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In Search of *L*-Fuzzy Contexts Adaptable to Variable Information: A Tool for Time-Varying Data Analysis

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Abstract: This paper sheds light on the study of a new structure, called *L*-fuzzy hypercontext, which provides an extension of the range of applications of fuzzy concept analysis. The advantage of working with *L*-fuzzy hypercontexts lies in the fact that we can establish relation among elements such that their values are, in turn, other relations. Thus, they are easily adaptable to different situations that vary over time. The usefulness of the developed theory is illustrated by a practical case in which the valuation made by the different clients of a hotel company is analyzed.

Keywords: *L*-fuzzy concept analysis; WOWA operators; linguistic variables; knowledge extraction

MSC: 06D72; 08A72; 94D05

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Academic Editor: Boldizsár Tüű-Szabó

Received: 4 December 2024
Revised: 24 January 2025
Accepted: 27 January 2025
Published: 29 January 2025

Citation: Alcalde, C.; Burusco, A. In Search of *L*-Fuzzy Contexts Adaptable to Variable Information: A Tool for Time-Varying Data Analysis. *Axioms* **2025**, *14*, 102. https://doi.org/10.3390/axioms14020102

1. Introduction

In 1982 R. Wille introduced the theory of formal concept analysis [1] as a for knowledge extraction from data tables representing binary relations defined between two sets.

With the purpose of giving us the opportunity to distinguish different levels of relationship between the elements, in 1994 Burusco and Fuentes-González developed the theory of *L*-fuzzy concept analysis [2]. The theory of *L*-fuzzy concept analysis extends the study of formal contexts presented in [3] considering values that vary in set *L* which is endowed with the structure of complete lattice. Hence, tools based on fuzzy logic are of huge help to achieve information from *L*-fuzzy contexts.

However, the analysis of *L*-fuzzy contexts is not efficient enough when the available information, be it objects, attributes or the relation between them, is different according to the time or the circumstances in which the information was gathered.

In order to have more flexible *L*-fuzzy contexts which can be adapted to these different situations, an extension of the structure of *L*-fuzzy context becomes necessary. The objective will be to establish a new framework that allows representing the relationship between each object and attribute by means of a family of values which, in turn, has the structure of *L*-fuzzy context. This new structure will be called *L*-fuzzy hypercontext. *L*-fuzzy hypercontexts can be seen as the extension to the fuzzy framework of certain multicontexts defined by Wille in [4]. They are also related to the heterogeneous formal contexts defined in [5].

It is worth mentioning that the approach given in this work, which is based on the use of WOWA operators and linguistic variables, is different from the study discussed in [4,5]. One of the advantages of this new proposal is that the double weighting vectors of WOWA operators are relevant in the performed aggregation processes and allow to delve deeper into the previous techniques developed in [6].

Some particular cases of hypercontexts that provide interesting results will be analyzed. One of them is the case of working with a set X which only has one element. In this situation, the obtained L-fuzzy hypercontext can be considered as a L-fuzzy context sequence which study was developed in [7].

Taking into account that most of the practical cases can be easily represented by a fuzzy relation, these new frameworks make an important contribution because they increase the range of fields where formal and fuzzy concept analysis can be applied.

This work is structured as follows. To start we introduce some necessary elements on the framework of L-fuzzy concept analysis, the operators WOWA introduced in [8] and the notion of linguistic labels [9]. In Section 2 the notion of L-fuzzy hypercontexts in a complete lattice L is recalled. Section 3 presents the WOWA operators as a new tool to carry out a more exhaustive scrutiny of these L-fuzzy hypercontexts when $L = [0,1]$. In Section 4 linguistic variables are proposed to improve the information obtained via L-fuzzy concepts. As an important point, in Section 5 a practical application of the developed methods is shown. Finally, last section is dedicated to detailing the conclusions of the work.

2. Preliminaries

We collect below some notions that will be necessary in the following sections.

2.1. L-Fuzzy Concept Analysis

The theory of Formal Concept Analysis, which principles were formulated by R. Wille [1] in 1982, extracts information from a triplet (X, Y, R) formed by two sets and the binary relation existing between them. The set X is the set of *objets* and the elements of Y are said to be the *attributes*. The information contained by the context can be visualized through the *formal concepts*, which are pairs (E, P), where $E \subseteq X$, $P \subseteq Y$, and they verify that $E^* = P$ and $P^* = E$, being $(\cdot)^*$ an operator, called *derivation*, which given any subset of objects E provides the set of all the attributes belonging to all the elements of E. Similarly, the set P^* is formed by the objects that are related to all the attributes in the subset P. Thus, every formal concept provides the group of all the objects sharing an attribute set, as well as the set of all shared attributes.

The generalization of this theory to the fuzzy framework (see [2]) considers a L-fuzzy context as a fourtuple (L, X, Y, R) consisting of complete lattice L, the sets X and Y, and the relation $R \in L^{X \times Y}$ which is defined between the two sets and takes values in L. Thus, the fact that the L-fuzzy contexts are defined from fuzzy relations allows working with different grades of relationship between objects and attributes. Fuzzy set theory will be necessary to analyze this issue.

Derivation operators are the tools that will help us to extract knowledge from the L-fuzzy contexts. These operators, denoted by 1 and 2, were defined by following expressions:

For all $E \in L^X$,

$$E_1(y) = \inf_{x \in X} \{\mathcal{I}(E(x), R(x,y))\}, \tag{1}$$

and, for all $P \in L^y$,

$$P_2(x) = \inf_{y \in Y} \{\mathcal{I}(P(y), R(x,y))\}, \tag{2}$$

where $\mathcal{I}$ represents a fuzzy implication taking values in the lattice L.

Also was defined the constructor operator φ such that given the set of object E calculates $\varphi(E) = (E_1)_2 = E_{12}$, which provides a tool to extract the information contained in the context. This information can be visualized through the L-fuzzy concepts. A L-fuzzy concept consists in a pair $(E, E_1) \in L^X \times L^Y$, being E a fixed point of the operator φ, and relates a collection of objects to the attributes shared by them. The fuzzy subset of objects in the first component of the pair is the extension of the concept, and the attribute subset in the second one forms the intension.

If we consider the order relation usually established between fuzzy sets as follows: Given $E, F \in L^X$,

$$E \leq F \Longleftrightarrow E(x) \leq F(x), \text{ for every } x \in X, \tag{3}$$

and denoting by $\text{fix}(\varphi)$ the set of fixed points of the constructor operator φ, it is possible to define in the set of pairs $\mathcal{L} = \{(E, E_1)/E \in \text{fix }(\varphi)\}$, the order relation $\preceq$ such that for every $(E, E_1), (F, F_1) \in \mathcal{L}$,

$$(E, E_1) \preceq (F, F_1) \text{ if } E \leq F \text{ (or, equivalently, if } E_1 \geq F_1). \tag{4}$$

As was showed by Burusco and Fuentes-González [2], $(\mathcal{L}, \preceq)$ is a complete lattice wich they named L-fuzzy concept lattice.

