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Application of the Averaging Method to the Optimal Control of Parabolic Differential Inclusions on the Semi-Axis

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Abstract: In this paper, we use the averaging method to find an approximate solution for the optimal control of parabolic differential inclusion with fast-oscillating coefficients on a semi-axis.

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1. Introduction

The intensive development of science and technology consistently drives the search for effective control methods for various natural, economic, social, and technical processes. Mathematical models of such processes are represented by optimal control problems for different classes of evolutionary systems [1–5].

There are many approaches used in the investigation of control problems for differential equations and inclusions, with asymptotic methods being used fairly extensively [6]. One of the most successful among these is the averaging method, which was originally developed and rigorously justified by Krylov and Bogolyubov for the approximate analysis of oscillating processes in non-linear mechanics, and then further refined for control-related problems (see, e.g., the monograph by Plotnikov [7]). Motivated by modern engineering control applications, the averaging method has been recently applied to the solution of optimal control problems for linear control systems with rapidly oscillating coefficients within a finite interval [8] and on the semi-axis [9]. The approximate solutions of the optimal control problems for non-linear systems of differential inclusions with fast-oscillating parameters were investigated in [10,11] for the cases of a finite interval and on the semi-axis, respectively. The optimal control problem on the semi-axis for the Poisson equation with nonlocal boundary conditions was studied in [12]. The averaging method can also be applied to the study of singularly perturbed systems [13,14] and optimal control problems for differential equations with rapidly oscillating coefficients, both on a finite time interval and on the half-line [15,16]. Further applications of the averaging method for parabolic systems with fast-oscillating coefficients were considered in [17–20].

In the present paper, we use the averaging method for the investigation of the optimal control problem for nonlinear parabolic differential inclusion with fast-oscillating coefficients (with respect to the time variable) on a semi-axis. In contrast to the generic non-linear case of [11], we specifically make use of parabolicity in our setup. With this, we prove that the optimal control for the averaged problem can be considered as an "approximately" optimal one for the original problem. It is noteworthy to say that the results of [17–19] differ significantly from those of the current paper since only optimal control problems for parabolic equations were considered in [17,18], and while the optimal control for parabolic inclusions was considered in [19], it was only considered on finite intervals. The current paper, however, addresses the optimal control for parabolic inclusions on infinite intervals, which poses substantial challenges and requires an essentially different method of investigation. In particular, differential inclusions require the proper treatment of multi-valued functions, and the infinite interval for the corresponding optimal control problem raises an additional challenge in estimating the "tale" in the cost functional.

2. Problem Statement

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded domain. In a cylinder $Q = (0, +\infty) \times \Omega$, let us consider an initial boundary-value problem for a parabolic inclusion

$$\begin{cases} \frac{\partial y}{\partial t} \in Ay + f\big(\frac{t}{\varepsilon}, y(t,x)\big) + g(y)u, \quad (t,x) \in Q, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x). \end{cases} \tag{1}$$

Here, $\varepsilon > 0$ is a small parameter, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to conv(\mathbb{R})$ (the space of nonempty, compact, convex subsets of $\mathbb{R}$) is a given multi-valued mapping, $g : \mathbb{R} \to \mathbb{R}$, $q : \Omega \times \mathbb{R} \to \mathbb{R}$ are given real-valued mappings, A is a Laplacian operator, y is an unknown state function, and u is an unknown control function, which satisfies the following constraints:

$$u \in U \subseteq L^2(Q), \tag{2}$$

$$J(y,u) = \int_Q e^{-\gamma t} q(x, y(t,x)) dt dx + \alpha \int_Q u^2(t,x) dt dx \to \inf, \tag{3}$$

where γ, α are positive constants. Later, in Section 3, under natural and mild conditions on f, g, u, q, we will show that the problem of optimal control (1)–(3) has a solution $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$, i.e., for every $u \in U$ and for any solution y^ε of (1) with control u, it holds that

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \le J(y^\varepsilon, u).$$

Despite proving the existence of a solution to (1)–(3), its construction is a challenging problem due to the presence of fast-varying coefficients. To address this issue, we consider the problem of finding an approximate solution of (1)–(3) by transitioning to the averaged coefficients. For this purpose, we assume there exists $\bar{f} : \mathbb{R} \to \mathbb{R}$ such that uniformly with regard to $y \in \mathbb{R}$

$$dist_H\left(\bar{f}(y), \frac{1}{T}\int_0^T f(s,y)ds\right) \to 0, \ T \to \infty, \tag{4}$$

where $dist_H(A,B)$ is a Hausdorff metric between sets A and B and the integral of the multivalued map is considered in the sense of Aumann [21]. Having the averaged version $\bar{f}$ of multi-valued mapping f at hand, we pose the following optimal control problem:

$$\begin{cases} \frac{\partial y}{\partial t} \in Ay + \bar{f}(y) + g(y)u, \quad (t,x) \in Q, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x), \end{cases} \tag{5}$$

$$u \in U \subseteq L^2(Q), \tag{6}$$

$$J(y,u) = \int\limits_Q e^{-\gamma t} q(x, y(t,x))dtdx + \alpha \int\limits_Q u^2(t,x)dtdx \to \inf. \tag{7}$$

The primary objective of the paper is thus to prove the convergence

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \to J(\bar{y}, \bar{u}), \quad \varepsilon \to 0, \tag{8}$$

where $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$ is the solution of (1)–(3), and $\{\bar{y}, \bar{u}\}$ is the solution of (5)–(7).

