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The Axiomatic Characterization of the Grey Shapley Value

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Abstract: One of the most significant solution concepts in cooperative grey game theory is the grey Shapley value. This value is a fascinating one among the models and methods of operations research, and has been the subject of extensive study by other researchers. The objective of this study is to characterize and redefine this value in cooperative games where coalition values are grey numbers. In this study, the grey Shapley value is characterized by the following axioms: $\mathcal{G}$-gain loss, $\mathcal{G}$-null player, and $\mathcal{G}$-differential marginality. Finally, this study concludes with an investigation of some applications involving production costs. This study is based on an investigation of the costs incurred when milk producers collaborate.

Keywords: grey system theory; cooperative grey games; grey Shapley value; characterization; cost management

MSC: 91A12; 91A80

1. Introduction

A grey system is defined as a system in which some information is known and some information is unknown [1,2]. Indeed, numerous researchers have addressed this ambiguity through the use of grey numbers, a pivotal concept in grey system theory. When decision-making involves an element of uncertainty, grey number games are a valuable tool for navigating this ambiguity [3]. With incomplete information, the theoretical and practical significance of employing grey games is evident. The authors of [3] conducted an in-depth examination of the intricacies inherent to grey matrix games based on pure strategies, ultimately formulating **necessary and sufficient** conditions that must be met for a solution to be reached in such a game. The authors of [4] focused on the matrix solution method of the grey matrix game, which is based on a full-rank grey payoff matrix. The challenge of identifying a potential optimal pure strategy solution for a matrix game with grey interval numbers has been explored [5]. Grey uncertainty is becoming increasingly prevalent across a range of disciplines, including the natural sciences, engineering, real-world applications, and operational research (OR) situations [6–9].

The comparison between stochastic approaches and the grey system framework, as employed in this paper, highlights the unique advantages of the latter in addressing uncertainties in cooperative game theory. Stochastic models typically rely on probabilistic distributions to quantify uncertainty, which requires detailed and precise knowledge of underlying probabilities. While effective when such information is available, stochastic methods can become impractical or unreliable in scenarios involving incomplete, ambiguous, or qualitative data. In contrast, the grey system approach [1] utilized in this study provides a robust alternative by employing grey numbers, which represent uncertainty through bounded intervals rather than relying on probabilistic assumptions. This

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Academic Editors: Boualem Djehiche and Valery Y. Glizer

Received: 14 November 2024
Revised: 19 December 2024
Accepted: 8 January 2025
Published: 10 January 2025

Citation: Gençtürk, M.; Öztürk, M.S.; Palancı, O. The Axiomatic Characterization of the Grey Shapley Value. *Axioms* **2025**, *14*, 51. https://doi.org/10.3390/axioms14010051

method is particularly advantageous for real-world scenarios where probabilities are not explicitly known, as it captures the ambiguity and incomplete nature of the information without over-specifying the uncertainty. For instance, as demonstrated in this paper, the grey Shapley value effectively characterizes cooperative games with grey coalition values, accommodating uncertainties in coalition worths through interval representations [10,11].

The axiomatic characterization of the grey Shapley value presented here further strengthens its applicability by establishing clear, systematic properties that allow for robust decision-making in grey environments. Compared to stochastic arguments, the grey approach is computationally efficient and aligns closely with scenarios where only qualitative or bounded data are available, avoiding potential biases or inaccuracies introduced by incorrect probabilistic assumptions. As shown in this paper, the application of the grey Shapley value to milk production enterprises demonstrates how cooperative strategies can reduce costs under uncertain conditions without relying on probability distributions. Therefore, including a comparison between stochastic and grey systems approaches in the introduction would underline the broader theoretical contributions of this work and its practical relevance to fields requiring decision-making under uncertainty.

The concept of transferable utility games (TU-games) was first proposed by [12]. The Shapley value is employed in resolving genuine, practical issues within the field of operational research (OR) and other contexts. A considerable number of studies on the Shapley value have been published recently [12–14]. Ref. [12] used additivity (ADD), efficiency (EFF), symmetry (SYM), and the property of null player (NULL) axioms. Ref. [14] characterizes the Shapley value by using EFF, SYM, and the strong monotonicity property (SMON). The Shapley value is characterized by [13], which uses a new axiom whose name is a coalitional strategic equivalence (CSE).

Subsequently, a solution for grey games is applied. In the game theory literature, the lower value of the grey set of a coalition can be interpreted as a pessimistic approach. In contrast, the upper value of the grey set of a coalition can be interpreted as an optimistic approach. All cooperative grey games are demonstrated to have a grey payoff, that is to say, the grey Shapley value. This value is the most influential grey solution concept in a cooperative grey game.

