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An Improvement of the Lower Bound on the Maximum Number of Halving Lines for Sets in the Plane with an Odd Number of Points

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Abstract: In this paper, we give examples that improve the lower bound on the maximum number of halving lines for sets in the plane with 35, 59, 95, and 97 points and, as a consequence, we improve the current best upper bound of the rectilinear crossing number for sets in the plane with 35, 59, 95, and 97 points, provided that a conjecture included in the literature is true. As another consequence, we also improve the lower bound on the maximum number of halving pseudolines for sets in the plane with 35 points. These examples, and the recursive bounds for the maximum number of halving lines for sets with an odd number of points achieved, give a new insight in the study of the rectilinear crossing number problem, one of the most challenging tasks in Discrete Geometry. With respect to this problem, it is conjectured that, for all n multiples of 3, there are 3-symmetric sets of n points for which the rectilinear crossing number is attained.

Keywords: discrete geometry; halving lines; rectilinear crossing number; optimization

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1. Introduction

The problem of finding the maximum number of halving lines for subsets of the plane with n points (h_n, see the definition below) has been widely treated in the Discrete Geometry literature.

In an informal way, a halving line of a set P is a line joining two points of P and that equally distributes the rest of the points of P in the two open half planes defined by the line (see the formal definition below).

The first asymptotic lower bound for the maximum number of these halving lines was given by Erdõs et al. in 1973 (see [1]). They achieved the bound $h_n \geq \frac{n}{4}\log_2(\frac{n}{3})$. Later, this bound was improved to $h_n \geq n\log_4(\frac{2n}{3})$ by Eppstein (see [2]).

More recently, Tóth found a lower bound that is asymptotically better than the bounds mentioned above, $h_n \geq \frac{n}{2}e^{0.744\sqrt{\log(\frac{n}{2})}-2.7}$ (see [3]). The constant in the exponent was improved by Nivasch [4].

The to-date best upper bound of h_n is $O(n^{\frac{4}{3}})$, according to Dey [5], with an improvement to the error term in [6].

A problem related to the halving line problem is the rectilinear crossing number problem. It aims to find the minimum number of crossings for planar sets of n points if every two points of the set are connected with a segment (the rectilinear crossing number is $cr(n)$; see the formal definition below).

The attempts to find sets minimizing the number of crossings have carried out interesting conjectures about the properties of these sets. Two of these properties are the three-decomposability and the three-symmetry. This last property is about invariance of the set with respect to rotations of angles $\frac{2}{3}\pi$, $\frac{4}{3}\pi$. The conjecture linking the three-symmetry with the rectilinear crossing number problem is that there are three-symmetric sets of n points that attains the rectilinear crossing number for every n multiple of 3; see [7] for more details.

A relation between the maximum number of halving lines and the rectilinear crossing number is given by the following conjecture of [8].

Conjecture 1. *Every set attaining $cr(n)$ maximizes the number of halving lines.*

The maximum number of halving lines is known for $n \leq 27$, and $cr(n)$ is known for $n \leq 27$ and $n = 30$ (see [9]). A table with the gaps between the best lower bound and the best upper bound of h_n for $28 \leq n \leq 33$ can be found in [9] and [10].

An improvement to the best lower bound of h_{32} from some of the authors of the present paper could yield the refutation of the conjecture (see [11]).

In this paper, we achieve an improvement to the current best lower bound of h_{35}, h_{59}, h_{95}, and h_{97}. These results give more evidence against the conjecture, but they do not imply its refutation by themselves. The result for $n = 35$ also implies an improvement by one of the current best lower bound of the maximum number of halving pseudolines for sets in the projective plane with 35 points ($\tilde{h}_{35}$; see [12] for a formal definition of halving pseudolines).

The examples that give the shifted lower bounds have been obtained by removing two points of the sets that attain the current best lower bound of h_{37}, h_{61}, h_{97}, and h_{99} included in the rectilinear crossing number web page by Aichholzer (see [10]). They are inspired by a relation between h_n and h_{n+2} included in this paper.

These kind of recursive bounds for h_n are also treated in [11] for n, an even number, and may give new insights for the task of finding h_n.

We give the following definitions.