3. Preliminaries and Notation

Subsequently, we assume the following assumptions for the parameters of problem (1)–(3) are fulfilled:

(f_1) Multi-valued function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to conv(\mathbb{R})$ is continuous and there exist $C, C_1 > 0$ such that

$$\forall t \geq 0 \quad \forall y \in \mathbb{R} \quad \|f(t,y)\|_+ := \sup_{\xi \in f(t,y)} \|\xi\|_{\mathbb{R}} \leq C + C_1 \|y\|_{\mathbb{R}}, \tag{9}$$

where $\|\xi\|_{\mathbb{R}}$ denotes the Euclidean norm of $\xi \in \mathbb{R}^n$;

(g_1) function $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $C_2 > 0$ such that

$$\forall y \in \mathbb{R} \ \|g(y)\|_{\mathbb{R}} \leq C_2; \tag{10}$$

(q_1) function $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodori function and there exists $C_3 > 0$ and functions $K_1 \in L^2(\Omega)$, $K_2 \in L^1(\Omega)$ such that

$$\|q(x,\xi)\|_{\mathbb{R}} \leq C_3 \|\xi\|^2_{\mathbb{R}} + K_1(x), \quad q(x,\xi) \geq K_2(x); \tag{11}$$

(U_1) $U \subseteq L_2(Q)$ is closed and convex, $0 \in U$;

(γ_1) $\gamma > 2C_1^2 + 1 + C_2$;

(A_1) uniformly with regard to $y \in \mathbb{R}$, there exists the limit (4).

For $u \in U$ and $y_0 \in L^2(\Omega)$, we understand the solution of (1) as a mild solution on every finite time interval; i.e., y is a solution of (1). If $y \in L^2_{loc}(0,+\infty; H^1_0(\Omega)) \bigcap L^\infty_{loc}(0,+\infty; L^2(\Omega))$ such that $\forall T > 0$, $\forall \varphi \in H^1_0(\Omega)$, $\forall \eta \in C_0^\infty(0,T)$ the following equality holds:

$$\begin{aligned} &-\int\limits_0^T (y,\varphi)_H \cdot \eta' dt + \int\limits_0^T (\nabla y, \nabla \varphi)_H \cdot \eta dt = \int\limits_0^T (l(t),\varphi)_H \cdot \eta dt + \\ &+ \int\limits_0^T (g(y)u,\varphi)_H \cdot \eta dt, \quad l(t) \in f\left(\tfrac{t}{\varepsilon}, y\right) \end{aligned} \tag{12}$$

and $l \in L^2_{loc}(0,+\infty; L^2(\Omega))$.

Hereafter, we denote by $\|\cdot\|_H$ and $(\cdot,\cdot)_H$ the classical norm and scalar product in $H = L^2(\Omega)$, by $\|\cdot\|_V$ the classical norm in $V := H^1_0(\Omega)$, and by V^* the dual space to V.

Note that due to assumptions (f_1) and (g_1) and the properties of operator A for y, from the definition of the mild solution we have

$$\frac{\partial y}{\partial t} \in L^2_{loc}(0,+\infty; V^*).$$

Following this, we denote by $\mathcal{F}^\varepsilon$ (or $\bar{\mathcal{F}}$) a set of all pairs $\{y,u\}$, where y is a solution of (1) (or (5)) with control u.

The following lemma gives us result about the solvability of the optimal control problem (1)–(3).

Lemma 1. *Let the conditions* (f_1), (g_1), (q_1), (U_1), *and* (γ_1) *hold true. Then, for every* $\varepsilon > 0$, *the problem* (1)–(3) *has a solution* $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$, *that is,*

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \le J(y, u) \quad \forall \{y, u\} \in \mathcal{F}^\varepsilon. \tag{13}$$

Proof. We fix arbitrary $\varepsilon > 0$ and drop index ε throughout the proof for readability. First of all, note that by Theorem 3.1 from [22], the set of admissible pairs $\mathcal{F}^\varepsilon$ is not empty. For further investigations, let us consider some a priori estimates for solutions. Taking into account the definition of the mild solution for parabolic inclusion, suppose that $\forall \varphi \in H_0^1(\Omega)$

$$\begin{aligned} &\tfrac{d}{dt}(y, \varphi) + (\nabla y, \nabla \varphi) = (f_1(t), \varphi) + (g(y)u, \varphi) \text{ for almost all (a.a.) } t > 0 \\ &f_1(t) \in f(t, y(t)) \end{aligned} \tag{14}$$

Thus, we can consider the following equality:

$$\int_0^s (y'(t), y(t))_H dt + \int_0^s (\nabla y, \nabla y)_H dt = \int_0^s (f_1(t), y(t))_H dt + \int_0^s (g(y)u, y(t)) dt. \tag{15}$$

Integrating by parts, and taking into account Young's inequality and the assumption (g_1), we obtain

$$\begin{aligned} &\|y(s)\|_H^2 + 2\hat{C}\int_0^s \|y(t)\|_V^2 dt \le \|y_0\|_H^2 + \int_0^s (\|f_1(t)\|_H^2 + \|y(t)\|_H^2) dt + \\ &+ C_2\left(\int_0^s \|u(t)\|_H^2 dt + \int_0^s \|y(t)\|^2 dt\right), \end{aligned} \tag{16}$$

where $\hat{C}$ is the constant from the inequality $\|\nabla y\|_H^2 \ge \hat{C}\|y\|_V^2$ for an arbitrary $y \in H_0^1(\Omega)$. Using (9), we have

$$\|f_1(t)\|_H^2 \le 2\left(C^2|\Omega| + C_1^2\|y(t)\|_H^2\right).$$

Then, from (16) we obtain

$$\begin{aligned} &\|y(s)\|_H^2 + 2\hat{C}\int_0^s \|y(t)\|_V^2 dt \le \|y_0\|_H^2 + \int_0^s 2C^2|\Omega| dt + \int_0^s 2C_1^2\|y(t)\|_H^2 dt + \\ &+ \int_0^s \|y(t)\|_H^2 dt + C_2\int_0^s \|u(t)\|_H^2 dt + C_2\int_0^s \|y(t)\|_H^2 dt = \\ &= \|y_0\|_H^2 + 2C^2|\Omega|s + C_2\int_0^s \|u(t)\|_H^2 dt + (2C_1^2 + 1 + C_2)\int_0^s \|y(t)\|_H^2 dt \end{aligned}$$

and using Gronwall's inequality, we arrive at

$$\|y(t)\|_H^2 \le \left(\|y_0\|_H^2 + 2C^2|\Omega|t + C_2\|u\|_{L^2(0,+\infty;H)}\right)e^{(2C_1^2+1+C_2)t}. \tag{17}$$