A particular case is that in wich the implication operator used $\mathcal{I}$ is residuated [10,11]. When this occurs, the constructor operator turns out to be a closure operator and therefore the L-fuzzy concept associated with a given object set is obtained by applying twice the derivation operator to that set. In the course of this work, residuated implication operators will be considered and, consequently, the L-fuzzy concept obtained from the set $E \in L^X$ will be done by the pair (E_{12}, E_1).

The constructor operator ϕ which for any set of attributes P calculates $\phi(P) = P_{21}$ allows to made a similar development focusing on attributes. In the same way the L-fuzzy concept associated with the attribute set P can be obtained as (P_2, P_{21}) when calculations are performed using a residuated implication operator.

There are other relevant papers that use residuated implicators in the generalization of Formal Concepts Analysis. Among the most important ones are those published by R. Belohlavek [12,13] and S. Pollandt [14].

2.2. WOWA Operators

Weighted OWA operators (WOWA) were introduced by Torra [8] as a generalization of OWA operators. Let us start with the definition of OWA operator established by Yager [15]:

Definition 1. *A function $F_z : [0,1]^n \longrightarrow [0,1]$ is an Ordered Weighted Averaging (OWA) operator of dimension n if there is a n-tuple of weights $z = (z_1, z_2, \ldots, z_n) \in [0,1]^n$ with $\sum\limits_{1 \leq i \leq n} z_i = 1$, such that F_z is given by*

$$F_z(\alpha_1, \alpha_2, \ldots, \alpha_n) = z_1.\beta_1 + z_2.\beta_2 + \cdots + z_n.\beta_n, \tag{5}$$

being β_i the ith largest element of the elements $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Weighted Ordered Weighted Averaging (WOWA) operators [8] combine the advantages of OWA operators [15] and weighted means [16,17]. These operators are defined using two different weighting vectors: $z = (z_1, z_2, \ldots, z_n)$ which corresponds the significance of the values (OWA operator) and $t = (t_1, t_2, \ldots, t_n)$ related to the importance of each expert.

Definition 2. *Let t and z be two weighting vectors, $t = (t_1, t_2, \ldots, t_n)$ and $z = (z_1, z_2, \ldots, z_n)$ in $[0,1]^n$ such that $t_i, z_i \in [0,1]$ and $\sum\limits_{1 \leq i \leq n} t_i = \sum\limits_{1 \leq i \leq n} z_i = 1$.*

Then, the map $F_{tz} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a n-dimensional WOWA operator if:

$$F_{tz}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_i \omega_i \alpha_{\sigma(i)} \tag{6}$$

being $\{\sigma(1), \ldots, \sigma(n)\}$ *a permutation of the values* $\{1, \ldots, n\}$ *fulfilling that* $\alpha_{\sigma(i-1)} \geq \alpha_{\sigma(i)}$ *for every i from 2 to n (that is,* $a_{\sigma(i)}$ *is the ith highest value among the elements* $\alpha_1, \ldots, \alpha_n$*) and* ω_i *the weight defined as:*

$$\omega_i = z^* \left(\sum_{j \leq i} t_{\sigma(j)} \right) - z^* \left(\sum_{j < i} t_{\sigma(j)} \right) \tag{7}$$

where z^* *is a monotonically increasing function used to interpolate the point set*

$$\left\{ \left(\frac{i}{n}, \sum_{j \leq i} z_j \right) \mid i = 1, \ldots, n \right\}, \tag{8}$$

together with the point $(0,0)$*. This interpolation function* z^* *is chosen to be a straight line whenever possible.*

Note that, as particular cases of WOWA operators, if for all i from 1 to n we choose $z_i = 1/n$ then the WOWA operator turns out to be a weighted mean with weighting vector t, and if $t_i = 1/n$ for every i, the resulting function is the OWA operator associated with z.

Using two different weighting vectors allows to improve the results obtained in aggregation processes and will be specially interesting elements in the development of our proposal.

In addition to WOWA operators, linguistic variables will be also used in this paper as a very helpful tool for extracting information.

2.3. Linguistic Variables Defined in the Interval $[0,1]$

Linguistic variables are those that are used to solve problems which involve data that do not take numerical values but are words or sentences.

The definition of linguistic variable was given by Zadeh in [9]. Formally, a *linguistic variable* consists on a tuple (V, T, U, G, M) in which V is the variable name, T is the set formed by the values or linguistic labels that the variable can take, U is the universe where the variable is defined, G contains the syntactic rules used to generate the different labels and the set M is formed by the semantic rules that assign a meaning to each linguistic value in T.

The interpretation of the linguistic label $l \in T$ is represented by a function $c_l : U \to [0,1]$ which associates every value in the universe U that its compatibility with the label l.

Linguistic variables that will be used in this paper are defined in the universe $U = [0,1]$ and the compatibility of the values of U with the label $l \in T$ is measured using a truncated symmetric trapezoidal fuzzy number. More concretely, for each label l we will use a compatibility function $c_l(x)$ represented in terms of a tern of parameters (a, b, p) such that $a, b \in [0,1]$ (with $a \leq b$) and $p > 0$, which mathematical formulation is given for every $x \in [0,1]$ by the function:

$$c_l(x) = \begin{cases} 1 + \frac{x-a}{p} & \text{if } x \in [a-p, a] \\ 1 & \text{if } x \in [a, b] \\ 1 - \frac{x-b}{p} & \text{if } x \in [b, b+p] \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The parameters (a, b, p) are considered to by assigned in the definition of the label $l \in T$. When no confusion can arise, and for the sake of simplicity, the value $c_l(x)$ will be named as x_l ($x_l = c_l(x)$).

Note that this definition of truncated trapezoidal number is the restriction to the interval $[0,1]$ of the usual trapezoidal fuzzy numbers defined in $\mathbb{R}$ [18]. In Figure 1 we show a graphic representing this kind of fuzzy numbers.

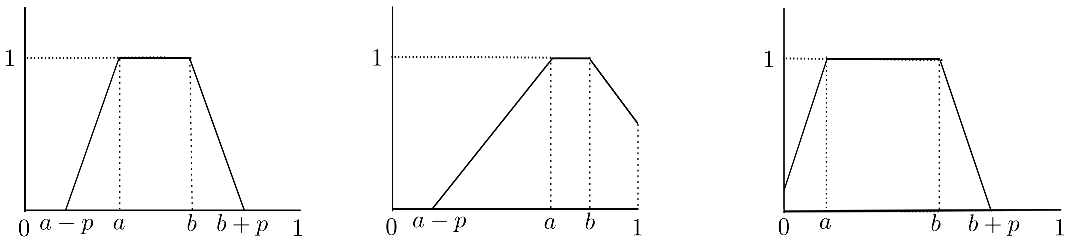

Figure 1. Examples of fuzzy numbers assigned to labels.

2.4. *L*-Fuzzy Hypercontexts

There are situations in which it is necessary to extend L-fuzzy contexts into more suitable object and attribute sets. L-fuzzy hypercontexts were introduced [6] to help model these new frameworks.