Ref. [15] combines grey system theory with the classic N-person game theory, establishing the N-person grey game with grey coalition values. The grey Shapley value was introduced as a means of establishing a new class of cooperative games, in which the set of players is finite and the coalition values are interval grey numbers [11]. Ref. [16] presents a characterization of the grey Shapley value, demonstrating how the fairness property can be applied in this context. The properties of efficiency, symmetry, and strong monotonicity are employed to characterize the grey Shapley value [6]. The principal objective of this paper is to present a novel axiomatic characterization of the grey Shapley value, which eschews the use of additivity and marginality, employing instead grey data. In this paper, we consider the grey Shapley value and its axiomatic characterization as inspired by [17].

Two of the most compelling characterizations of the Shapley value are primarily based on one of the following axioms: the EFF and ADD. The objective of this study is to present a novel axiomatic characterization of the grey Shapley value that does not rely on EFF or ADD.

The aim is to redefine a value using alternative axioms. This approach enables us to gain a new perspective on the concept of a grey value. In essence, an axiomatic characterization entails assigning a grey value and adopting a distinct standpoint, with the aid of a set of axioms. These axioms facilitate the analysis of characterizations. In this study, our objective is to characterize the grey Shapley value. Consequently, we have identified new grey properties and characterized this value in conjunction with several properties.

The grey Shapley value has been effectively applied in various real-world scenarios where uncertainty and incomplete information play a significant role. In supply chain and logistics management, it has been utilized for cost allocation among collaborative enterprises. For instance, in transportation scenarios, the grey Shapley value helps allocate logistics costs fairly among companies under uncertain demand and cost conditions, ensuring equitable and efficient cooperation [11]. Ref. [18] introduced two families of point-valued solutions that generalize the Shapley value in global cooperative games, providing a broader context for our work. Ref. [19] analyzed cooperative game theory solutions, focusing on the core and Shapley value in the Cartesian product of two sets, which aligns with our exploration of the grey Shapley value.

In the energy sector, the grey Shapley value has been employed to address uncertainties in renewable energy production and distribution. It facilitates the allocation of benefits among investors or energy producers working together in renewable projects, accounting for fluctuations in energy generation due to environmental conditions. This application demonstrates its utility in promoting sustainable energy collaborations while ensuring fair benefit sharing. Another prominent area of application is disaster management. The grey Shapley value has been used to allocate resources and share costs in post-disaster recovery efforts. For example, in housing solutions after natural disasters, it aids in fair resource distribution among affected stakeholders, even when the data on damages and costs are imprecise [20]. Additionally, public policy and infrastructure projects have benefited from the grey Shapley value, particularly in water management and resource distribution. It enables fair cost-sharing among regions or stakeholders in large-scale projects, where exact costs and benefits are uncertain. These applications highlight the grey Shapley value's adaptability and effectiveness in addressing cooperative decision-making under uncertainty across diverse fields. By integrating the grey Shapley value into these domains, decision-makers can achieve equitable outcomes, improve resource efficiency, and enhance cooperation among stakeholders, demonstrating its broad applicability and practical significance in real-world problems.

The primary objectives are to develop an axiomatic characterization of the grey Shapley value within the framework of cooperative grey games and to demonstrate its applicability in real-world decision-making scenarios characterized by uncertainty. In terms of contributions, this paper introduces a novel framework for cooperative games in which coalition values are represented as grey numbers, thus extending the scope of traditional game theory to encompass uncertain environments. Furthermore, this study establishes a rigorous axiomatic foundation for the grey Shapley value, ensuring its theoretical robustness while enabling its practical application in diverse fields such as economics and operations research. Finally, the practical utility of the proposed approach is illustrated through detailed examples that highlight how the grey Shapley value can be used to facilitate fair and equitable decision-making in uncertain contexts.

Cost analysis in production enterprises is a systematic examination of the costs associated with produced goods or services, to evaluate them from a managerial perspective. The primary objectives of cost analysis include facilitating cost control for current production activities and estimating future production costs. Furthermore, cost analysis serves to inform business managers about existing business expenses and potential cost trends. It constitutes a crucial element within all economic evaluation methodologies, providing businesses with a valuable tool for self-assessment and strategic planning [21].

Based on the importance of calculating and analyzing costs, this study investigates the analyses of the costs of enterprises with the game theory approach. In this direction, the changes in the costs of milk production enterprises as a result of their cooperation with each other are calculated and the results are evaluated.

The remainder of this paper is structured as follows. Section 2 refers to the concepts of classical and cooperative grey games. Section 3 introduces and axiomatically characterizes the grey Shapley value, presenting a set of axioms that define its properties. A proposal for cooperative grey games is presented in Section 4. This paper concludes with an outlook for future studies.

2. Preliminaries

This section presents preliminary concepts from the fields of cooperative game theory and grey calculus.

A cooperative game in coalitional form is defined as an ordered pair $< N, v >$, where $N = \{1, 2, ..., n\}$ represents the set of players and $v : 2^N \to \mathbb{R}$ is a function that assigns a real number to each coalition $S \in 2^N$, with $v(\emptyset) = 0$. This function v is known as the game's characteristic function, and $v(S)$ denotes the worth (or value) of coalition S. Thus, a cooperative game $< N, v >$ is identified by its characteristic function v. The set of all coalitional games with player set N is denoted as G^N [22].