Definition 1. *Given a finite set of points in the plane P, assume that we join each pair of points of P with a straight line segment. The rectilinear crossing number of P ($cr(P)$) is the number of intersections out of the vertices of said segments. The rectilinear crossing number of n ($cr(n)$) is the minimum of $cr(P)$ over all the sets P with n points.*

Definition 2. *Given a set of points $P = \{p_1, ..., p_n\}$, a k-edge of P is a line R that joins two points of P and leaves k points of P in one of the open half planes. We call it the k-half plane.*

Definition 3. *Given a set of points $P = \{p_1, ..., p_n\}$, a halving line of P is a line R that joins two points of P and leaves $\left[\frac{n-2}{2}\right]$ points of P in one of the open half planes (so a halving line of P partitions P in two equally sized or almost equally sized subsets).*

Definition 4. *Given a set of points $P = \{p_1, ..., p_n\}$, the graph of the halving lines of P is the graph $G = (V, E)$, with $V = P$ and $\{p_i, p_j\} \in E$, if the line that joins p_i; p_j is a halving line of P.*

Notation: pq stands for the line joining the points p, q.

We assume that all the sets in the paper are in general position (no three points in a line).

The outline of the rest of the paper is as follows: In Section 2, we give the upper bound and the lower bound of h_n in terms of h_{n+2}, for n, an odd number. In Section 3 we give the examples that improve the lower bound of h_{35}, h_{59}, h_{95}, and h_{97}. In Section 4, we give a lower bound of h_n in terms of h_{n+1}, which implies an improvement in the multiplicative constant in the asymptotic best lower bound of h_n for odd values of n, and in Section 5, we give some concluding remarks.

2. The Relation Between h_n, h_{n+2}

Let us see a result similar to the one included in Lemma 2.3 of [11] for an even n, but more generally because we do not need additional conditions assumed there.

Proposition 1. *For n an odd number, $n \geq 5$, it is satisfied that $h_{n+2} \geq h_n + 5$.*

Proof. Consider a set $P = \{p_1, ..., p_n\}$ in which h_n is attained, for n as an odd number, $n \geq 5$. Since $n \geq 5$, there exist $\frac{n-5}{2}$-edges of P. Then, we take one of them, R', and define a line R parallel to R', in the $\frac{n-5}{2}$-half plane, so that R does not contain any point of P. If we consider two points, p_{n+1} and p_{n+2}, in R, such that p_{n+1} is in the intersection of the upper half planes defined by halving lines of P and p_{n+2} is in the intersection of the lower half planes defined by halving lines of P, then we have that the halving lines of P are still halving lines of $Q := P \cup \{p_{n+1}, p_{n+2}\}$ because they separate the points p_{n+1}, p_{n+2}. See Figure 1 for the case $n = 5$. We also have that R' is now a halving line of Q, because it leaves $\frac{n-5}{2} + 2 = \frac{(n+2)-3}{2}$ points of Q in one half plane. Moreover, since $p_{n+1}p_{n+2}$ is not a halving line of Q, because it is parallel to a halving line of Q (R'), there are at least other four halving lines of Q, two containing p_{n+1} and two containing p_{n+2}, because they must have an even (and positive) degree in the graph of the halving lines of Q, as a consequence of Corollary 2.6 of [1]. This implies that $h_{n+2} \geq h(Q) \geq h(P) + 1 + 4 = h_n + 5$, as desired. □