Definition 3. *The tuple* $(L, X, Y, (Q_x)_{x\in X}, (S_y)_{y\in Y}, R)$*, in which L has the structure of complete lattice, X and Y are two sets of objects and attributes respectively,* $(Q_x)_{x\in X}$ *and* $(S_y)_{y\in Y}$ *are set families linked to the elements of X and Y, and the relation R is such that, for every* $(x, y) \in X \times Y$*, the value* $R(x, y) = R_{xy} \in L^{Q_x \times S_y}$ *is a new relation, is called L-fuzzy hypercontext. Each relation* R_{xy} *characterizes another new L-fuzzy context* (L, Q_x, S_y, R_{xy}) *whose objects and attributes are said to be* subobjects *and* subattributes *of the hypercontext.*

Remark 1. *The subobjects of* Q_x *associated with* $x \in X$ *are not necessarily elements belonging to X. The same goes for the elements of* S_y*.*

Note that, as the original relation R does not define a L-fuzzy context, it is not possible to extract information using L-fuzzy concept analysis tools. In [6], these new structures are transformed in order to aggregate those parts of the hypercontexts which represent elements that are not essential for the treated information. In that study, the analysis was carried out using a complete lattice L and OWA operators. The main intention of the present work is to improve this study using the lattice $L = [0, 1]$ and WOWA operators. The advantage of these operators is that they use two weighting vectors, one of them is associated with the values (z) and the other one (t) is needed to prioritize the sources from which these values have been obtained. These operators open up new expectations for information extraction as we will see with a practical application in Section 5.

3. Aggregated *L*-Fuzzy Contexts

This section is devoted to transform L-fuzzy hypercontexts aggregating any subset of objects and subattributes depending on the kind of information that is expected. The goal of this process is to obtain a more manageable L-fuzzy context.

By the definition of L-fuzzy hypercontext, we can observe that the main difference with L-fuzzy contexts is the emergence of two new sets in addition to the previously existing sets of objects and attributes. Hence, aggregation processes become very important when it comes to diminishing the size of the context. The key point under this process is to achieve a suitable aggregation function that maintains the relevant information.

This study would not be possible using only OWA operators, as they do not allow to take into account the variability of the data depending, for example, on the time. For this reason, our proposal consists in using WOWA operators defined in the lattice $L = [0, 1]$.

Definition 4. *Let $(L, X, Y, (Q_x)_{x\in X}, (S_y)_{y\in Y}, R)$ be a L-fuzzy hypercontext. For each pair $(x, y) \in X \times Y$ we choose a family of subobjects $E_x \subseteq Q_x$ and a set of subattributes $P_y \subseteq S_y$ and we denote $\check{X} = \{\dot{E}_x \mid x \in X\}$ and $\check{Y} = \{\dot{P}_y \mid y \in Y\}$, where:*

$$\dot{E}_x = \begin{cases} (x, E_x) \cup (\{x\} \times (Q_x \backslash E_x)) & \text{if } E_x \neq \emptyset \\ \{x\} \times Q_x & \text{if } E_x = \emptyset \end{cases} \tag{10}$$

and, similarly,

$$\dot{P}_y = \begin{cases} (y, P_y) \cup \left(\{y\} \times (S_y \backslash P_y)\right) & \text{if } P_y \neq \emptyset \\ \{y\} \times S_y & \text{if } P_y = \emptyset \end{cases} \tag{11}$$

Given any pair $(\check{x}, \check{y}) \in \check{X} \times \check{Y}$, we can consider $F_{t_{\check{x}\check{y}} z_{\check{x}\check{y}}}$ the WOWA operator associated with the weighting vectors $t_{\check{x}\check{y}} = (t_1, t_2, \ldots, t_{|\check{x}|\cdot|\check{y}|})$, and $z_{\check{x}\check{y}} = (z_1, z_2, \ldots, z_{|\check{x}|\cdot|\check{y}|})$, being $|\check{x}|$ the number of subobjects included in the definition of the element $\check{x}$ and $|\check{y}|$ the number of subattributes included in the element $\check{y}$.

The aggregated L-fuzzy context *is defined as the L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$ in which the relation $\check{R} : \check{X} \times \check{Y} \longrightarrow L$ verifies, for every pair $(\check{x}, \check{y}) \in \check{X} \times \check{Y}$, that $\check{R}(\check{x}, \check{y})$ is the aggregated value obtained applying the WOWA operator $F_{t_{\check{x}\check{y}} z_{\check{x}\check{y}}}$ to the elements of the subrelation of R corresponding to the subobjects and subattributtes included in $(\check{x}, \check{y})$.*

This definition allows to select sets of subobjects and subattributes taking into account the different aspects that are of interest in each case.

We will aggregate those sets of subobjects or subattributes for which it is not necessary to distinguish the information related to each one, but the information provided by the total set of them. This will reduce the size of the initial context and simplify the calculations considerably.

Among the different aggregated L-fuzzy contexts that can be defined from a given L-fuzzy hypercontext the ones that appear in the following subsections stand out for their applicability.

3.1. Transforming L-Fuzzy Hypercontexts into L-Fuzzy Contexts

In a first approximation, we will try to maintain the complete information existing in the context. Therefore, we will not aggregate any subobjects or subattributes, that is, in the aggregated L-fuzzy context we will consider $E_x = \emptyset$ and $P_y = \emptyset$ for every element $(x, y) \in X \times Y$.

The aggregated L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$ is such that the object set $\check{X}$ is the collection of pairs:

$$\check{X} = \{(\{x\} \times Q_x) \mid x \in X\} \tag{12}$$

the attribute set $\check{Y}$ is defined as,

$$\check{Y} = \left\{\left(\{y\} \times S_y\right) \mid y \in Y\right\} \tag{13}$$

and, for all $((x, q), (y, s)) \in \check{X} \times \check{Y}$ the relation takes the value

$$\check{R}((x, q), (y, s)) = R_{xy}(q, s) \tag{14}$$

Proposition 1. *The extension of the derivation operators to the aggregated context defined above is done by the following expressions:*

Given $E \in L^{\check{X}}$, for all $(y,s) \in \check{Y}$,

$$E_1(y,s) = \inf_{(x,q)\in\check{X}}\{\mathcal{I}(E(x,q), \check{R}((x,q)(y,s))\} = \inf_{(x,q)\in\check{X}}\{\mathcal{I}(E(x,q), R_{xy}(q,s)\} \quad (15)$$

And, if $P \in L^{\check{Y}}$, for all $(x,q) \in \check{X}$,

$$P_2(x,q) = \inf_{(y,s)\in\check{Y}}\{\mathcal{I}(P(y,s)), \check{R}((x,q)(y,s))\} = \inf_{(y,s)\in\check{Y}}\{\mathcal{I}(P(y,s)), R_{xy}(q,s)\} \quad (16)$$

being $\mathcal{I}$ a L-fuzzy implication operator which is defined in the lattice $(L, \leq)$.

The derived set E_1 is formed by the attributes associated with all the elements of E, and the objects of the set P_2 share all the attributes of P.

Proof. The proof is straightforward applying Equations (1) and (2) to the above defined aggregated L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$. □

This aggregated L-fuzzy context keeps the complete original information but its size is usually very large. To reduce this size, we can choose those objects and attributes we are interested in, and after aggregating the others, extract the corresponding information helped by the L-fuzzy concepts obtained from departure sets that represent each situation that we want to analyze. Furthermore, we can establish different points of view using different weighing vectors.