A grey number is defined as a quantity with an uncertain but bounded value, meaning its exact value is unknown, yet it lies within a known range. This type of number is denoted by w and typically represented as an interval or general set of numbers [10].

There are various types of grey numbers, among which interval grey numbers are especially useful. A grey number with both a lower limit $\underline{x}$ and an upper limit $\overline{x}$ is called an interval grey number and is denoted by $w \in [\underline{a}, \overline{a}]$. In this study, we focus on interval grey numbers.

Now, we look at the operations of interval grey numbers.

Let

$$w_1 \in [a, b],\ a < b \text{ and } w_2 \in [c, d],\ c < d.$$

The sum is given by

$$w_1 + w_2 \in [a + c, b + d].$$

The additive inverse is given by

$$-w_1 \in [-b, -a].$$

Therefore, the subtraction is given by

$$w_1 - w_2 = w_1 + (-w_2) \in [a - d, b - c].$$

The multiplication is defined as follows:

$$w_1 \cdot w_2 \in [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}].$$

The reciprocal of $w_1 \in [a, b]$, $a < a$, $a, b \neq 0$, $ab > 0$ is defined as

$$w_1^{-1} \in \left[\frac{1}{b}, \frac{1}{a}\right].$$

Let $c, d \neq 0$, $cd > 0$. The division is defined as follows:

$$w_1 / w_2 = w_1 \cdot w_2^{-1} = \left[\min\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}, \max\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}\right].$$

Let $w \in [a, b]$, where $a < b$, and let k be a positive real number. The scalar multiplication of k and w_1 is defined as follows:

$$kw_1 \in [ka, kb].$$

The k-th power of the grey number w_1 is given by

$$w_1^k \in \left[a^k, b^k\right]$$

where k is a positive real number [23].

In general, the difference between $w_1 \in [a, b]$ and $w_2 \in [c, d]$ is defined as follows:

$$w_1 \ominus w_2 = w_1 + (-w_2) \in [a - d, b - c],$$

(see [24]).

For example, let $w_1 \in [4, 7]$ and $w_2 \in [8, 9]$; then, we have

$$\begin{aligned} w_1 \ominus w_2 &\in [4 - 9, 7 - 8] = [-5, -1], \\ w_2 \ominus w_1 &\in [8 - 7, 9 - 4] = [1, 5]. \end{aligned}$$

Unlike the subtraction defined previously, we employ a partial subtraction operator. We define $w_1 - w_2$ only if $|b - a| \geq |d - c|$, in which case $w_1 - w_2 \in [a - c, b - d]$. It is important to note that $a - c \leq b - d$. We say that the interval $[a, b]$ is weakly preferred to the interval $[c, d]$, denoted as $[a, b] \succcurlyeq [c, d]$, if and only if $a \geq c$ and $b \geq d$. Conversely, we denote $[a, b] \preccurlyeq [c, d]$ if and only if $a \leq c$ and $b \leq d$ [11].

Notice that, if we make a comparison to the example above, then in our case, $[8, 9] - [4, 7]$ is not defined. However, $[4, 7] - [8, 9]$ is defined.

Let $w_1 \in [4, 7]$ and $w_2 \in [8, 9]$; $w_1 - w_2$ is defined, since $|7 - 4| \geq |9 - 8|$, but $w_2 - w_1$ is not defined, since $|9 - 8| = 1 \ngeq 3 = |7 - 4|$. Then, we have

$$w_1 - w_2 \in [4 - 8, 7 - 9] = [-4, -2].$$

A cooperative grey game is defined as an ordered pair $< N, w >$, where $N = \{1, \ldots, n\}$ represents the set of players and $w : 2^N \to \mathcal{G}(\mathbb{R})$ serves as the characteristic function. This function satisfies $w(\emptyset) \in [0, 0]$. The grey payoff function $w(S) \in [\underline{A_S}, \overline{A_S}]$ refers to the value of the grey expected benefit associated with a coalition $S \in 2^N$, where $\overline{A_S}$ and $\underline{A_S}$ denote the maximum and minimum potential profits of coalition S, respectively. Consequently, a cooperative grey game can be viewed as a classical cooperative game characterized by grey profits w.

Grey solutions are particularly beneficial for addressing reward/cost-sharing problems involving grey data, utilizing cooperative grey games as a methodological framework. The foundational elements of grey solutions are grey payoff vectors, which are defined as vectors whose components belong to $\mathcal{G}(\mathbb{R})$. We denote the set of all such grey payoff vectors by $\mathcal{G}(\mathbb{R})^N$ and the family of all cooperative grey games by $\mathcal{G}G^N$.

We begin by recalling the definition of the grey Shapley value and its associated properties. A game $< N, w >$ is termed grey size monotonic if $< N, |w| >$ is monotonic, meaning that $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ such that $S \subset T$. For future reference, we denote the class of all grey size monotonic games with player set N as $SM\mathcal{G}G^N$.