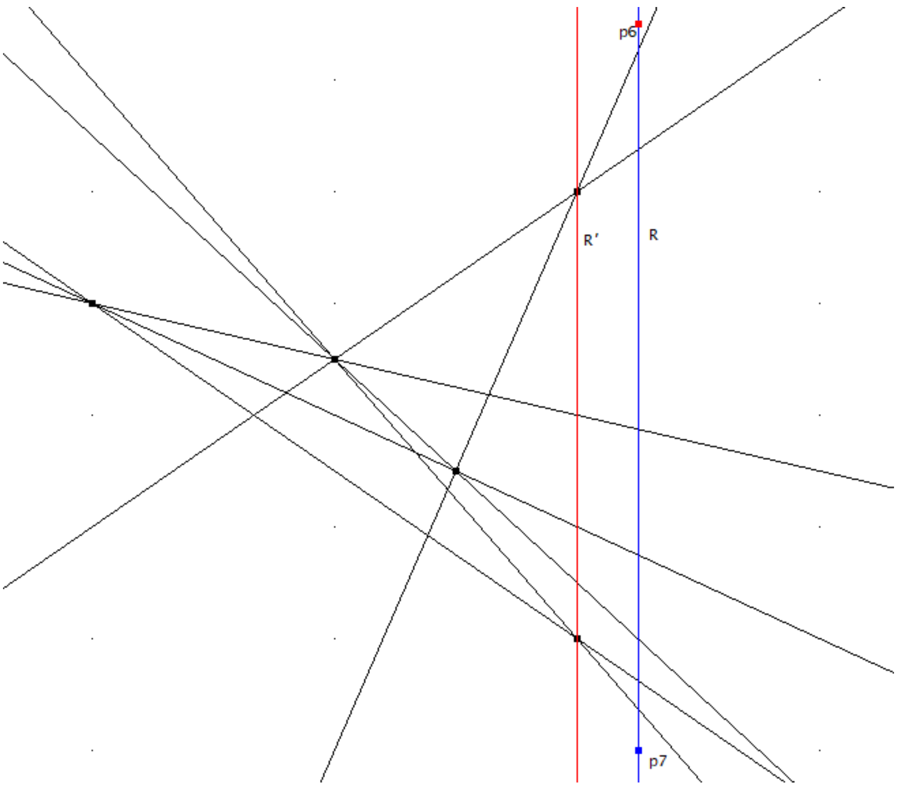

Figure 1. Graphical representation of the proof of Proposition 1 for $n = 5$.

Now we see the lower bound of h_n in terms of h_{n+2}

Proposition 2. *For n an odd number, $n \geq 3$, it is satisfied that*

$$h_n \geq \frac{7n^2+4n-3}{8n^2+24n+16}h_{n+2} - \frac{n^2+4n+3}{4n^2+12n+8}.$$

Proof. Consider a set $P = \{p_1, ..., p_n\}$ in which h_n is attained, for n as an odd number, $n \geq 5$. Then, we have that the number of pairs of points of P with one of the points in the $\frac{n-3}{2}$-half plane, and the other one in the $\frac{n-1}{2}$-half plane of some halving line of P, or in the $\frac{n-1}{2}$-half plane of some halving line of P or in the $\frac{n+1}{2}$-half plane of some $\frac{n-5}{2}$-edge of P, allowing repetitions, is

$$(\frac{n-3}{2})(\frac{n-1}{2})h_n + \binom{\frac{n-1}{2}}{2}h_n + \binom{\frac{n+1}{2}}{2}e_{\frac{n-5}{2}}(P)\,,$$

where $e_{\frac{n-5}{2}}(P)$ is the number of $\frac{n-5}{2}$-edges of P, so there exists a pair of points of P, say $p_{n-1},\ p_n$, that belongs to s of said half planes, with

$$s \geq \frac{\frac{n-3}{2}\frac{n-1}{2}\,h_n + \binom{\frac{n-1}{2}}{2}\,h_n + \binom{\frac{n+1}{2}}{2}\,e_{\frac{n-5}{2}}(P)}{\binom{n}{2}}.$$

If we remove p_{n-1}, p_n, then we obtain a set $Q := \{p_1,\ ...,\ p_{n-2}\}$ such that the halving lines and $\frac{n-5}{2}$-edges corresponding to the s half planes become halving lines of Q: the halving lines for which we have removed a point in the $\frac{n-3}{2}$-half plane now have $\frac{n-3}{2} - 1 = \frac{(n-2)-3}{2}$ points of Q in one of the half planes, the halving lines for which we have removed two points in the $\frac{n-1}{2}$-half plane now have $\frac{n-1}{2} - 2 = \frac{(n-2)-3}{2}$ points of Q in one of the half planes, and the $\frac{n-5}{2}$-edges for which we have removed two points in the $\frac{n+1}{2}$-half plane still have $\frac{n-5}{2} = \frac{(n-2)-3}{2}$ points of Q in the other half plane, so they are halving lines of Q. Thus,

$$\begin{aligned} h_{n-2} &\geq h(Q) \geq s \geq \frac{\frac{n-3}{2}\frac{n-1}{2}\,h_n + \binom{\frac{n-1}{2}}{2}\,h_n + \binom{\frac{n+1}{2}}{2}\,e_{\frac{n-5}{2}}(P)}{\frac{n^2-n}{2}} \\ &= \frac{3\,\frac{n-3}{2}\frac{n-1}{2}h_n + \frac{n+1}{2}\frac{n-1}{2}\,e_{\frac{n-5}{2}}(P)}{n^2-n}, \end{aligned}$$

where $h(Q)$ is the number of halving lines of Q.