Note that from (16) and (17), we can conclude that $\exists M > 0$:

$$\|y\|_{L^2(0,T;V)}^2 \le M \ \forall T > 0. \tag{18}$$

Due to the inclusion from (1), (9), and (17), we obtain

$$\|f_1(t)\|_H \le \sqrt{2}(C^2|\Omega| + C_1^2 M_1)^{1/2} \quad \text{for a.a. } t > 0 \tag{19}$$

$$\|y'(t)\|_{V^*} \le \hat{\hat{C}}\|y(t)\|_V + C|\Omega| + C_1\sqrt{M_1} + C_2\|u(t)\|_H \quad \text{for a.a. } t > 0, \tag{20}$$

where $M_1 = \Big(\|y_0\|_H^2 + 2C^2|\Omega|T + C_2\|u\|^2_{L^2(0,+\infty;H)} \Big) e^{(2C_1^2+1+C_2)T}$, and as a consequence, there exist $M_2 := \big(2(C^2|\Omega| + C_1^2 M_1)T\big)^{1/2}$ and $M_3 > 0$ such that

$$\|f_1\|_{L^2(0,T;H)} \leq M_2, \quad \|y'\|_{L^2(0,T;V^*)} \leq M_3 \quad \forall T > 0. \tag{21}$$

Taking into account (q_1), we obtain

$$J(y,u) \leq \int\limits_Q e^{-\gamma t}(C_3\|y\|_{\mathbb{R}} + K_1(x))dtdx + \alpha \int\limits_Q u^2(t,x)dtdx. \tag{22}$$

In view of (17), we have

$$\begin{aligned} &\int\limits_Q C_3 e^{-\gamma t}\|y\|^2_{\mathbb{R}} dtdx \leq \\ &\leq C_3 \int\limits_0^{+\infty} e^{-\gamma t}\Big(\|y_0\|_H^2 + 2C^2|\Omega|t + C_2\|u\|^2_{L^2(0,+\infty;H)}\Big) e^{(2C_1^2+1+C_2)t} dt =: I_1 + I_2 + I_3. \end{aligned}$$

Due to (γ_1) and (q_1), we obtain

$$\begin{aligned} I_1 &:= C_3 \int\limits_0^{+\infty} e^{(-\gamma+2C_1^2+1+C_2)t}\|y_0\|_H^2 dt = \\ &= \frac{C_3\|y_0\|_H^2}{-\gamma+2C_1^2+1+C_2} e^{(-\gamma+2C_1^2+1+C_2)t}\Big|_0^{+\infty} = \\ &= -\frac{C_3\|y_0\|_H^2}{-\gamma+2C_1^2+1+C_2} = \frac{C_3\|y_0\|_H^2}{\gamma-(2C_1^2+1+C_2)}, \end{aligned} \tag{23}$$

$$\begin{aligned} I_2 &= C_3 \int\limits_0^{+\infty} 2C^2|\Omega| t e^{(-\gamma+2C_1^2+1+C_2)t} dt = \\ &= \frac{C_3 2C^2|\Omega| t e^{(-\gamma+2C_1^2+1+C_2)t}}{-\gamma+2C_1^2+1+C_2}\Big|_0^{+\infty} - C_3 2C^2|\Omega| \int\limits_0^{+\infty} \frac{e^{(-\gamma+2C_1^2+1+C_2)t}}{-\gamma+2C_1^2+1+C_2} dt = \\ &= -\frac{C_3 2C^2|\Omega| e^{(-\gamma+2C_1^2+1+C_2)t}}{(-\gamma+2C_1^2+1+C_2)^2}\Big|_0^{+\infty} = \frac{C_3 2C^2|\Omega|}{(-\gamma+2C_1^2+1+C_2)^2}, \end{aligned} \tag{24}$$

$$\begin{aligned} I_3 &= C_3 C_2 \int\limits_0^{+\infty} \|u\|^2_{L^2(0,+\infty;H)} e^{(-\gamma+2C_1^2+1+C_2)t} dt = \\ &= \frac{C_3C_2\|u\|^2_{L^2(0,+\infty;H)}}{-\gamma+2C_1^2+1+C_2} e^{(-\gamma+2C_1^2+1+C_2)t}\Big|_0^{+\infty} = \frac{C_3C_2\|u\|^2_{L^2(0,+\infty;H)}}{\gamma-(2C_1^2+1+C_2)}. \end{aligned} \tag{25}$$

Further we have that

$$\begin{aligned} &\int_Q e^{-\gamma t} K_1(x) dtdx = \int\limits_0^{+\infty} e^{-\gamma t} dt \cdot \int\limits_\Omega K_1(x)dx = \\ &= \frac{e^{-\gamma t}}{-\gamma}\Big|_0^{+\infty} \cdot \int\limits_\Omega K_1(x)dx = \frac{1}{\gamma}\int\limits_\Omega K_1(x)dx \leq \\ &= \frac{1}{\gamma}\Bigg(\int\limits_\Omega dx\Bigg)^{1/2} \cdot \Bigg(\int\limits_\Omega K_1^2(x)dx\Bigg)^{1/2} = \frac{|\Omega|^{1/2}}{\gamma} \cdot \|K_1\|_{L^2(\Omega)}. \end{aligned} \tag{26}$$