The grey marginal operators and the grey Shapley value are defined on the set $SM\mathcal{G}G^N$. Let $\Pi(N)$ represent the set of permutations $\sigma : N \to N$ of N. The *grey marginal operator* $m^\sigma : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ associated with σ assigns to each $w \in SM\mathcal{G}G^N$ the *grey marginal vector* $m^\sigma(w)$ of w with respect to σ, defined as $m_i^\sigma(w) = w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i)) \in$

$[m_i^\sigma(\underline{w}), m_i^\sigma(\overline{w})]$ for each $i \in N$. Here, $P^\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, and $\sigma^{-1}(i)$ denotes the entrance number of player i.

The *grey Shapley value* $\Phi' : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is defined by

$$\Phi'(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w) \in \frac{1}{n!}[\sum_{\sigma \in \Pi(N)} m^\sigma(\underline{w}), \sum_{\sigma \in \Pi(N)} m^\sigma(\overline{w})].$$

for each $w \in SM\mathcal{G}G^N$ [11].

Let $S \in 2^N \setminus \{\emptyset\}$, $w \in \mathcal{G}(\mathbb{R})$ and let u_S represent the classical unanimity game based on S [22]. The cooperative grey game $< N, wu_S >$ is defined by $(wu_S)(T) = wu_S(T)$ for each $T \in 2^N \setminus \{\emptyset\}$. The Shapley value is expressed as follows:

$$\Phi'_i(wu_S) = \begin{cases} w/|S|, & i \in S, \\ [0,0], & i \notin S. \end{cases}$$

We denote by $K\mathcal{G}G^N$ the additive cone generated by the set

$$K = \left\{w_S u_S | S \in 2^N \setminus \{\emptyset\}, w_S \in \mathcal{G}(\mathbb{R})\right\}.$$

Consequently, each element of the cone is a finite sum of elements from K. It is important to observe that $K\mathcal{G}G^N \subset SM\mathcal{G}G^N$, and we axiomatically characterize the restriction of the grey Shapley value to the cone $K\mathcal{G}G^N$ [11]. Additionally, it is noteworthy that the classical game is denoted by v, while the grey game is represented by w.

A table of the terminology is given in Table 1.

Table 1. Table of terminology.

Symbol	Meaning
N	The set of players
2^N	The collection of subsets of N
$\mathbb{R}$	Real numbers
S	The coalition: each element of 2^N
$v(S)$	The value of coalition S
v	Classical cooperative game
G^N	The set of classical cooperative games
w	Cooperative grey game
$\mathcal{G}(\mathbb{R})^N$	The set of grey payoff vectors
$\mathcal{G}G^N$	The family of all cooperative grey games
Φ'	The grey Shapley value
K	The base of the cooperative grey game
$K\mathcal{G}G^N$	The additive cone generated by the set K
$SM\mathcal{G}G^N$	The class of all grey size monotonic games

3. The Characterization

In this section, we present a new characterization of the grey Shapley value as defined in cooperative grey games. We propose that this characterization is based on certain axioms, lemmas, and theorems.

The (single-valued) grey solution is defined as a function $f : \mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$, which assigns an $|N|$-dimensional real vector to each grey game on N. This vector represents the distribution of grey payoffs that can be achieved through cooperation among the individual players within the game.

We first state the well-known axioms for solutions $f : \mathcal{G}G^N \to I(\mathbb{R})^N$.

Axiom 1 **($\mathcal{G}$-EFF)**: For all $w \in \mathcal{G}G^N$, it holds that

$$\sum_{i \in N} f_i(w) = w(N).$$

A player $i \in N$ is considered a $\mathcal{G}$-null player in $w \in \mathcal{G}G^N$ if $w(S) = w(S \backslash \{i\})$ for all $S \subset N$.

Axiom 2 **($\mathcal{G}$-GL)**: For all $\omega, w \in \mathcal{G}G^N$ and $i \in N$ such that $\omega(N) = v(N)$ and $f_i(\omega) \succ f_i(w)$, and there is some $j \in N$ such that

$$f_j(\omega) \prec f_j(w).$$

A player $i \in N$ is a $\mathcal{G}$-null player in $w \in \mathcal{G}G^N$ if $w(S) = w(S \backslash \{i\})$ for all $S \subset N$.

Axiom 3 **($\mathcal{G}$-NULL):** If $i \in N$ is a $\mathcal{G}$-null player in the game $w \in \mathcal{G}G^N$, then $f_i(w) \in [0, 0]$. If

$$w(S \cup \{i\}) = w(S \cup \{j\})$$

for all $S \subseteq N \backslash \{i, j\}$, then the two players $i, j \in N$ are referred to as $\mathcal{G}$-symmetric in $w \in \mathcal{G}G^N$.

Axiom 4 **($\mathcal{G}$-SYM):** If i and j are $\mathcal{G}$-symmetric in $w \in \mathcal{G}G^N$, then $f_i(w) = f_j(w)$.