By Corollary 1 of [9], we have that $e_{\frac{n-5}{2}}(P) \geq \frac{1}{2}h_n - 1$; so,

$$h_{n-2} \geq \frac{3\,\frac{n^2-4n+3}{4}h_n + \frac{n^2-1}{8}h_n - \frac{n^2-1}{4}}{n^2-n} = \frac{7\,n^2 - 24\,n + 17}{8\,(n^2-n)}h_n - \frac{n^2-1}{4\,(n^2-n)},$$

and we obtain the desired result by changing n by $n+2$. □

The multiplicative factor of the bound has limit $\frac{7}{8}$ as $n \to \infty$. Since it is close to 1, it gives us the following intuition: by removing two points, in all the possible ways, of a set for which h_{n+2} is attained, we can obtain a set of n points with many halving lines. We apply this procedure in the following section to improve the current best lower bound of h_{35}, h_{59}, h_{95}, and h_{97}.

3. The Improvement of the Lower Bound

The inequality of Proposition 2 is a worst-case one. In particular cases, it may be improved to obtain better lower bounds for h_n when n is an odd number. Concretely, to achieve an improvement for $n = 35$, we use the example of [10] of a set with 37 points that gives the current best upper bound of $cr(37)$, with 148 halving lines. The set is

$$P = \left\{ \begin{array}{rlllll} (3217,5509), & (3261,5598), & (3134,5775), & (3158,5661), & (3143,5742), \\ (3617,5403), & (3143,5744), & (3140,5767), & (3052,5889), & (2995,5981), \\ (3039,5915), & (3277,5343), & (3101,5305), & (3091,5283), & (2819,6251), \\ (2789,4636), & (3054,5005), & (2562,4321), & (2454,4168), & (2046,3453), \\ (0,0), & (1631,2754), & (205,346), & (1924,3251), & (3363,5471), \\ (3438,5377), & (3436,5375), & (3444,5380), & (2867,6177), & (3542,5433), \\ (3582,5413), & (3695,5410), & (3664,5410), & (3791,5417), & (3896,5423), \\ (3682,5409), & (3265,5594) & & & \end{array} \right\}$$

By the adequate removal of two points, we obtain the following result:

Proposition 3. *It is satisfied that $h_{35} \geq 137$*

Proof. If we remove the points $p_5 = (3143,5742)$, $p_{26} = (3438,5377)$ from the set P defined above, then we obtain a set Q with 35 points and 137 halving lines, so $h_{35} \geq h(Q) = 137$, as desired.

The set is

$$Q = \left\{ \begin{array}{lllll} (3217,5509), & (3261,5598), & (3134,5775), & (3158,5661), & (3617,5403), \\ (3143,5744), & (3140,5767), & (3052,5889), & (2995,5981), & (3039,5915), \\ (3277,5343), & (3101,5305), & (3091,5283), & (2819,6251), & (2789,4636), \\ (3054,5005), & (2562,4321), & (2454,4168), & (2046,3453), & (0,0), \\ (1631,2754), & (205,346), & (1924,3251), & (3363,5471), & (3436,5375), \\ (3444,5380), & (2867,6177), & (3542,5433), & (3582,5413), & (3695,5410), \\ (3664,5410), & (3791,5417), & (3896,5423), & (3682,5409), & (3265,5594) \end{array} \right\}$$

□

Remark 1.