3.2. Aggregating All the Subobjects and Subattributes

In the case of being interested in analyzing the behavior of objects and attributes but without distinguishing their subobjects and subattributes, we can define an aggregated L-fuzzy context, considering for every $(x,y) \in X \times Y$ the subsets $E_x = Q_x$ and $P_y = S_y$, that is, aggregating all the values of the subrelation R_{xy}.

The obtained L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$ is such that $\check{X} = \{(x, Q_x) \mid x \in X\}$, $\check{Y} = \{(y, S_y) \mid y \in Y\}$ and the relation $\check{R} : \check{X} \times \check{Y} \longrightarrow L$, where for all $(x,y) \in X \times Y$:

$$\check{R}((x,Q_x),(y,S_y)) = F_{t_{xy}z_{xy}}(R_{xy}(q_{x_1}, s_{y_1}), R_{xy}(q_{x_1}, s_{y_2}), \ldots, R_{xy}(q_{x_{|Q_x|}}, s_{y_{|S_y|}})) \quad (17)$$

3.3. Aggregating All the Subobjects or All the Subattributes

Another interesting particular situation is that in which we want to take into account the information associated with the different objects without distinguishing subobjects but we are interested in maintaining the subattributes.

Now, we will construct the aggregated L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$ where, for every $(x,y) \in X \times Y$, $E_x = Q_x$ and $P_y = \emptyset$ and the relation $\check{R}$ is given for all $x \in X, y \in Y, s \in S_y$, by:

$$\check{R}((x,Q_x),(y,s)) = F_{t_x z_x}(R_{xy}(q_{x_1}, s), R_{xy}(q_{x_2}, s), \ldots, R_{xy}(q_{x_{|Q_x|}}, s) \quad (18)$$

In this occasion, the objects of set X can be analyzed despite the lack of information about the elements of Q_x.

A similar procedure can be carried out if the information related to the subobjects is important no matter which subattributes assumes each value. The aggregated L-fuzzy context considered to represent this situation is the one associated with the subsets $E_x = \emptyset$ and $P_y = S_y$, for all $x \in X$ and $y \in Y$.

3.4. Information Associated with the Set of Subobjects or Subattributes

When all the objects share a set of subobjects or if the set of subattributes is the same for every attribute, more comprehensive studies can be carried out and it is possible extract information related to these subobjects or subattributes.

Let us suppose first that in the L-fuzzy hypercontext $(L, X, Y, (Q_x)_{x\in X}, (S_y)_{y\in Y}, R)$ the set of subattributes is kept constant for all the attributes, that is, $S_y = S$ for all $y \in Y$. In this case, the L-fuzzy hypercontext can be rewritten as $(L, X, Y, (Q_x)_{x\in X}, S, R)$.

For the sake of analyzing the information gathered by the set of subattributes, we could transform this hypercontext into another one where attributes and subattributes switch roles and apply the procedure developed in previous subsection.

In this way, we can consider a new L-fuzzy hypercontext $(L, X, S, (Q_x)_{x\in X}, Y, R')$ such that for each $x \in X$, $y \in Y$ and $s \in S$, $R'_{xs}(q_x, y) = R_{xy}(q_x, s)$ for all $q_x \in Q_x$. Then, taking $E_x = \emptyset$ and $P_s = Y$ for every $x \in X$ and $s \in S$, one can define the corresponding aggregated L-fuzzy context $(L, \check{X}, \check{S}, \check{R})$ where for all $(x, q) \in \check{X}$ and $s \in S$,

$$\check{R}((x, q), (s, Y)) = F_{t_Y z_Y}(R'_{xs}(q_x, y_1), R'_{xs}(q_x, y_2), \ldots, R'_{xs}(q_x, y_{|Y|})) \tag{19}$$

Similar developments can be obtained in the case of having $Q_x = Q$ for any $x \in X$. This would allow us to study in depth the subobjects of the context.

4. Using Linguistic Variables in Departure Sets

Once the aggregated L-fuzzy context $(L, \check{X}, \check{Y}, \check{R})$ has been defined, it is possible to obtain knowledge from a departure set. This departure set of either objects or attributes, will represent the situation that we want to analyze at each moment. Corresponding information will be given through the associated L-fuzzy concept obtained by means of the derivation operators.

Although this process produces good results when the considered objects or attributes present high membership values, there are problems in applying it to the low ones since the results could not be significant in these cases. Furthermore, sometimes we will be interested not only in studying high or low values of objects or attributes but also in other ones: medium, medium-high, very low etc. In these cases, linguistic variables can be a good tool to obtain more accurate information.

In Appendix A is given the Algorithm A1 which provides the procedure for obtaining those attributes that can be expected when the approximate values of some objects are known.

A similar algorithm (see Algorithm A2) was developed with the purpose of getting the objects to be shared by a given fuzzy set of attributes.

5. Practical Application: Evaluation of a Hotel Company

There are many fields in which the available information can vary over time and for which the L-fuzzy hypercontexts analysis can be of great utility. Let us think, for example, of domains such as health, finance or environmental conditions related to climate change.

To illustrate the applicability of the theory developed above, we present in this section the case of a hotel company interested in analyzing the valuation made by the different customers over the last three years. The company have some establishments that are regular hotels and others that operate as Bed and Breakfast accommodations (*B&B*). In addition, the company has been changing the survey and the questions answered by customers were different depending on the year of stay.

Before starting with the analysis of the survey, it is necessary to establish the L-fuzzy hypercontext $(L, X, Y, (Q_x)_{x\in X}, (S_y)_{y\in Y}, R)$ to represent the situation. In this occasion the set of objects $X = \{x_1, x_2\}$ will be formed by the different types of establishments that the company has: $x_1 = hotels$ and $x_2 = B\&B$. We will consider the set of attributes $Y = \{y_1, y_2, y_3\}$ each of whose elements corresponds with one of the three years in which the survey was carried out.

For every type of establishment the set of subobjects represents the different accommodation regimes.

- For x_1 = *hotels*, we consider the set of subobjects $Q_{x_1} = \{q_{x_1}1, q_{x_1}2, q_{x_1}3\}$ whose elements represent *breakfast included*, *half-board room* and *full board*, respectively.
- For $x_2 = B\&B$, we will take the set $Q_{x_2} = \{q_{x_2}1, q_{x_2}2\}$ formed by the elements *breakfast included* and *without breakfast*.

For each year $y \in Y$, the corresponding family of subattributes $(S_y)_{y\in Y}$ is set up by the different categories the satisfaction survey was based on: *environment and decoration, comfort, equipment and facilities, reception, quality/price relation*. Specifically,

- The elements of $S_{y_1} = \{s_{y_1}1, s_{y_1}2, s_{y_1}3\}$ represent *quality/price relation*, *reception* and *comfort*, respectively.
- $S_{y_2} = \{s_{y_2}1, s_{y_2}2, s_{y_2}3\}$ is formed by the categories *quality/price relation*, *environment and decoration* and *equipment and facilities*.
- $S_{y_3} = \{s_{y_3}1, s_{y_3}2\}$ whose elements represent, respectively, *quality/price relation* and *comfort*.