Taking into account (23)–(26), we have

$$\begin{aligned} J(y,u) &\leq \frac{C_3\|y_0\|_H^2}{\gamma-(2C_1^2+1+C_2)} + \frac{C_3 2C^2|\Omega|}{(-\gamma+2C_1^2+1+C_2)^2} + \frac{C_3C_2\|u\|^2_{L^2(0,+\infty;H)}}{\gamma-(2C_1^2+1+C_2)} + \\ &+ \frac{|\Omega|1/2}{\gamma} \cdot \|K_1\|_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(0,+\infty;H)} \leq \\ &\leq L\Big(\|y_0\|_H^2 + \|u\|^2_{L^2(0,+\infty;H)} + \|K_1\|_{L^2(\Omega)} + 1\Big) < \infty, \end{aligned} \tag{27}$$

where $L = \max\{\frac{C_3}{\gamma-(2C_1^2+1+C_2)}; \frac{C_3C_2}{\gamma-(2C_1^2+1+C_2)} + \alpha; \frac{|\Omega|^{1/2}}{\gamma}; \frac{C_3 2C^2}{(-\gamma+2C_1^2+1+C_2|\Omega|)^2}\}$.

Now, let $\{y_n, u_n\}$ be a minimizing sequence, that is,

$$\lim_{n\to\infty} J(y_n, u_n) = \inf_{\{y,u\}\in\mathcal{F}^\varepsilon} J(y,u) =: \bar{J}^\varepsilon. \tag{28}$$

Note that in view of (q_1) $\forall\{y,u\} \in \mathcal{F}^\varepsilon$, it holds that

$$J(y,u) \geq -\frac{\|K_2\|_{L^1(\Omega)}}{\gamma} \Rightarrow \bar{J}^\varepsilon \geq -\frac{\|K_2\|_{L^1(\Omega)}}{\gamma} > -\infty.$$

From (28), for rather large n

$$J(y_n, u_n) \leq \bar{J}^\varepsilon + 1. \tag{29}$$

On the other hand,

$$J(y_n, u_n) \geq -\frac{\|K_2\|_{L_1(\Omega)}}{\gamma} + \alpha\|u_n\|^2_{L^2(0,+\infty;H)}. \tag{30}$$

Inequalities (29), (30) imply that $\{u_n\}$ is bounded in $L^2(0,+\infty;H)$ and thus for subsequence

$$u_n \to u \text{ weakly in } L^2(0,+\infty;H) \tag{31}$$

In view of the convexity of U, we have that $u \in U$. From (17), (18) we obtain that $\forall T > 0$ $\{y_n\}$ is bounded in $L^2(0,T;V)\bigcap L^\infty(0,T;H)$, from (21) we have that $\left\{\frac{\partial y_n}{\partial t}\right\}$ is bounded in $L^2(0,T;V^*)$. Due to the Compactness Lemma from [23], we conclude that up to subsequence $\forall T > 0$

$$\begin{aligned} &y_n \to y \text{ weakly in } L^2(0,T;V),\\ &y_n \to y \text{ in } L^2(0,T;H),\\ &\forall t \geq 0\ y_n(t) \to y(t) \text{ weakly in } H,\\ &y_n(t,x) \to y(t,x) \text{ a.e. in } Q. \end{aligned} \tag{32}$$

Let us consider y_n to be a mild solution of the problem

$$\begin{cases} \frac{\partial y_n}{\partial t} = Ay_n + f_{1n}(t) + g(y_n)u_n \\ y_n|_{\partial\Omega} = 0 \\ y_n|_{t=0} = y_0(x) \end{cases} \tag{33}$$

with $f_{1n}(t) \in f(t, y_n(t))$.

From (21), (32), and Lemma 3.2 from [24], we have that $f_{1n} \to f_1$ weakly in $L^2(0,T;H)$, $y_n \to y$ in $C(0,T;H)$ $\forall T > 0$, where y is the solution of (1) and $f_1(t) \in f(t, y(t))$. Thus, from (32) and Lebesgue's Dominated Convergence Theorem, we can pass to the limit in the equality (12), which we can apply to $\{y_n, u_n\}$ and receive that $\{y,u\} \in \mathcal{F}^\varepsilon$. In view of to the pointwise convergence,

$$e^{-\gamma t}\cdot q(x, y_n(t,x)) \to e^{-\gamma t} q(x, y(t,x)) \text{ a.e. in } Q,$$

Given Fatou's lemma and the weak convergence in (31), we obtain

$$\begin{aligned} &\bar{J}^\varepsilon = \lim_{n\to\infty} J(y_n,u_n) \geq \varliminf_{n\to\infty} \int_Q e^{-\gamma t} q(x, y_n(t,x))dtdx + \\ &+ \varliminf_{n\to\infty} \alpha \int_Q u^2(t,x)dtdx \geq J(y,u). \end{aligned}$$

Therefore, $\{y,u\}$ is a solution of (1)–(3). □

Remark 1. *In view of the properties of the Hausdorff metric, we have that Condition (f_1) is satisfied for the averaged function $\bar{f}(y)$ with the same constants as for $f(t,y)$. Indeed,*

$$\forall \xi \in f(t,y): \ \|\xi\|_{\mathbb{R}} \le C + C_1\|y\|_{\mathbb{R}}.$$