1. *A program that calculates the 127 halving lines is available upon petition to the authors;*
2. *Another three sets Q with 35 points and 137 halving lines can be obtained by removing the following pairs of points from P: $\{p_7, p_{26}\}$, $\{p_5, p_{27}\}$, $\{p_7, p_{27}\}$;*
3. *The crossing number of the four obtained sets with 137 halving lines is 18,810;*
4. *Since the set P attaining the current best upper bound of $cr(35)$ satisfies that $cr(P) = 18{,}808$, we have that if conjecture 1 was true, then $cr(35) \leq 18{,}807$.*
5. *Since $\tilde{h}_n \geq h_n$, Proposition 3 implies that $\tilde{h}_{35} \geq 137$. This improves by one the current best lower bound of $\tilde{h}_{35}$ included in [12] (it is conjectured that $\tilde{h}_n = h_n$).*

Now, to obtain an improvement for $n = 59$, we use the example from [10] of a set with 61 points that gives the current best upper bound of $cr(61)$, with 302 halving lines and remove two points from it. The set is

$$P = \left\{ \begin{array}{rllll} (1,024,145; 0), & (0; 83), & (521,191; 600,976), & (521,183; 600,677), \\ (521,077; 596,217), & (519,335; 558,846), & (519,148; 554,877), & (518,980; 551,154), \\ (518,511; 541,832), & (519,285; 519,786), & (519,330; 513,554), & (513,980; 449,368), \\ (513,689; 445,654), & (519,584; 404,307), & (519,984; 401,339), & (523,099; 379,101), \\ (524,700; 375,014), & (506,277; 334,803), & (505,146; 331,878), & (504,722; 331,393), \\ (527,285; 313,597), & (527,606; 313,329), & (498,443; 293,741), & (498,433; 293,734), \\ (444,675; 258,623), & (442,279; 255,702), & (420,822; 239,424), & (420,124; 238,898), \\ (284,110; 161,531), & (278,189; 158,159), & (91,228; 51,881), & (30,278; 17,263), \\ (136; 160), & (100,793; 57,127), & (154,890; 87,712), & (210,921; 117,721), \\ (369,540; 203,129), & (383,671; 210,077), & (489,888; 263,627), & (493,067; 263,996), \\ (493,448; 263,954), & (493,979; 264,133), & (529,456; 278,173), & (557,740; 289,242), \\ (558,417; 289,507), & (559,997; 286,271), & (560,022; 286,228), & (587,194; 248,097), \\ (590,198; 246,536), & (622,468; 233,995), & (665,215; 217,390), & (667,431; 215,993), \\ (682,342; 206,149), & (744,974; 164,414), & (751,907; 160,516), & (830,364; 116,804), \\ (846,398; 107,121), & (888,013; 81,836), & (897,965; 76,012), & (998,122; 15,650), \\ (530,090; 278,445) & & & \end{array} \right\}$$

Proposition 4. *It is satisfied that* $h_{59} \geq 286$.

Proof. If we remove the points $p_{11} = (519,330; 513,554)$, $p_{41} = (493,448; 263,954)$ from the set P defined above, then we obtain a set Q with 59 points and 286 halving lines, so $h_{59} \geq h(Q) = 286$, as desired.

The set is

$$Q = \left\{ \begin{array}{rl} (1,024,145; 0), & (0; 83), \\ (521,191; 600,976), & (521,183; 600,677), \\ (521,077; 596,217), & (519,335; 558,846), \\ (519,148; 554,877), & (518,980; 551,154), \\ (518,511; 541,832), & (519,285; 519,786), \\ (513,980; 449,368), & (513,689; 445,654), \\ (519,584; 404,307), & (519,984; 401,339), \\ (523,099; 379,101), & (524,700; 375,014), \\ (506,277; 334,803), & (505,146; 331,878), \\ (504,722; 331,393), & (527,285; 313,597), \\ (527,606; 313,329), & (498,443; 293,741), \\ (498,433; 293,734), & (444,675; 258,623), \\ (442,279; 255,702), & (420,822; 239,424), \\ (420,124; 238,898), & (284,110; 161,531), \\ (278,189; 158,159), & (91,228; 51,881), \\ (30,278; 17,263), & (136; 160), \\ (100,793; 57,127), & (154,890; 87,712), \\ (210,921; 117,721), & (369,540; 203,129), \\ (383,671; 210,077), & (489,888; 263,627), \\ (493,067; 263,996), & (493,979; 264,133), \\ (529,456; 278,173), & (557,740; 289,242), \\ (558,417; 289,507), & (559,997; 286,271), \\ (560,022; 286,228), & (587,194; 248,097), \\ (590,198; 246,536), & (622,468; 233,995), \\ (665,215; 217,390), & (667,431; 215,993), \\ (682,342; 206,149), & (744,974; 164,414), \\ (751,907; 160,516), & (830,364; 116,804), \\ (846,398; 107,121), & (888,013; 81,836), \\ (897,965; 76,012), & (998,122; 15,650), \\ (530,090; 278,445) & \end{array} \right\}$$

□

Remark 2.