Finally, for each different establishment type x the relation $R_{xy} \in L^{Q_x \times S_y}$, represents the score obtained in the categories that where analyzed in the year y. The complete relation R_1 is given in Table 1.

Table 1. Hotel guest evaluation.

R_1	y_1				y_2				y_3		
	$R_{x_1y_1}$	$s_{y_1}1$	$s_{y_1}2$	$s_{y_1}3$	$R_{x_1y_2}$	$s_{y_2}1$	$s_{y_2}2$	$s_{y_2}3$	$R_{x_1y_3}$	$s_{y_3}1$	$s_{y_3}2$
x_1	$q_{x_1}1$	0.3	1	0.1	$q_{x_1}1$	0.6	0.9	1	$q_{x_1}1$	0.4	0.5
	$q_{x_1}2$	0.7	0.3	0.8	$q_{x_1}2$	1	0	0.2	$q_{x_1}2$	0	1
	$q_{x_1}3$	0.9	0.2	0	$q_{x_1}3$	0.6	0.8	1	$q_{x_1}3$	0.9	1
	$R_{x_2y_1}$	$s_{y_1}1$	$s_{y_1}2$	$s_{y_1}3$	$R_{x_2y_2}$	$s_{y_2}1$	$s_{y_2}2$	$s_{y_2}3$	$R_{x_2y_3}$	$s_{y_3}1$	$s_{y_3}2$
x_2	$q_{x_2}1$	0.7	0.8	1	$q_{x_2}1$	0.3	0.9	0	$q_{x_2}1$	0.4	0.2
	$q_{x_2}2$	1	0	0.2	$q_{x_2}2$	1	0.2	0.8	$q_{x_2}2$	0	0.6

The techniques developed in this work allow us to approach the analysis of different situations as we will show below.

To start, we are interested in analyzing the opinion that customers of each establishment type have done distinguishing two categories: *quality/price relation* and *other aspects*, but without taking into account the accommodation regime. To do this we can consider the following subsets of subobjects and subattributes:

$$E_{x_1} = \{q_{x_1}1, q_{x_1}2, q_{x_1}3\}$$
$$E_{x_2} = \{q_{x_2}1, q_{x_2}2\}$$
$$P_{y_1} = \{s_{y_1}2, s_{y_1}3\}$$
$$P_{y_2} = \{s_{y_2}2, s_{y_2}3\}$$

Then we can transform this L-fuzzy hypercontext into an aggregated L-fuzzy context (see Definition 4) by aggregating the values of the relation that correspond to the sets of subobjects and subattributes above. Value sets to be aggregated are highlighted in Table 2.

Table 2. Values to be aggregated.

R_1	y_1				y_2				y_3		
	$R_{x_1y_1}$	$s_{y_1}1$	$s_{y_1}2$	$s_{y_1}3$	$R_{x_1y_2}$	$s_{y_2}1$	$s_{y_2}2$	$s_{y_2}3$	$R_{x_1y_3}$	$s_{y_3}1$	$s_{y_3}2$
x_1	$q_{x_1}1$	0.3	1	0.1	$q_{x_1}1$	0.6	0.9	1	$q_{x_1}1$	0.4	0.5
	$q_{x_1}2$	0.7	0.3	0.8	$q_{x_1}2$	1	0	0.2	$q_{x_1}2$	0	1
	$q_{x_1}3$	0.9	0.2	0	$q_{x_1}3$	0.6	0.8	1	$q_{x_1}3$	0.9	1
	$R_{x_2y_1}$	$s_{y_1}1$	$s_{y_1}2$	$s_{y_1}3$	$R_{x_2y_2}$	$s_{y_2}1$	$s_{y_2}2$	$s_{y_2}3$	$R_{x_2y_3}$	$s_{y_3}1$	$s_{y_3}2$
x_2	$q_{x_2}1$	0.7	0.8	1	$q_{x_2}1$	0.3	0.9	0	$q_{x_2}1$	0.4	0.2
	$q_{x_2}2$	1	0	0.2	$q_{x_2}2$	1	0.2	0.8	$q_{x_2}2$	0	0.6

To determine the WOWA operators that are used to calculate the aggregated L-fuzzy context, we choose the weighting vectors so that, with the aim of improving service, we will give greater importance to the lowest scores. Hence, depending on the value of $|\check{x}| \cdot |\check{y}|$, that is, depending on whether the quantity of numbers to be aggregated is 2, 3, 4 or 6, we will consider:

$$t_{\check{x}\check{y}} = \begin{cases} (0.4, 0.6) & \text{when } |\check{x}| \cdot |\check{y}| = 2 \\ (0.2, 0.4, 0.4) & \text{when } |\check{x}| \cdot |\check{y}| = 3 \\ (0.2, 0.2, 0.3, 0.3) & \text{when } |\check{x}| \cdot |\check{y}| = 4 \\ (0.1, 0.1, 0.2, 0.2, 0.2, 0.2) & \text{when } |\check{x}| \cdot |\check{y}| = 6 \end{cases}$$

On the other hand, we will prioritize the opinions of customers who hired more complete services. To achieve it the chosen second weighting vector will be:

$$z_{\check{x}\check{y}} = \begin{cases} (0.3, 0.7) & \text{when } |\check{x}| \cdot |\check{y}| = 2 \\ (0.2, 0.3, 0.5) & \text{when } |\check{x}| \cdot |\check{y}| = 3 \\ (0.1, 0.2, 0.3, 0.4) & \text{when } |\check{x}| \cdot |\check{y}| = 4 \\ (0.1, 0.1, 0.1, 0.2, 0.2, 0.3) & \text{when } |\check{x}| \cdot |\check{y}| = 6 \end{cases}$$

As interpolating function, we will use the interpolation polynomial $z^*(x)$ such that $z^*(0) = 0$ and $z^*\left(\frac{i}{n}\right) = \sum_{j \leq i} z_j$ for every $i = 1, \ldots, n$, which is a monotonic increasing function in the interval $[0, 1]$.

The relation of the obtained aggregated L-fuzzy context $(L, \check{X}, \check{Y}, \check{R}_2)$ can be seen in Table 3.

Table 3. Aggregated L-fuzzy relation.