Due to the condition $dist_H\left(\bar{f}, \frac{1}{T}\int_0^T f(s,y)ds\right) \to 0,\ T \to \infty$, *we obtain that* $\forall \varepsilon > 0\ \exists T_0 :$ $\forall T \ge T_0$ *we have* $\bar{f}(y) \in O_\varepsilon\left(\frac{1}{T}\int_0^T f(s,y)ds\right)$. *Furthermore, we obtain*

$$\|\bar{f}(y)\|_{\mathbb{R}} \le \left\| \frac{1}{T}\int_0^T f(s,y)ds \right\|_+ + \varepsilon \le \frac{1}{T}\int_0^T \|f(s,y)\|_+ ds + \varepsilon \le C + C_1\|y\|_{\mathbb{R}} + \varepsilon.$$

The existence of a solution $\{\bar{y}, \bar{u}\}$ to (5)–(7) can be proved following similar arguments to the proof of the existence of $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$ for problem (1)–(3). Note, however, that the construction of $\{\bar{y}, \bar{u}\}$ is much simpler than that of $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$ due to the averaged nature of the coefficients involved.

4. Main Result

Theorem 1. *Suppose that the assumptions (f_1), (g_1), (q_1), (u_1), (γ_1), and (A_1) are fulfilled and, moreover, that for every $u \in U$ there exists a unique solution of the problem (5). We additionally assume that* $\forall \eta > 0\ \exists \delta > 0\ \forall t \ge 0\ \forall y, z \in \mathbb{R}$

$$\|y - z\|_{\mathbb{R}} < \delta \Rightarrow dist(f(t,y), f(t,z)) < \eta. \tag{34}$$

Let $\{\bar{y}^\varepsilon, \bar{u}^\varepsilon\}$ be a solution of (1)–(3). Then,

$$J(\bar{y}^\varepsilon, \bar{u}^\varepsilon) \to J(\bar{y}, \bar{u}),\ \ \varepsilon \to 0 \tag{35}$$

and up to subsequence

$$\bar{y}^\varepsilon \to \bar{y} \text{ in } L^2(0, +\infty; H), \tag{36}$$

$$\bar{u}^\varepsilon \to \bar{u} \text{ weakly in } L^2(0, +\infty; H). \tag{37}$$

Here, $\{\bar{y}, \bar{u}\}$ is a solution of (5)–(7).

Proof. Let $\varepsilon_n \to 0$ and $\{\bar{y}^n, \bar{u}^n\}$ be a solution of (1)–(3) for $\varepsilon = \varepsilon_n$. Since $\{\bar{y}^n, \bar{u}^n\}$ is an optimal pair, we obtain

$$J(\bar{y}^n, \bar{u}^n) \le J(y_n, 0),$$

where y_n is a solution of (1) with $\varepsilon = \varepsilon_n$ and $u \equiv 0$. Therefore, from (27) and (30), we obtain

$$-\frac{\|K_2\|_{L^1(\Omega)}}{\gamma} + \alpha\|\bar{u}^n\|_{L^2(0,+\infty;H)} \le L(\|y_0\|_H^2 + \|K_1\|_{L^2(\Omega)} + 1). \tag{38}$$

Applying similar reasoning as in the proof of Lemma 1, we conclude that on subsequence for some $\hat{y}, \hat{u}$:

$$\bar{u}^n \to \hat{u} \text{ weakly in } L^2(0, +\infty; H),\ n \to \infty, \tag{39}$$

$$\bar{y}^n \to \hat{y} \text{ in the sense of (32)},\ n \to \infty. \tag{40}$$

Let us show that $\{\hat{y}, \hat{u}\} \in \bar{\mathcal{F}}$, i.e., $\hat{y}$ is a solution of the corresponding averaged problem (5) with control $\hat{u}$. In order to achieve this goal, we have to make a limit transition in the equality

$$\begin{aligned}(\bar{y}^n(T),\varphi)_H-(y_0,\varphi)_H+\int_0^T(\nabla\bar{y}^n,\nabla\varphi)_H dt=\\=\int_0^T(f_1^{\varepsilon_n}(t),\varphi)_H dt+\int_0^T(g(\bar{y}^n)\bar{u}^n,\varphi)_H dt\end{aligned}\tag{41}$$

for arbitrary $\varphi\in V$ and $T>0$, where $f_1^{\varepsilon_n}(t)\in f\left(\frac{t}{\varepsilon^n},\bar{y}^n\right)$.

As for the left-hand side of the equality (41), the limit transition is a direct consequence of (40). From the Dominated Convergence Theorem, we have that

$$\begin{aligned}&g(\bar{y}^n)\to g(\hat{y})\text{ in }L^2(0,T;H),\ n\to\infty\\&\int_\Omega\int_0^T f_1^{\varepsilon_n}(t)\varphi dtdx-\int_\Omega\int_0^T\bar{f}(\hat{y})\varphi dtdx=\\&dist_H\left(\int_\Omega\int_0^T f_1^{\varepsilon_n}(t)\varphi dtdx,\int_\Omega\int_0^T\bar{f}(\hat{y})\varphi dtdx\right)\le\\&\le dist_H\left(\int_\Omega\int_0^T f\left(\frac{t}{\varepsilon_n},\bar{y}^n(t,x)\right)\varphi dtdx,\int_\Omega\int_0^T\bar{f}(\hat{y})\varphi dtdx\right).\end{aligned}$$

We still have to prove that

$$dist_H\left(\int_\Omega\int_0^T f\left(\frac{t}{\varepsilon_n},\bar{y}^n(t,x)\right)\varphi dtdx,\int_\Omega\int_0^T\bar{f}(\hat{y})\varphi dtdx\right)\to 0,\ n\to\infty.\tag{42}$$

First of all, let us note that due to (A_1) and [25] $\forall 0<a<b\ \forall\varphi\in H$, it holds that

$$dist_H\left(\int_\Omega\int_a^b f\left(\frac{t}{\varepsilon_n},\psi(x)\right)\varphi dtdx,\int_\Omega\int_a^b\bar{f}(\psi(x))\varphi dtdx\right)\to 0,\ n\to\infty\tag{43}$$

In view of Egorov's theorem [26] $\forall\delta>0\ \exists Q_1^\delta\subset Q_T$ such that $\mu(Q_1^\delta)<\delta$ and

$$\bar{y}^n\to\hat{y}\text{ uniformly on }Q_T\setminus Q_1^\delta,\text{ as }n\to\infty.\tag{44}$$

Here, μ is Lebesgue's measure on $\mathbb{R}^2$.