1. *Another four sets Q with* 59 *points and* 286 *halving lines can be obtained by removing the following pairs of points from P: $\{p_{17}, p_{41}\}$, $\{p_{11}, p_{42}\}$, $\{p_{29}, p_{45}\}$, $\{p_{25}, p_{51}\}$;*
2. *The crossing number of the five obtained sets but the second one is* 167,510. *The crossing number of the second set is* 167,526*;*
3. *Since the set P attaining the current best upper bound of $cr(59)$ satisfies that $cr(P) = 167,506$, we have that if conjecture 1 was true, then $cr(59) \leq 167{,}505$.*

Proposition 5. *It is satisfied that $h_{97} \geq 553$.*

Proof. If we remove the points p_3, p_{26} from the set P included in [10] that attains the current best lower bound for h_{99}, then we obtain a set Q with 97 points and 553 halving lines, so $h_{97} \geq h(Q) = 553$, as desired (see Appendix A). □

Remark 3.

1. *Another two sets Q with 97 points and 553 halving lines can be obtained by removing the following pairs of points from P: p_3, p_{58}, and p_{87}, p_{98}.*
2. *The crossing number of the three obtained sets except the last one is 1,292,450; the crossing number of the last set is 1,292,418.*

Proposition 6. *It is satisfied that $h_{95} \geq 539$.*

Proof. If we remove the points p_{35}, p_{70} from the set Q described in Proposition 5 that attains the bound $h_{97} \geq 553$, then we obtain a set R with 95 points and 539 halving lines, so $h_{95} \geq h(R) = 539$, as desired (see Appendix B). □

Remark 4.

1. *Another set R with 95 points and 539 halving lines can be obtained by removing the following pair of points from Q: p_{34} and p_{70}. In the same way, other two sets with 95 points and 539 halving lines can be obtained by removing the following pairs of points from the first set in Remark 3 of Proposition 5: p_{35}, p_{70} and p_{34}, p_{70}.*
2. *The crossing number of the four obtained sets is 1,187,073. The best upper bound for the minimum crossing number for sets of 95 points is 1,186,887.*

We summarize all of the results in Table 1.

Table 1. Results of the paper; $h_n^{old} \geq$ stands for the current best lower bound of h_n, $h_n^{new} \geq$ stands for the lower bound of h_n obtained in this paper, and $\widetilde{h}_n^{current} \geq$ stands for the current best lower bound for the maximum number of halving pseudolines.

$n\backslash$ Results	$h_n^{old} \geqslant$	$h_n^{new} \geqslant$	$\widetilde{h}_n^{current} \geqslant$	# of Sets Attaining $h_n^{new} \geqslant$	# of Basis Sets/References
35	136	137	136	4	1/[10]
59	285	286	286	5	1/[10]
95	532	539	546	4	2/this paper
97	546	553	558	3	1/[10]

4. An Asymptotic Improvement

In this section, we apply the technique of the proof of Proposition 2 to shift, by a factor of $\frac{3}{2}$, the multiplicative constant of the current best asymptotic bound of h_n for odd numbers n by relating h_n with h_{n+1}.