$\check{R}_2$	$(y_1, s_{y_1}1)$	(y_1, P_{y_1})	$(y_2, s_{y_2}1)$	(y_2, P_{y_2})	$(y, s_{y_3}1)$	$(y_3, s_{y_3}2)$
(x_1, E_{x_1})	0.63	0.25	0.7	0.43	0.31	0.87
(x_2, E_{x_2})	0.83	0.26	0.61	0.32	0.1	0.38

Once the aggregated context is obtained, information extraction is carried out by analyzing the L-fuzzy concepts. In order to calculate the derived set we use the Lukasiewicz implication wich is a residuated operator [10,11] and is defined as:

$$\mathcal{I}(x, y) = min(1, 1 - x + y), \text{ for all } x, y \in [0, 1] \tag{20}$$

From this aggregated context it is possible, for example, to extract information about those establishments that achieved a very good score in the category *quality/price relation* in the first year. This information is provided by the L-fuzzy concept reached from the attribute set that we use to represent the situation to be analyzed:

$$\{(y_1, s_{y_1 1})/1, (y_1, P_{y_1})/0, (y_2, s_{y_2 1})/0, (y_2, P_{y_2})/0, (y, s_{y_3 1})/0, (y_3, s_{y_3 2})\}$$

Taking this set of attributes as starting point, the obtained L-fuzzy concept is:

$$\{(x_1, E_{x_1})/0.63, (x_2, E_{x_2})/0.83\},$$
$$\{(y_1, s_{y_1 1})/1, (y_1, P_{y_1})/0.43, (y_2, s_{y_2 1})/0.78, (y_2, P_{y_2})/0.49, (y, s_{y_3 1})/0.27, (y_3, s_{y_3 2})/0.55\}$$

and, attending to the membership values we can conclude a good score in the first year was obtained more commonly by a *B&B* establishment. We also see that this score has been decreasing in the following years. On the other hand, the valuation obtained in the other categories has presented a slight growth over time.

For a more exhaustive analysis, we could convert the initial L-fuzzy hypercontext into a L-fuzzy context maintaining the complete information (see Section 3.1). The relation that defines this new context is the one represented in Table 4.

Table 4. Obtained L-fuzzy relation.

$\check{R}_3$	$(y_1, s_{y_1 1})$	$(y_1, s_{y_1 2})$	$(y_1, s_{y_1 3})$	$(y_2, s_{y_2 1})$	$(y_2, s_{y_2 2})$	$(y_2, s_{y_2 3})$	$(y_3, s_{y_3 1})$	$(y_3, s_{y_3 2})$
$(x_1, q_{x_1 1})$	0.3	1	0.1	0.6	0.9	1	0.4	0.5
$(x_1, q_{x_1 2})$	0.7	0.3	0.8	1	0	0.2	0	1
$(x_1, q_{x_1 3})$	0.9	0.2	0	0.6	0.8	1	0.9	1
$(x_2, q_{x_2 1})$	0.7	0.8	1	0.3	0.9	0	0.4	0.2
$(x_2, q_{x_2 2})$	1	0	0.2	1	0.2	0.8	0	0.6

This context maintains all available information and allows to answer more concrete questions by relating hotel types, analyzed characteristics and periods of time.

For example, we can analyze here which of the establishments obtained a good score in the question related to *comfort* when this category was included in the survey. In this occasion we will depart from the set representing the attributes that measure comfort:

$$\{(y_1, s_{y_1 1})/0, (y_1, s_{y_1 2})/0, (y_1, s_{y_1 3})/1, (y_2, s_{y_2 1})/0, y_2, s_{y_2 2})/0, (y_2, s_{y_2 3})/0, (y_3, s_{y_3 1})/0, (y_3, s_{y_3 2})/1\}$$

the obtained L-fuzzy concept is:

$$\{(x_1, q_{x_1 1})/0.1, (x_1, q_{x_1 2})/0.8, (x_1, q_{x_1 3})/0, (x_2, q_{x_2 1})/0.2, (x_2, q_{x_2 2})/0.2\},$$
$$\{(y_1, s_{y_1 1})/0.9, (y_1, s_{y_1 2})/0.5, (y_1, s_{y_1 3})/1, (y_2, s_{y_2 1})/1, (y_2, s_{y_2 2})/0.2, (y_2, s_{y_2 3})/0.4, (y_3, s_{y_3 1})/0.2, (y_3, s_{y_3 2})/1\}$$

from where we can deduce that the good score in the question related to *comfort* was given to *hotels* by customers in *half-board room* accommodation. These costumers also rated with a high score the *quality/price relation* of the hotels during the first two years but the score was not good in the third one.

Let us suppose now that we want to study the general score obtained by the hotel company every year in its two types of accommodation regardless the accommodation regime or the category analyzed in the survey. With this purpose we will define a L-fuzzy context aggregating all the subobjects and subattributes in the hypercontext (see Section 3.2).

In this occasion, we want to give more relevance to the rates that are closest to 1, therefore will use the weighting vector z such that $z_i = \frac{2(n-i+1)}{n(1+n)}$ for every i.

In addition, since we consider that the contribution to this score of all the establishments and all the categories considered in the survey is the same, we will take a constant weighting vector t. Thus, the aggregation process will be carried out in this case by means of the OWA operator defined from the vector z. Relation in Table 5 defines the aggregated L-fuzzy context.

Table 5. L fuzzy relation obtained aggregating all the subobjects and subattributes.

$\check{R}_4$	(y_1, S_{y_1})	(y_2, S_{y_2})	(y_3, S_{y_3})
(x_1, Q_{x_1})	0.66	0.84	0.80
(x_2, Q_{x_2})	0.80	0.71	0.40

At this point, we can get information with the naked eye. It is easy to conclude that customer satisfaction increased for hotels (x_1) in the second year and, although it has decreased slightly, it has remained high. In contrast, the overall score for $B\&B$ establishments (x_2) has been decreasing rapidly over time.

In order to extract more complete information, we can now establish different departure sets representing the situation we are interested in and calculate the associated L-fuzzy concept.

So, for instance, we can study the establishments that have obtained a good score in the last two years. For this, we consider the starting set

$$\{(y_1, S_{y_1})/0, (y_2, S_{y_2})/1, (y_3, S_{y_3})/1\}$$

and obtain the following L-fuzzy concept:

$$\{(x_1, Q_{x_1})/0.8, (x_2, Q_{x_2})/0.4\},$$
$$\{(y_1, S_{y_1})/0.86, (y_2, S_{y_2})/1, (y_3, S_{y_3})/1\}$$

This result can be interpreted by saying that the establishments that scored well in the last two years were mainly *hotels* (x_1), and that the score obtained by those hotels in the first year was also quite good.

Another interesting particular situation is described in Section 3.3 where we want to maintain the subattributes but not the subobjects. That is, we are interested in analyzing the results making a distinction between the types of establishments but without taking into account the selected accommodation. We can choose the weighting vectors so that the highest membership degrees are prioritized and suppose that we have ordered the establishments from the least to the greatest relevance. Then, to aggregate the values, we will use $t_{x_1} = (1/6, 2/6, 3/6)$, $z_{x_1} = (3/6, 2/6, 1/6)$, $t_{x_2} = (1/3, 2/3)$ and $z_{x_2} = (2/3, 1/3)$, and the piecewise linear interpolation of the points $(i/n, \sum_{j \leq i} z_j)$ in both cases. The obtained result is given in Table 6.

Table 6. Relation obtained aggregating all the subobjects.

$\check{R}_5$	$(y_1, s_{y_1}1)$	$(y_1, s_{y_1}2)$	$(y_1, s_{y_1}3)$	$(y_2, s_{y_2}1)$	$(y_2, s_{y_2}2)$	$(y_2, s_{y_2}3)$	$(y_3, s_{y_3}1)$	$(y_3, s_{y_3}2)$
(x_1, Q_{x_1})	0.79	0.44	0.42	0.8	0.69	0.87	0.66	0.95
(x_2, Q_{x_2})	0.93	0.35	0.55	0.85	0.51	0.62	0.18	0.51

Now we can analyze the score obtained by the two sections of the hotel company in all the considered categories over the three years.