On the other hand, there exists a sequence of step functions

$$y^m(t,x)=\sum_{k=1}^m y_k^m(x)\chi_{A_k^m}(t),\ \{y_k^m\}\subset H$$

with $\{A_k^m=(a_k^m,b_k^m)\}$ being a covering of $(0,T)$ such that

$$y^m\to\hat{y}\text{ in }L^2(0,T;H)\text{ and almost everywhere in }Q_T.$$

Moreover, $\forall\delta>0\ \exists Q_2^\delta\subset Q_T$ such that $\mu(Q_2^\delta)<\delta$ and

$$y^m\to\hat{y}\text{ uniformly on }Q_T\setminus Q_2^\delta\text{ as }m\to\infty.\tag{45}$$

Furthermore, we have that

$$\begin{aligned}
& dist_H\left(\int\limits_{Q_T} f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right)\varphi dtdx, \int\limits_{Q_T} \bar{f}(\hat{y}(t,x))\varphi dtdx\right) \leq \\
& \leq dist_H\left(\int\limits_{Q_T} f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right)\varphi dtdx, \int\limits_{Q_T} f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\varphi dtdx\right) + \\
& + dist_H\left(\int\limits_{Q_T} f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\varphi dtdx, \int\limits_{Q_T} \bar{f}(\hat{y}(t,x))\varphi dtdx\right) =: I_1^{(n)} + I_2^{(n)}.
\end{aligned}$$

Due to (44), Hölder's inequality, (9), and (17), we have

$$\begin{aligned}
& I_1^{(n)} \leq \int\limits_{Q_T} dist_H\left[f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right]\varphi dtdx \leq \\
& \leq \int\limits_{Q_T\setminus Q_1^\delta} dist_H\left[f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right]\varphi dtdx + \\
& + \int\limits_{Q_1^\delta} dist_H\left[f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right]\varphi dtdx \leq \\
& \leq \int\limits_{Q_T\setminus Q_1^\delta} dist_H\left[f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right]\|\varphi\|_{\mathbb{R}} dtdx + \\
& + 2C\int\limits_{Q_1^\delta}\varphi dtdx + 2C_1\int\limits_{Q_1^\delta}\|y\|_{\mathbb{R}}\varphi dtdx \leq \\
& \leq \int\limits_{Q_T\setminus Q_1^\delta} dist\left[f\left(\tfrac{t}{\varepsilon_n}, \bar{y}^n(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right]\|\varphi\|_{\mathbb{R}} dtdx + \\
& + 2C\|\varphi\|_H\cdot\delta^{1/2}T^{1/2} + 2C_1\sqrt{M_1}\|\varphi\|_H\cdot\delta^{1/2}\cdot T.
\end{aligned} \tag{46}$$

Due to (34), for a given $\delta > 0$ $\exists\lambda$ $\forall n \geq 1$ $\forall t \geq 0$

$$\|y - z\|_{\mathbb{R}} < \lambda \Rightarrow dist\left(f\left(\frac{t}{\varepsilon_n}, y\right), f\left(\frac{t}{\varepsilon_n}, z\right)\right) \leq \delta^{1/2}.$$

Therefore, choosing n_1 such that $\forall n \geq n_1$

$$\sup_{(t,x)\in Q_T\setminus Q_1^\delta}\|\bar{y}^n(t,x) - \hat{y}(t,x)\| < \lambda,$$

we have from (46) that $\forall n \geq n_1$

$$\begin{aligned}
& I_1^{(n)} \leq \delta^{1/2}\mu^{1/2}(Q_T)\|\varphi\|_H\sqrt{T} + 2C\|\varphi\|_H\delta^{1/2}\sqrt{T} + \\
& + 2C_1\sqrt{M_1}\|\varphi\|_H\delta^{1/2}T \leq \tilde{C}(T)\delta^{1/2}.
\end{aligned} \tag{47}$$

On the other hand, for every step function $y^m(t,x)$, we have, using (43), that $\forall m \geq 1$

$$\begin{aligned}
& dist_H\left(\int\limits_{Q_T} f\left(\tfrac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx, \int\limits_{Q_T} \bar{f}(y^m(t,x))\varphi dtdx\right) = \\
& = dist_H\left(\sum_{k=1}^m\int\limits_\Omega\int\limits_{A_k^m} f\left(\tfrac{t}{\varepsilon_n}, y_k^m(t,x)\right)\varphi dtdx, \sum_{k=1}^m\int\limits_\Omega\int\limits_{A_k^m}\bar{f}(y_k^m(x))\varphi dtdx\right) \leq \\
& \leq \sum_{k=1}^m dist_H\left(\int\limits_\Omega\int\limits_{A_k^m} f\left(\tfrac{t}{\varepsilon_n}, y_k^m(t,x)\right)\varphi dtdx, \int\limits_\Omega\int\limits_{A_k^m}\bar{f}(y_k^m(x))\varphi dtdx\right) \to 0,\ n\to\infty.
\end{aligned} \tag{48}$$