Axiom 5 **($\mathcal{G}$-ADD):** If

$$f(\omega + w) = f(\omega) + f(w)$$

for $\forall \omega$, then $w \in \mathcal{G}G^N$, where $(\omega + w) \in \mathcal{G}G^N$ is given by

$$(\omega + w)(S) = \omega(S) + w(S)$$

for all $S \subseteq N$.

Axiom 6 **($\mathcal{G}$-DM):** For $\forall \omega, w \in \mathcal{G}G^N$ and $i, j \in N$ such that

$$\omega(S \cup \{i\}) - \omega(S \cup \{j\}) = w(S \cup \{i\}) - w(S \cup \{j\})$$

for all $S \subseteq N \backslash \{i, j\}$,

$$f_i(\omega) - f_j(\omega) = f_i(w) - f_j(w)$$

The concept of fairness demands that the payoffs of two players change by the same amount whenever a symmetric game is added. This requirement is plausible because the addition of such a game does not alter the difference in productivity between these players, as measured by their marginal contributions. The concept of differential marginality directly imposes this condition, asserting that productivity differences should result in equal differences in payoffs. In other words, the difference in payoffs between two players should be determined solely by their differences in productivity.

Before presenting the theorem, let us recall the meaning of the axioms we have employed. $\mathcal{G}$-**GL** ensures that any gains or losses in coalition values are distributed proportionally among members, maintaining balance and reflecting the principles of equity.

$\mathcal{G}$-**NULL** guarantees that a player who does not contribute to any coalition receives a payoff of zero, adhering to the fairness criterion. $\mathcal{G}$-**DM** addresses the consistency of payoff distribution based on the marginal contributions of players, ensuring that differences in productivity are accurately reflected in their respective payoffs.

Theorem 1. *$f : K\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is a unique solution satisfying the properties of $\mathcal{G}$-**GL**,$\mathcal{G}$-**NULL**, and $\mathcal{G}$-**DM**.*

Proof. It is clear that the Shapley value satisfies $\mathcal{G}$-**NULL** and $\mathcal{G}$-**GL**. Additionally, ref. [11] shows that the Shapley value satisfies $\mathcal{G}$-**DM**. $\square$

Let the value g obey $\mathcal{G}$-**GL**,$\mathcal{G}$-**NULL**, and $\mathcal{G}$-**DM**. If $|N| = 1$, then $\mathcal{G}$-**NULL** already entails $g = \Phi'$.

Consider now $|N| > 1$. For $w \in K\mathcal{G}G^N$, set

$$T_1(w) := \{T \subseteq N \mid |T| > 1 \text{ and } \lambda_T(w) \neq 0\}. \tag{3}$$

For $w \in K\mathcal{G}G^N$ and $T \in T_1(w)$, let $w^T \in K\mathcal{G}G^N$ be given by

$$w^T := w - \lambda_T(w) \cdot \left(v_T - |T|^{-1} \cdot \sum_{i \in T} v_{\{i\}}\right). \tag{4}$$

By using $\mathcal{G}$-**DM**, this implies

$$g_i(w) - g_i\left(w^T\right) = g_j(w) - g_j\left(w^T\right) \tag{5}$$

for all $i, j \in T$ and all $i, j \in N \backslash T$.

We show that $g = \Phi'$ by induction on $|T_1(w)|$.

Induction basis: If $|T_1(w)| = 0$ for $w \in K\mathcal{G}G^N$, the claim follows from $\mathcal{G}$-**NULL**. We now turn to $|T_1(w)| = 1$, i.e., $T_1(w) = \{T\}$ for some $T \subseteq N$, $|T| > 1$, i.e.,

$$w = \rho \cdot v_T + \sum_{k \in N} \rho_k \cdot v_{\{k\}}$$

for some $\rho \in \mathbb{R} \backslash \{0\}$ and $\rho_k \in \mathbb{R}$, $k \in N$. By using $\mathcal{G}$-**DM** and by using the definition of the grey Shapley value, we have

$$g_i(w) = g_i\left(w^T\right) = \Phi'_i(w) \tag{6}$$

for all $i \in N \backslash T$.

Suppose $g_i(w) \succ \Phi'_i(w)$ for some $i \in T$. By using $\mathcal{G}$-**NULL** and by using the definition of the grey Shapley value, then

$$g_i(w) \succ \Phi'_i(w) = \Phi'_i\left(w^T\right) = g_i\left(w^T\right).$$

By $w^T(N) = w(N)$, $\mathcal{G}$-**GL** , and (6), there is some $j \in T$ such that $g_j(w) \prec g_j(w^T)$, contradicting (5). Analogously, one excludes $g_i(w) \prec \Phi'_i(w)$ for $i \in T$. Hence, $g_i(w) = \Phi'_i(w)$ for all $i \in T$. By (6), we thus have $g(w) = \Phi'(w)$.