Proposition 7. *For an odd number n, $n > 1$, it is satisfied that $h_n \geq \frac{3n-1}{2n+2} h_{n+1}$.*

Proof. Let P be a set in which h_m is attained, where $m > 2$ is an even number. Then, we have that there are $(m-2)h_m + \frac{m}{2}e_{\frac{m-4}{2}}(P)$ points of P (allowing repetitions) in the $\frac{m-2}{2}$-half planes generated by the halving lines of P, or in the $\frac{m}{2}$-half planes generated by the $\frac{m-4}{2}$-edges of P. Therefore, there exists a point $p \in P$ which belongs to s of said half planes, with

$$s \geq \frac{(m-2)h_m + \frac{m}{2}e_{\frac{m-4}{2}}(P)}{m} = \frac{m-2}{m}h_m + \frac{1}{2}e_{\frac{m-4}{2}}(P).$$

So, if we remove p, then we obtain a set $P - \{p\}$, for which the halving lines are either the halving lines of P not containing p or the $\frac{m-4}{2}$-edges of P such that p is contained in their $\frac{m}{2}$-half planes. Thus, if we call $h(P - \{p\})$ the number of halving lines of $P - \{p\}$, we have that

$$h_{m-1} \geq h(P - \{p\}) = s \geq \frac{m-2}{m}h_m + \frac{1}{2}e_{\frac{m-4}{2}}(P).$$

As we have the lower bound $e_{\frac{m-4}{2}}(P) \geq h_m$ (see the proof of Corollary 2 in [9]), we obtain

$$h_{m-1} \geq \frac{m-2}{m}h_m + \frac{1}{2}h_m = \frac{3m-4}{2m}h_m$$

and we obtain the desired result by substituting m by $n+1$. □

5. Conclusions

We have improved the current lower bound on the maximum number of halving lines for planar sets of $n = 35$, $n = 59$, $n = 95$, and $n = 97$ points. To do this, we have considered as basis sets the sets P that attains the current best lower bound of h_{37}, h_{61}, h_{97}, and h_{99} and we have removed two points of P in all the possible ways. This way, we have obtained four different sets with the new lower bound for the case $n = 35$, five sets for the case $n = 59$, three sets for the case $n = 97$, four sets for the case $n = 95$. They are not combinatorially equivalent for the case $n = 35$ because, despite they having the same crossing number, if we remove two points of each one of the four sets in all the possible ways, we obtain different sequences of number of halving lines. The same applies for $n = 59$, $n = 95$, and $n = 97$.

We have also given a lower bound of h_n in terms of h_{n+2} that can be considered as a generalization of the aforementioned examples, and also a lower bound of h_n in terms of h_{n+1} for n an odd number.

A future line of work could be to try to obtain more examples that shift the current best lower bound of h_{35} by applying slight perturbations to the points of the four examples. The new examples could yield an improvement in the lower bound of h_{33} with the technique of removing two points performed in this paper. We could obtain the same for h_{57}, h_{95}, and h_{97}.

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Appendix A

A set that attains the lower bound of h_n for $n = 97$ (Proposition 5) follows.

x	y
0	2,410,549
11,742	2,410,712
253,995	2,414,075
1,769,588	2,435,106
2,092,353	2,439,715
3,534,845	2,458,357
3,536,391	2,458,377
3,673,262	2,461,902
3,707,815	2,462,787
3,740,573	2,463,626
3,766,348	2,464,286
3,952,658	2,463,758
3,952,880	2,463,758
3,980,918	2,463,946
4,030,513	2,471,051
4,030,578	2,471,057
4,071,932	2,474,928
4,072,082	2,474,945
4,072,244	2,474,995
4,145,805	2,465,045
4,145,824	2,465,044
4,149,887	2,464,495
4,154,742	2,463,840
4,180,988	2,482,601
4,180,990	2,482,602
4,181,341	2,482,777
4,184,429	2,485,599
4,184,613	2,485,689
4,272,230	2,458,459
4,272,252	2,458,458
4,272,402	2,458,453
4,272,412	2,458,450
4,315,942	2,550,204
4,315,958	2,550,209
4,317,690	2,550,771
4,317,767	2,550,796
4,369,521	2,665,953
4,369,528	2,665,960
4,373,996	2,652,282
4,374,000	2,652,269
4,394,626	2,588,858
4,394,719	2,588,904
4,395,656	2,589,368
4,400,037	2,577,757
4,400,683	2,576,398
4,418,495	2,714,930
4,418,502	2,714,939
4,418,595	2,715,695
4,421,102	2,735,149
4,421,457	2,737,904
4,423,165	2,785,294
4,429,299	2,798,762
4,430,659	2,801,992
4,435,870	2,451,150
4,435,931	2,450,926
4,436,108	2,450,318
4,437,682	2,444,906
4,439,924	2,413,591
4,441,169	2,413,373
4,446,838	2,412,376
4,447,160	2,412,291
4,447,199	2,412,282
4,481,666	2,406,068
4,482,364	2,405,382
4,485,650	2,402,155
4,485,674	2,402,120
4,494,421	2,389,578
4,496,330	2,388,737
4,499,503	2,955,253
4,499,662	2,955,724
4,500,206	2,957,226
4,501,829	2,386,267
4,501,902	2,386,244
4,502,207	2,386,148
4,524,142	2,379,167