Thus, $\forall m \geq 1\ \exists n_2 = n_2(m)\ \forall n \geq n_2$

$$dist_H\left(\int\limits_{Q_T} f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx, \int\limits_{Q_T} \bar{f}(y^m(t,x))\varphi dtdx\right) < \delta. \tag{49}$$

Furthermore, $\exists m_0\ \forall m \geq m_0\ \forall n \geq 1$

$$\begin{aligned} & dist_H\left(\int\limits_{Q_T\setminus Q_2^\delta} f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\varphi dtdx, \int\limits_{Q_T\setminus Q_2^\delta} f\left(\tfrac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx\right) \leq \\ & \int\limits_{Q_T\setminus Q_2^\delta} dist_H\left(f\left(\tfrac{t}{\varepsilon_n}, \hat{y}(t,x)\right), f\left(\tfrac{t}{\varepsilon_n}, y^m(t,x)\right)\right)\|\varphi\|_{\mathbb{R}} dtdx \leq \\ & \leq \delta^{1/2}\mu^{1/2}(Q_T)\|\varphi\|_H\sqrt{T}, \end{aligned} \tag{50}$$

$$\begin{aligned} & dist_H\left(\int\limits_{Q_T\setminus Q_2^\delta} \bar{f}(\hat{y}(t,x))\varphi dtdx, \int\limits_{Q_T\setminus Q_2^\delta} \bar{f}(y^m(t,x))\varphi dtdx\right) \leq \\ & \int\limits_{Q_T\setminus Q_2^\delta} dist_H\left(\bar{f}(\hat{y}(t,x))\varphi, \bar{f}(y^m(t,x))\right)\|\varphi\|_{\mathbb{R}} dtdx \leq \\ & \leq \delta^{1/2}\mu^{1/2}(Q_T)\|\varphi\|_H\sqrt{T}. \end{aligned} \tag{51}$$

Combining (48)–(51), we obtain $\forall m \geq m_0\ \forall n \geq n_2(m)$

$$I_2^{(n)} \leq 2\delta^{1/2}\mu^{1/2}(Q_T)\|\varphi\|_H\sqrt{T} + \delta \leq \tilde{\tilde{C}}(T)\delta^{1/2}. \tag{52}$$

Inequalities (47), (52) imply (42). Thus, we can pass to the limit in (41) and obtain that $(\hat{y}, \hat{u}) \in \bar{\mathcal{F}}$.

Let us now show that $\{\hat{y}, \hat{u}\}$ is an optimal process in (5)–(7).

Due to Fatou's lemma, we have

$$\underline{\lim}_{n\to\infty} J(\bar{y}^n, \bar{u}^n) \geq J(\hat{y}, \hat{u}). \tag{53}$$

On the other hand, for every $u \in U$ and any y_n—a solution of (1) with control u and $\varepsilon = \varepsilon_n$—we obtain

$$J(\bar{y}^n, \bar{u}^n) \leq J(y_n, u).$$

Applying similar reasoning as in proof of the Lemma 1 for $\{y_n\}$, we obtain that $y_n \to y$ in the sense of (32), where y is a unique solution of (5) with control u.

Let us show that

$$\int\limits_Q e^{-\gamma t} q(x, y_n(t,x))dtdx \to \int\limits_Q e^{-\gamma t} q(x, y(t,x))dtdx. \tag{54}$$

Indeed, due to (q_1), we have

$$\left|e^{-\gamma t} q(x, y_n(t,x))\right| \leq C_3 e^{-\gamma t}\|y_n(t,x)\|_{\mathbb{R}}^2 + e^{-\gamma t}K_1(x). \tag{55}$$

Since $y_n \to y$ in $L^2(0,T;H)$ and a.e. in Q, in view of (γ_1), (17), (23)–(26), we deduce from Lebesgue's Dominated Convergence Theorem:

$$\forall T > 0 \int\limits_{Q_T} e^{-\gamma t} q(x, y_n(t,x))dtdx \to \int\limits_{Q_T} e^{-\gamma t} q(x, y(t,x))dtdx,\ n \to \infty. \tag{56}$$

On the other hand, from (17) and (55),

$$\begin{aligned}&\int_T^{+\infty}\int_\Omega e^{-\gamma t}|q(x,y_n(t,x))|dtdx \le\\ &\le \int_T^{+\infty} e^{-\gamma t}\Big(C_3\|y_n(t)\|_H^2+|\Omega|^{1/2}\|K_1\|_{L^2(\Omega)}\Big)dt\le\\ &\le \int_T^{+\infty} e^{-\gamma t}((C_3\|y_0\|_H^2+2C^2C_3|\Omega|t+C_3C_2\|u\|^2_{L^2(0,+\infty;H)})e^{(2C_1^2+1+C_2)t}+\\ &+|\Omega|^{1/2}\|K_1\|_{L^2(\Omega)})dt =: J_1+J_2+J_3+J_4.\end{aligned} \tag{57}$$