Induction hypothesis: $g(w) = \Phi'(w)$ for all $w \in K\mathcal{G}G^N$ such that $|T_1(w)| \leq k$.

Induction step: Let $w \in K\mathcal{G}G^N$ be such that $|T_1(w)| = k + 1 > 1$. By (3) and (4), we have $\left|T_1\left(w^T\right)\right| = |T_1(w)| - 1$, and therefore

$$g\left(w^T\right) = \Phi'\left(w^T\right) = \Phi'(w) \tag{7}$$

for all $T \in T_1(w)$.

By (5) and (7), we have

$$g_i(v) - \Phi'_i(v) = g_j(v) - \Phi'_j(v) \tag{8}$$

for all $i, j \in N$ such that there is some $T \in T_1(w)$ with $i, j \in T$ or $i, j \in N \backslash T$. We now deal with players $i, j \in N$, for whom there is no such $T \in T_1(w)$.

Case 1: $T_1(w) \neq \{T, N\backslash T\}$ for all $T \subseteq N$, $T \neq \emptyset$, $N\backslash T \neq \emptyset$. One of the following conditions is met:

(i) There are $S, T \in T_1(w)$, $S \neq T$ such that $S \cap T \neq \emptyset$.

(ii) There are $S, T \in T_1(w)$, $S \neq T$ such that $S \cup T \neq N$.

We note that these subcases may not be mutually exclusive.

Case 1 (i): Since $S \neq T$, w.l.o.g., $S\backslash T \neq \emptyset$. Let $i \in S \cap T, j \in S\backslash T, k \in T$, and $l \in N\backslash(S \cup T)$. Note that such an l might not exist. By (8), we have

$$\begin{aligned} g_l(w) - \Phi'_l(w) &\overset{j,l \notin T}{=} g_j(w) - \Phi'_j(w) \overset{i,j \in S}{=} g_i(w) - \Phi'_i(w) \\ &\overset{i,k \in T}{=} g_k(w) - \Phi'_k(w), \end{aligned} \tag{9}$$

where the leftmost equation applies only if $N\backslash(S \cup T) \neq \emptyset$.

Case 1 (ii): Since $S \neq T$, w.l.o.g., $S\backslash T \neq \emptyset$. Let $l \in S \cap T, j \in S\backslash T, k \in T\backslash S$, and $i \in N\backslash(S \cup T)$. Note that such k or l might not exist. By (8), we have

$$\begin{aligned} g_l(w) - \Phi'_l(w) &\overset{j,l \in S}{=} g_j(w) - \Phi'_j(w) \overset{i,j \notin T}{=} g_i(w) - \Phi'_i(w) \\ &\overset{i,k \notin S}{=} g_k(w) - \Phi'_k(w), \end{aligned} \tag{10}$$

where the leftmost or the rightmost equation applies only if $S \cap T \neq \emptyset$ or $T\backslash S \neq \emptyset$.

Case 2: $T_1(w) = \{T, N\backslash T\}$ for some $T \subseteq N$, $T \neq \emptyset$, $N\backslash T \neq \emptyset$.

Fix $i \in T$ and $j \in N\backslash T$. We have

$$w = \rho_T \cdot u_T + \rho_{N\backslash T} \cdot u_{N\backslash T} + \sum_{k \in N} \rho_k \cdot u_{\{k\}}$$

for some $\rho_T, \rho_{N\backslash T} \in \mathbb{R}\backslash\{0\}$, $\rho_k \in \mathbb{R}$, $k \in N$. Let $w \in K\mathcal{G}G^N$ be given by

$$w = \rho_T \cdot u_T + \rho_{N\backslash T} \cdot u_{((N\backslash T)\backslash\{j\})\cup\{i\}} + \sum_{k \in N} \rho_k \cdot u_{\{k\}}$$

Note that $T_1(\omega) = \{T, ((N\backslash T)\backslash\{j\}) \cup \{i\}\}$ and $T \cap ((N\backslash T)\backslash\{j\}) \cup \{i\} = \{i\}$, i.e., (*) w is as in Case 1 (i). Thus, by using $\mathcal{G}$-**DM** and (*), we have

$$\begin{aligned} g_i(w) - g_j(w) = g_i(\omega) - g_j(\omega) &= \Phi'_i(\omega) - \Phi'_j(\omega) \\ &= \Phi'_i(w) - \Phi'_j(w) \end{aligned} \tag{11}$$

By (8)–(11), we have

$$g_i(w) - \Phi'_i(w) = g_j(w) - \Phi'_j(w) \tag{12}$$

for all $i, j \in N$.

Suppose $g_i(w) \succ \Phi'_i(w)$ for some $i \in N$. Then,

$$g_i(w) \succ \Phi'_i(w) = \Phi'_i\left(w^T\right).$$

By $w^T(N) = w(N)$ and using $\mathcal{G}$-**GL**, there is some $j \in N$ such that $g_j(w) \prec g_j(w^T) = \Phi'_j(w)$, contradicting (12). Analogously, one excludes $g_i(w) \prec \Phi'_i(w)$ for $i \in N$. Hence, $g(w) = \Phi'(w)$.