For instance, if we want to study when and in which of the categories were the two types of establishments well scored, we take the set $\{(x_1, Q_{x_1})/1, (x_2, Q_{x_2})/1\}$ and, applying the derivation operators, obtain the fuzzy intension of the L-fuzzy concept:

$$\{(y_1, s_{y_1 1})/0.8, (y_1, s_{y_1 2})/0.35, (y_1, s_{y_1 3})/0.42, (y_2, s_{y_2 1})/0.8, (y_2, s_{y_2 2})/0.5, (y_2, s_{y_2 3})/0.62, (y_3, s_{y_3 1})/0.18, (y_3, s_{y_3 2})/0.5\}$$

We can conclude that both *hotels* and *B&B* were well rated in the question regarding *quality/price relation* in the first two years ($s_{y_1 1}$ and $s_{y_2 1}$), and the result was also quite good for *equipment and facilities* ($s_{y_2 3}$).

The next study that can be carried out consists of the evolution over time of the general score obtained by each of the different accommodations in the two types of establishment. In this situation we will not make a distinction among the subattributes (see Section 3.3). Here we are going to give a higher value to highest marks and assume that the questions in the survey were ordered from most to least important. Hence, to aggregate the values in the relation, we will use the weighting vectors $t_{y_1} = t_{y_2} = (0.5, 0.3, 0.2)$, $t_{y_3} = (0.6, 0.4)$, $z_{y_1} = z_{y_2} = (0.5, 0.3, 0.2)$ and $z_{y_3} = (0.6, 0.4)$. Interpolation function is the same that in the previous case. The obtained aggregated relation is shown in Table 7.

Table 7. Relation obtained aggregating all the subattributes.

$\check{R}_6$	(y_1, S_{y_1})	(y_2, S_{y_2})	(y_3, S_{y_3})
$(x_1, q_{x_1 1})$	0.59	0.83	0.45
$(x_1, q_{x_1 2})$	0.67	0.68	0.48
$(x_1, q_{x_1 3})$	0.63	0.79	0.95
$(x_2, q_{x_2 1})$	0.83	0.53	0.34
$(x_2, q_{x_2 2})$	0.68	0.82	0.29

We can examine now, for example, how the general opinion of customers who selected *breakfast included* accommodation has evolved. Our departure set to represent this situation is:

$$\{(x_1, q_{x_1 1})/1, (x_1, q_{x_1 2})/0, (x_1, q_{x_1 3})/0, (x_2, q_{x_2 1})/1, (x_2, q_{x_2 2})/0\}$$

and the obtained associated L-fuzzy concept has the intension:

$$\{(y_1, S_{y_1})/0.59, (y_2, S_{y_2})/0.53, (y_3, S_{y_3})/0.34\}$$

From which can be determined that the general score given by customers who chose *breakfast included* accommodation is not good and, moreover, it has worsened over the years.

If the satisfaction survey were the same every year, it would be possible to carry out an analysis of each of the questions regardless of time, as explained in Section 3.4. For example, let us consider that each year we have collected the scores obtained in the question regarding *quality/price relation* (s_1) and, under the heading of *amenities* (s_2) we have registered the mean of the punctuations given for the other questions. The resulting relation is the one shown in Table 8.

As the set S does not vary with year, we can exchange the role of attributes and subattributes and consider the new L-fuzzy hypercontext given by relation R_8 in Table 9 (see Section 3.4).

Table 8. Coincident sets of subattributes

R_7		y_1		y_2		y_3	
		s_1	s_2	s_1	s_2	s_1	s_2
x_1	$q_{x_1 1}$	0.3	0.55	0.6	0.95	0.4	0.5
	$q_{x_1 2}$	0.7	0.55	1	0.1	0	1
	$q_{x_1 3}$	0.9	0.1	0.6	0.9	0.9	1
x_2	$q_{x_2 1}$	0.7	0.9	0.3	0.45	0.4	0.2
	$q_{x_2 2}$	1	0.1	1	0.5	0	0.6

Table 9. Relation of the new L-fuzzy hypercontext

R_8		s_1			s_2		
		y_1	y_2	y_3	y_1	y_2	y_3
x_1	$q_{x_1 1}$	0.3	0.6	0.4	0.55	0.95	0.5
	$q_{x_1 2}$	0.7	1	0	0.55	0.1	1
	$q_{x_1 3}$	0.9	0.6	0.9	0.1	0.9	1
x_2	$q_{x_2 1}$	0.7	0.3	0.4	0.9	0.45	0.2
	$q_{x_2 2}$	1	1	0	0.1	0.5	0.6

At this time we can aggregate the marks obtained in the three years to study the general score obtained by the company's establishments in each question of the survey. With the aim of considering the highest values more important, we will employ the weighting vector $z = (0.5, 0.3, 0.2)$. Using the vector $t = (0.2, 0.3, 0.5)$ we give more relevance to the most recent values. We obtain the aggregated L-fuzzy context defined by the relation given in Table 10.

Table 10. Relation of the aggregated L-fuzzy context.

$\check{R}_9$	(s_1, Y)	(s_2, Y)
$(x_1, q_{x_1 1})$	0.48	0.71
$(x_1, q_{x_1 2})$	0.59	0.76
$(x_1, q_{x_1 3})$	0.85	0.87
$(x_2, q_{x_2 1})$	0.47	0.5
$(x_2, q_{x_2 2})$	0.65	0.52

From this context we can extract information about which are the accommodation establishments that maintain over time a good score in the different categories.

For instance, if we select the departure set to represent the situation where the question about *quality/price relation* (s_1, Y) has achieved a good score:

$$\{(s_1, Y)/1, (s_2, Y)/0\}$$

we obtain the following L-fuzzy concept:

$$\{(x_1, q_{x_1 1})/0.48, (x_1, q_{x_1 2})/0.59, (x_1, q_{x_1 3})/0.85, (x_2, q_{x_2 1})/0.47, (x_2, q_{x_2 2})/0.65\},$$
$$\{(s_1, Y)/1, (s_2, Y)/0.87\}$$

Hence, we can conclude that throughout these three years *quality/price relation* was well scored by costumers of *hotels* in *full board* accommodation $(x_1, q_{x_1 3})$ and the punctuation given by guests of *B&B* establishments in room *without breakfast* $(x_2, q_{x_2 2})$ was also quite good. In addition, also the question regarding *amenities* (s_2, Y) received a good score in these two groups of clients, but the average mark was in this case a little worse.

Finally, the introduction of linguistic variables in starting sets will provide us with more accurate results.

Going back to $(L, \check{X}, \check{Y}, \check{R}_6)$, the aggregated L-fuzzy context from which we extracted information regarding the evolution of the general scores over time (See Table 7), and using the *IEA* Algorithm explained in Appendix A, we can now analyze the results introducing different punctuation levels. So, for example, we can answer questions such as whether there have been accommodations that, having obtained medium-high scores in the first two years, have worsened to low values in the third one.