Let us consider each term of the right hand side of (57) separately:

$$\begin{aligned}J_1 &:= \int_T^{+\infty} C_3\|y_0\|_H^2 e^{(-\gamma+2C_1^2+1+C_2)t}dt=\\ &=\frac{C_3\|y_0\|_H^2}{-\gamma+2C_1^2+1+C_2}e^{(-\gamma+2C_1^2+1+C_2)t}\Big|_T^{+\infty}=\\ &=\frac{C_3\|y_0\|_H^2}{\gamma-(2C_1^2+1+C_2)}e^{(-\gamma+2C_1^2+1+C_2)T};\end{aligned} \tag{58}$$

$$\begin{aligned}&J_2 := \int_T^{+\infty} 2C^2C_3|\Omega|te^{(-\gamma+2C_1^2+1+C_2)t}dt = \frac{2C^2C_3|\Omega|te^{(-\gamma+2C_1^2+1+C_2)t}}{-\gamma+2C_1^2+1+C_2}\Big|_T^{+\infty}-\\ &-2C^2C_3|\Omega|\int_T^{+\infty}\frac{e^{(-\gamma+2C_1^2+1+C_2)t}}{-\gamma+2C_1^2+1+C_2}dt=\\ &=\frac{2C^2C_3T|\Omega|e^{(-\gamma+2C_1^2+1+C_2)T}}{\gamma-(2C_1^2+1+C_2)}-2C^2C_3|\Omega|\frac{e^{(-\gamma+2C_1^2+1+C_2)t}}{(-\gamma+2C_1^2+1+C_2)^2}\Big|_T^{+\infty}=\\ &=\frac{2C^2C_3T|\Omega|e^{(-\gamma+2C_1^2+1+C_2)T}}{\gamma-(2C_1^2+1+C_2)}+\frac{2C^2C_3|\Omega|e^{(-\gamma+2C_1^2+1+C_2)T}}{(-\gamma+2C_1^2+1+C_2)^2};\end{aligned} \tag{59}$$

$$\begin{aligned}J_3 &:= \int_T^{+\infty} C_3C_2\|u\|^2_{L^2(0,+\infty;H)}e^{(-\gamma+2C_1^2+1+C_2)t}dt=\\ &=\frac{C_3C_2\|u\|^2_{L^2(0,+\infty;H)}}{-\gamma+2C_1^2+1+C_2}e^{(-\gamma+2C_1^2+1+C_2)t}\Big|_T^{+\infty}=\\ &=\frac{C_3C_2\|u\|^2_{L^2(0,+\infty;H)}}{\gamma-(2C_1^2+1+C_2)}e^{(-\gamma+2C_1^2+1+C_2)T};\end{aligned} \tag{60}$$

$$\begin{aligned}&J_4 := \int_T^{+\infty} e^{-\gamma t}|\Omega|^{1/2}\|K_1\|_{L^2(\Omega)}dt = |\Omega|^{1/2}\|K_1\|_{L^2(\Omega)}\frac{e^{-\gamma t}}{-\gamma}\Big|_T^{+\infty}=\\ &=\frac{|\Omega|^{1/2}\|K_1\|_{L^2(\Omega)}e^{-\gamma T}}{\gamma}.\end{aligned} \tag{61}$$

Combining (58)–(61), we obtain (54).

From (54) we obtain the following inequality: $\forall\{y,u\}\in\bar{\mathcal{F}}$

$$J(\hat{y},\hat{u}) \le \underline{\lim}_{n\to\infty} J(\bar{y}^n,\bar{u}^n) \le \underline{\lim}_{n\to\infty} J(y_n,u) = J(y,u). \tag{62}$$

This means that $\{\hat{y},\hat{u}\}$ is a solution of (5)–(7).

Let us substitute $u=\hat{u}$ in previous arguments. Then, $y=\hat{y}$ in view of uniqueness. Thus, from (62), we obtain

$$J(\hat{y},\hat{u}) \le \underline{\lim}_{n\to\infty} J(\bar{y}^n,\bar{u}^n) \le J(\hat{y},\hat{u}). \tag{63}$$

These inequalities imply that up to subsequence

$$J(\bar{y}^n,\bar{u}^n) \to J(\hat{y},\hat{u}),\ n\to\infty. \tag{64}$$

Since $J(\hat{y}, \hat{u}) = \inf_{\{y,u\} \in \bar{\mathcal{F}}} J(y, u)$, then the convergence in (64) holds for the whole sequence. Therefore, (35) is proved. □

Let us consider an example of an investigated problem. Let $\Omega = (0, l)$, $Ay = \frac{\partial^2 y}{\partial x^2}$ and consider the following problem:

$$\begin{cases} \frac{\partial y}{\partial t} - Ay \in \left[e^{-y^2} \cdot \sin^2(2\frac{t}{\varepsilon}), e^{-y^2} \cdot \sin^2(\frac{t}{\varepsilon})\right] + u, \ (t, x) \in Q, \\ y|_{x=0} = y|_{x=l} = 0, \\ y|_{t=0} = y_0(x), \\ J(y, u) = \int\limits_Q e^{-\gamma t} y^2(t, x) dt dx + \alpha \int\limits_Q u^2(t, x) dt dx \to \inf. \end{cases} \tag{65}$$

The corresponding averaged problem is

$$\begin{cases} \frac{\partial y}{\partial t} - Ay = \frac{1}{2} + u, \ (t, x) \in Q, \\ y|_{x=0} = y|_{x=l} = 0, \\ y|_{t=0} = y_0(x), \\ J(y, u) = \int\limits_Q e^{-\gamma t} y^2(t, x) dt dx + \alpha \int\limits_Q u^2(t, x) dt dx \to \inf. \end{cases} \tag{66}$$

We consider a control $u \in L^2(Q)$ such that $\|u(t, x)\|_{\mathbb{R}} \leq 1$ a.e. We can see that Lemma 1 (Remark 1) and Theorem 1 can be applied to problems (65) and (66).

5. Discussion

Our aim was to establish a theoretical result illustrating the effectiveness of the averaging method in finding approximate solutions for the optimal control problem of a nonlinear parabolic differential inclusion with rapidly oscillating parameters. Specifically, we demonstrated that the optimal control of the averaged problem can be regarded as "approximately" optimal for the original perturbed system. Importantly, this was achieved under fairly mild and natural assumptions regarding the system's parameters. To further highlight the significance and utility of the averaging method in such contexts, we intend to extend our research to its application in control problems involving hyperbolic differential inclusions.

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