x	y
4,531,003	2,374,822
4,539,611	2,369,374
4,540,118	2,369,013
4,553,159	2,360,710
4,553,830	2,360,375
4,641,459	3,375,977
4,718,104	3,584,665
4,907,437	4,100,176
5,253,488	5,045,870
5,267,137	2,005,294
5,304,811	1,986,425
5,504,908	5,732,954
5,597,316	1,839,411
5,744,445	6,387,564
5,926,553	1,675,034
6,161,456	1,557,755
6,805,853	9,288,324
6,806,512	9,290,125
6,927,949	1,175,071
8,222,187	528,126
9,139,085	69,865
9,278,867	0

Appendix B

A set that attains the lower bound of h_n for $n = 95$ (Proposition 6) follows.

x	y
0	2,410,549
11,742	2,410,712
253,995	2,414,075
1,769,588	2,435,106
2,092,353	2,439,715
3,534,845	2,458,357
3,536,391	2,458,377
3,673,262	2,461,902
3,707,815	2,462,787
3,740,573	2,463,626
3,766,348	2,464,286
3,952,658	2,463,758
3,952,880	2,463,758
3,980,918	2,463,946
4,030,513	2,471,051
4,030,578	2,471,057
4,071,932	2,474,928
4,072,082	2,474,945
4,072,244	2,474,995
4,145,805	2,465,045
4,145,824	2,465,044
4,149,887	2,464,495
4,154,742	2,463,840
4,180,988	2,482,601
4,180,990	2,482,602
4,181,341	2,482,777
4,184,429	2,485,599
4,184,613	2,485,689
4,272,230	2,458,459
4,272,252	2,458,458
4,272,402	2,458,453
4,272,412	2,458,450
4,315,958	2,550,209
4,317,690	2,550,771
4,317,767	2,550,796
4,369,521	2,665,953
4,369,528	2,665,960
4,373,996	2,652,282
4,374,000	2,652,269
4,394,626	2,588,858
4,394,719	2,588,904
4,395,656	2,589,368

x	y
4,400,037	2,577,757
4,400,683	2,576,398
4,418,495	2,714,930
4,418,502	2,714,939
4,418,595	2,715,695
4,421,102	2,735,149
4,421,457	2,737,904
4,423,165	2,785,294
4,429,299	2,798,762
4,430,659	2,801,992
4,435,870	2,451,150
4,435,931	2,450,926
4,436,108	2,450,318
4,437,682	2,444,906
4,439,924	2,413,591
4,441,169	2,413,373
4,446,838	2,412,376
4,447,160	2,412,291
4,447,199	2,412,282
4,481,666	2,406,068
4,482,364	2,405,382
4,485,650	2,402,155
4,485,674	2,402,120
4,494,421	2,389,578
4,496,330	2,388,737
4,499,503	2,955,253
4,499,662	2,955,724
4,500,206	2,957,226
4,501,902	2,386,244
4,502,207	2,386,148
4,524,142	2,379,167
4,531,003	2,374,822
4,539,611	2,369,374
4,540,118	2,369,013
4,553,159	2,360,710
4,553,830	2,360,375
4,641,459	3,375,977
4,718,104	3,584,665
4,907,437	4,100,176
5,253,488	5,045,870
5,267,137	2,005,294
5,304,811	1,986,425
5,504,908	5,732,954
5,597,316	1,839,411
5,744,445	6,387,564
5,926,553	1,675,034
6,161,456	1,557,755
6,805,853	9,288,324
6,806,512	9,290,125
6,927,949	1,175,071
8,222,187	528,126
9,139,085	69,865
9,278,867	0

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