4. An Application

In this section, we present some interesting applications concerning the grey game. Finally, we give the calculation of the grey Shapley value.

In the case discussed within the scope of this study, there are three different milk production enterprises. Milk is a product that spoils quickly and is difficult to store. Enterprises must have hygienic conditions to store the milk produced. The storage area must comply with high technological possibilities and cooling conditions. The milk-producing enterprises in this study do not have the desired storage conditions. Milk-producing enterprises must deliver the milk they produce to a specific collection center. The milk collected at the collection center is stored for a certain period and then taken to milk processing centers. Storage facilities are available at the collection center under desired conditions. In addition to storage, businesses have limited opportunities to sell the milk they produce themselves. Since the milk collected at the collection center can be sold in bulk, it can be sold at a higher price. For all these reasons, three different milk production enterprises will deliver the milk they produce to the collection center. The costs that the three businesses will incur if they deliver milk to the collection center and each other are listed below.

The numbers in Figure 1 represent the following businesses:

1: First Business

2: Second Business

3: Third Business

0: Collection Center

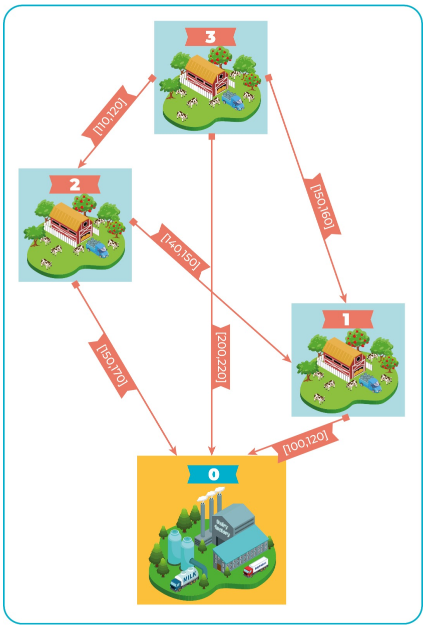

Figure 1. Example of an application.

The cost of the first enterprise delivering the milk to the collection center is between 100 and 120 units. The cost of the second enterprise delivering the milk to the collection center is between 150 and 170 units. The cost of the third enterprise delivering the milk to the collection center is between 200 and 220 units. The cost of the second enterprise delivering milk to the first enterprise is between 240 and 270 units. The cost of the third enterprise delivering milk to the first enterprise is between 250 and 280 units. The cost of the third enterprise delivering the milk to the second enterprise is between 260 and 290 units. Within the specified costs, it will be determined which methods the enterprises should use to deliver the milk to the collection center.

The example that follows illustrates the transformed form of the previously discussed application, which has been reimagined as a cooperative grey game theory framework.

Example 1. *Let* $< N, w >$ *be a cooperative grey game with* $N = \{1, 2, 3\}$ *and*

$$\begin{aligned} w(1) &\in [100, 120] \\ w(2) &\in [150, 170] \\ w(3) &\in [200, 220] \\ w(12) &\in [240, 270] \\ w(13) &\in [250, 280] \\ w(23) &\in [260, 290] \\ w(N) &\in [350, 390] \end{aligned}$$

Then, the grey marginal vectors are given in Table 2, where $\sigma : N \to N$ *is identified with* $(\sigma(1), \sigma(2), \sigma(3))$.

Table 2. Grey marginal vectors.

σ	$m_1^\sigma(w)$	$m_2^\sigma(w)$	$m_3^\sigma(w)$
$\sigma_1 = (1,2,3)$	$m_1^{\sigma_1}(w) \in [100, 120]$	$m_2^{\sigma_1}(w) \in [140, 150]$	$m_3^{\sigma_1}(w) \in [110, 120]$
$\sigma_2 = (1,3,2)$	$m_1^{\sigma_2}(w) \in [100, 120]$	$m_2^{\sigma_2}(w) \in [100, 110]$	$m_3^{\sigma_2}(w) \in [150, 160]$
$\sigma_3 = (2,1,3)$	$m_1^{\sigma_3}(w) \in [140, 150]$	$m_2^{\sigma_3}(w) \in [150, 170]$	$m_3^{\sigma_3}(w) \in [60, 70]$
$\sigma_4 = (2,3,1)$	$m_1^{\sigma_4}(w) \in [40, 50]$	$m_2^{\sigma_4}(w) \in [150, 170]$	$m_3^{\sigma_4}(w) \in [160, 170]$
$\sigma_5 = (3,1,2)$	$m_1^{\sigma_5}(w) \in [50, 60]$	$m_2^{\sigma_5}(w) \in [100, 110]$	$m_3^{\sigma_5}(w) \in [200, 220]$
$\sigma_6 = (3,2,1)$	$m_1^{\sigma_6}(w) \in [90, 100]$	$m_2^{\sigma_6}(w) \in [60, 70]$	$m_3^{\sigma_6}(w) \in [200, 220]$

