{8(4\vartheta+3)\mathfrak{C}_1^2(x) - 16(\mathfrak{C}_2(x)-\mathfrak{C}_1(x))(\vartheta+2)^2}. \tag{24}$$

Using the $|p_2| \le 2$ and $|q_2| \le 2$, and using the initial values (3) and (4), we obtain the desired bound for the modulus of a_2.

Now, we look for the bound on $|a_3|$. In order to do this, we subtract Equation (19) from Equation (17), which gives

$$4(4\vartheta+3)(a_3-a_2^2) = \frac{\mathfrak{C}_1(x)}{2}(p_2-q_2) + \frac{(\mathfrak{C}_2(x)-\mathfrak{C}_1(x))}{4}(p_1^2-q_1^2).$$

In view of Equation (20), we obtain

$$a_3 = \frac{\mathfrak{C}_1(x)(p_2-q_2)}{8(4\vartheta+3)} + a_2^2 \tag{25}$$

If follows from Equation (22) that

$$a_3 = \frac{\mathfrak{C}_1(x)(p_2-q_2)}{8(4\vartheta+3)} + \frac{\mathfrak{C}_1^2(x)(p_1^2+q_1^2)}{32(\vartheta+2)^2}.$$

Therefore, using the initial values (3) and (4) the facts $|p_2| \le 2$, $|q_2| \le 2$, we obtain the desired bound for the modulus of a_3. This completes the proof of Theorem 1. □

3. Fekete–Szegö Inequality for the Class $\mathcal{S}_{sc}^{c}(\vartheta, \Xi(x))$

In this section, we maximize the modulus of the functional $\Upsilon_\zeta(\mathcal{F}) = a_3 - \zeta a_2^2$ for $\zeta \in \mathbb{R}$ and for $\mathcal{F} \in \mathcal{S}_{sc}^{c}(\vartheta, \Xi(x))$. The following lemma (see, for details [15,17,18]) is a well-known fact, but it is crucial for our presented work.

Lemma 2. *Let $k, l \in \mathbb{R}$ and $x, y \in \mathbb{C}$. If $|\eta_1| < r$ and $|\eta_2| < r$,*

$$|(k+l)\eta_1 + (k-l)\eta_2| \le \begin{cases} 2|k|r, & \text{if } |k| \ge |l| \\ 2|l|r, & \text{if } |k| \le |l| \end{cases}$$

Theorem 2. *Let the function $\mathcal{F}$ given by (1) be in the class $\mathcal{S}_{sc}^{c}(\vartheta, \Xi(x))$. Then, for some $\zeta \in \mathbb{R}$ and for $x \in [-1, 1]$*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{3|x|}{4(4\vartheta+3)}, & \text{if } |\Delta(\zeta,\vartheta)| \le \frac{1}{8(4\vartheta+3)} \\ 12|x||\Delta(\zeta,\vartheta)|, & \text{if } |\Delta(\zeta,\vartheta)| \ge \frac{1}{8(4\vartheta+3)}. \end{cases} \tag{26}$$

where

$$\Delta(\zeta,\vartheta) = \frac{(1-\zeta)(3x)^2}{8(4\vartheta+3)(3x)^2 - 16(\vartheta+2)^2|18x^2-3x-1|}.$$

Proof. For some $\zeta \in \mathbb{R}$, using Equation (25), we obtain

$$a_3 - \zeta a_2^2 = \frac{\mathfrak{C}_1(x)(p_2-q_2)}{4(4\beta+3)} + (1-\zeta)a_2^2.$$

Using Equation (24), we obtain

$$a_3 - \zeta a_2^2 = \frac{\mathfrak{C}_1(x)(p_2 - q_2)}{8(4\vartheta+3)} + \frac{(1-\zeta)\mathfrak{C}_1^3(x)(p_2+q_2)}{8(4\vartheta+3)\mathfrak{C}_1^2(x) - 16(\vartheta+2)^2(\mathfrak{C}_2(x) - \mathfrak{C}_1(x))}$$
$$= \mathfrak{C}_1(x)\left\{\left(\Delta(\zeta,\vartheta) + \frac{1}{8(4\vartheta+3)}\right)p_2 + \left(\Delta(\zeta,\vartheta) - \frac{1}{8(4\vartheta+3)}\right)q_2\right\},$$

where

$$\Delta(\zeta,\vartheta) = \frac{(1-\zeta)\mathfrak{C}_1^2(x)}{8\mathfrak{C}_1^2(x)(4\vartheta+3) - 16(\mathfrak{C}_2(x) - \mathfrak{C}_1(x))(\vartheta+2)^2}.$$

Thus, using Lemma 2, we obtain

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{3|x|}{4(4\vartheta+3)}, & \text{if } |\Delta(\zeta,\vartheta)| \leq \frac{1}{8(4\vartheta+3)} \\ 12|x||\Delta(\zeta,\vartheta)|, & \text{if } |\Delta(\zeta,\vartheta)| \geq \frac{1}{8(4\vartheta+3)}. \end{cases} \tag{27}$$

Therefore, using the initial values (3) and (4), then simplifying (27), we obtain the desired result (26). This completes the proof of Theorem 2. □

4. Concluding Remarks

We defined a new subclasses of Σ in $\Delta_{\mathcal{D}}$ related to Lucas–Balancing Polynomials (LBPs) and found initial coefficients of functions further determined the Fekete–Szegö inequalities. By fixing the values of ϑ, one can determine new results for the subclasses presented in Definitions 2 and 3. The results of the research could be expanded to develop q—fractional calculus, extending the results for bi-univalent functions. Furthermore, we can use the q—differential and integral operator and the q—fractional differential and integral operator to construct a subclass of $\mathcal{S}_{sc}^{c}(\vartheta, \Xi(x))$. The approximated coefficient constraints can be used in image processing, specifically texture analysis. The work can also be extended for coloured images and to investigate various image-processing techniques like enhancement, sharpening, pattern identification, restoration, and retrieval. Mathematically, future research can be carried out with the results of Fekete inequality obtained for inverse functions and can be applied in image processing.

Author Contributions: Conceptualization, G.M., L.-I.C., D.B. and S.M.E.-D.; Methodology, G.M., L.-I.C., D.B. and S.M.E.-D.; Software, L.-I.C.; Investigation, G.M., D.B. and S.M.E.-D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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