Table 2 illustrates the grey marginal vectors of the cooperative grey game in Example 1. The average of the six grey marginal vectors is the grey Shapley value of this game, which can be shown as follows:

$$f(w) \in ([86\frac{2}{3}, 100], [116\frac{2}{3}, 130], [146\frac{2}{3}, 160]).$$

According to the above results obtained as a result of the application of the game theory approach to the case of dairy enterprises, the cost of the first enterprise decreases from the $[100, 120]$ unit range to the $[86\frac{2}{3}, 100]$ unit range, the cost of the second enterprise decreases from the $[150, 170]$ unit range to the $[116\frac{2}{3}, 130]$ unit range, and the cost of the third enterprise decreases from the $[200, 220]$ unit range to the $[146\frac{2}{3}, 160]$ unit range. As can be seen, the costs of all three enterprises decrease when they cooperate.

Remark 1. *The background of the example has been expanded to provide a clearer explanation of the challenges faced by milk production enterprises, including the logistical and cost-related complexities of delivering milk to a central collection center. A more comprehensive description of the cooperative framework and its implications for cost reduction has been incorporated. The computation of grey marginal vectors and their role in determining the grey Shapley value have been detailed in a* *step-by-step manner. Each calculation, including the interpretation of grey intervals, has been explicitly outlined to ensure that the methodology is easy to follow. The broader significance of the results has been discussed, highlighting how the grey Shapley value facilitates fair cost allocation and promotes cooperation among enterprises.*

Remark 2. *The sensitivity analysis conducted on the grey Shapley value provides valuable insights into the robustness and adaptability of the model under varying cost scenarios. The initial analysis establishes the cost-sharing framework, demonstrating how cooperative strategies reduce individual costs for the enterprises involved. To test the robustness of these results, two sensitivity scenarios were introduced. In the first scenario, a 10 percent reduction in the lower bounds of the cost ranges was considered, while the second scenario involved a 10 percent increase in the upper bounds. The resulting grey Shapley values in both scenarios exhibit proportional adjustments, maintaining the fairness and consistency of the allocation despite changes in input intervals. This demonstrates the model's resilience to moderate variations in cost estimates and highlights its practical applicability in real-world settings where uncertainty in data is prevalent. These findings further validate the grey Shapley value as a robust and reliable tool for equitable decision-making in uncertain cooperative environments.*

5. Conclusions and Outlook

This study aimed to provide an axiomatic characterization of the grey Shapley value using the above axioms. We also consider that these axioms uniquely characterize the grey Shapley value.

This paper surveys cooperative grey game theory in the literature, and it has two special subtraction operators. These operators are Moore's subtraction operator and the special subtraction operator. There are now many different axiomatic characterizations of the grey Shapley value using the special subtraction operator. Shortly, we plan to provide new axiomatic characterizations of the grey Shapley value using Moore's subtraction operator.

The dividends of the classical game are introduced by [25]. The dividends are useful to characterize the classical Shapley value. This concept can be extended to cooperative games where their coalition values are compact real greys. The grey Shapley value can be characterized using grey dividends. Finally, this idea could be a promising area for future studies.

Regular and accurate calculation and analysis of costs, especially in production enterprises, increase the profitability of enterprises, provide better planning and forecasting for the future, and increase productivity. Therefore, enterprises should continuously control their costs by making cost analyses and conduct research to reduce costs. Enterprises operating in the same sector can reduce costs by cooperating among themselves. In this study, changes in costs as a result of cooperation between enterprises were investigated by using the game theory approach. Three different enterprises engaged in milk production were considered in this study. A study was conducted on the costs incurred by these enterprises while delivering the milk they produce to the collection center. According to the results of the game theory approach used in this study, it was determined that if the enterprises engage in milk production cooperate among themselves, the costs of all three enterprises will decrease and they can earn more profit. Based on the results of this study, it has been seen that the cooperation of enterprises in their activities within the scope of the game theory approach can reduce the costs of enterprises and make a significant positive contribution to enterprises.

It was observed that the costs of all three enterprises decreased when the enterprises transmitted the milk to the milk center collectively through each other instead of transmitting the milk separately. The cost of the first enterprise decreased by approx. 17%, the cost of the second enterprise by approx. 24%, and the cost of the third enterprise by approx. 27%.

Thanks to the decrease in costs, the sustainability of the companies will increase and all segments of the society, especially consumers, will benefit. Consumers can buy cheaper products, as the decrease in costs may lead to a decrease in prices. With the decrease in production costs, competition increases and development in local economies may occur. As the long-term financial performance of enterprises increases, the development of local economies can be achieved. With the decrease in costs and increase in sustainability, an increase in public budgets may occur and the state may give more importance to environmental policies.

Author Contributions: Writing—original draft, M.G., M.S.Ö. and O.P.; Writing—review & editing, O.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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