In order to apply the exposed process we need, first to all, to define a linguistic variable V. The considered label set is in this case

$$T = \{high, medium - high, medium, medium - low, low\},$$

where the labels $medium - high$ and low are associated with the truncated symmetric trapezoidal fuzzy sets defined, respectively, by the terns $(0.6, 0.9, 0.35)$ and $(0, 0.3, 0.4)$. The graphical representation of these labels is shown in Figure 2.

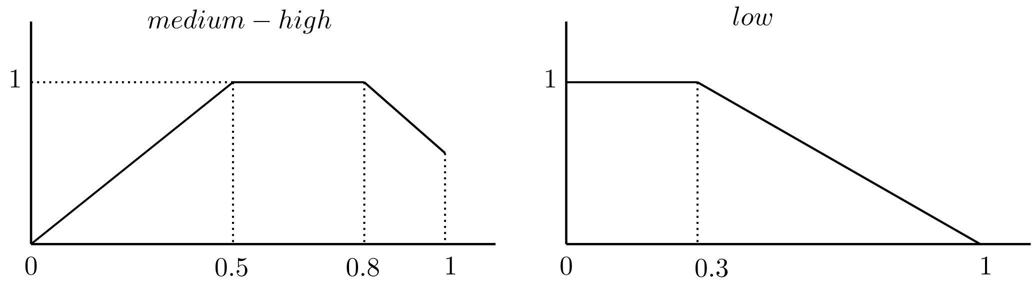

Figure 2. Fuzzy sets corresponding to the considered labels.

The compatibility of value $x \in [0,1]$ with the linguistic label $medium - high$ is measured by the function:

$$c_{medium-high}(x) = \begin{cases} 0 & \text{if } x < 0.25 \\ 1 + \frac{1}{0.35}(x - 0.6) & \text{if } x \in [0.25, 0.6] \\ 1 & \text{if } x \in [0.6, 0.9] \\ 1 - \frac{1}{0.35}(x - 9) & \text{if } x > 0.9 \end{cases}$$

In the case of label low, the compatibility function is defined as:

$$c_{low}(x) = \begin{cases} 1 & \text{if } x < 0.3 \\ 1 - 2.5(x - 0.3) & \text{if } x \in [0.3, 0.7] \\ 0 & \text{if } x > 0.7 \end{cases}$$

The initial situation to be studied will be represented here by the set of pairs

$$\mathcal{U}_{\check{Y}} = \{(\check{y}_1, medium - high), (\check{y}_2, medium - high), (\check{y}_3, low)\}.$$

Next, it is necessary to define the l-labeled L-fuzzy contexts $(L, \check{X}, \check{Y}, \check{R}_{medium-high})$ and $(L, \check{X}, \check{Y}, \check{R}_{low})$ associated with the considered labels. The relations of these contexts are the ones showed in Table 11.

Table 11. Relations defined from labels.

$\check{R}_{medium-high}$	(y_1, S_{y_1})	(y_2, S_{y_2})	(y_3, S_{y_3})
$(x_1, q_{x_1 1})$	0.97	1	0.57
$(x_1, q_{x_1 2})$	1	1	0.66
$(x_1, q_{x_1 3})$	1	1	0.86
$(x_2, q_{x_2 1})$	1	0.8	0.26
$(x_2, q_{x_2 2})$	1	1	0.11
$\check{R}_{low}$	(y_1, S_{y_1})	(y_2, S_{y_2})	(y_3, S_{y_3})
$(x_1, q_{x_1 1})$	0.28	0	0.63
$(x_1, q_{x_1 2})$	0.07	0.05	0.55
$(x_1, q_{x_1 3})$	0.18	0	0
$(x_2, q_{x_2 1})$	0	0.43	0.9
$(x_2, q_{x_2 2})$	0.05	0	1

From the departure set

$$P = \{(y_1, S_{y_1})/1, (y_2, S_{y_2})/1, (y_3, S_{y_3})/0\},$$

and considering the l-labeled L-fuzzy context $(L, \check{X}, \check{Y}, \check{R}_{medium-high})$, we calculate the associated L-fuzzy concept to determine those accommodations that got medium-high scores in the first and second years. The extension of the resulting concept is:

$$\{(x_1, q_{x_1 1})/0.97, (x_1, q_{x_1 2})/1, (x_1, q_{x_1 3})/1, (x_2, q_{x_2 1})/0.8, (x_2, q_{x_2 2})/1\}$$

On the other hand, from the set

$$P = \{(y_1, S_{y_1})/0, (y_2, S_{y_2})/0, (y_3, S_{y_3})/1\},$$

and obtaining the L-fuzzy concept in the l-labeled L-fuzzy context $(L, \check{X}, \check{Y}, \check{R}_{low})$, we get the set of establishments that scored low values in the third year. The result is in this case:

$$\{(x_1, q_{x_1 1})/0.63, (x_1, q_{x_1 2})/0.55, (x_1, q_{x_1 3})/0, (x_2, q_{x_2 1})/0.9, (x_2, q_{x_2 2})/1\}$$

Finally, in order to guess the accommodations fulfilling the two required conditions, we need to calculate the intersection of the obtained two sets of objects applying the intersection associated with the implication operator used in the derivation processes. In this occasion, we will use the bounded difference intersection which is associated with the used Lukasiewicz implicator, and is defined as:

$$i(x, y) = \max(0, x + y - 1), \text{ for all } x, y \in [0, 1] \tag{21}$$

The resulting intersection set is:

$$\{(x_1, q_{x_1 1})/0.6, (x_1, q_{x_1 2})/0.55, (x_1, q_{x_1 3})/0, (x_2, q_{x_2 1})/0.7, (x_2, q_{x_2 2})/1\}$$

In view of the result, where the objects with high membership values are those associated with object x_2 (*B*&*B*), we can conclude that were the clients of the *B*&*B* establishments in both types of accommodation, with or without breakfast, the ones who having issued a medium-high punctuation in the first two years, gave a low score in the last one.

6. Conclusions

The use of L-fuzzy hypercontexts has allowed to work with L-fuzzy contexts formed from variable sets objects and attributes, in which the application of WOWA operators and linguistic variables has provided good tools to extract relevant information.

Different situations have been analyzed depending on different interests. Firstly we have carried out a general study where new L-fuzzy contexts are defined aggregating some parts of the L-fuzzy hypercontext. After that, we study those L-fuzzy contexts that stand out for their applicability.

We have also seen how, using fixed sets of sub-objects or sub-attributes, it is possible to carry out a more detailed analysis.

Our further work focuses on a deeper extension to the fuzzy framework of the study of multicontexts that were defined to be used in the analysis of formal concepts.

From a more practical point of view, we are currently working on issues related to financial credit rating using L-fuzzy hypercontexts.

Author Contributions: C.A. and A.B. have contributed equally to the conception, design and draft of the manuscript, revising it critically. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been partially supported by the research group from the Basque Government "Intelligent Systems and Energy (SI+E)" under grant IT1644-22.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Algorithm A1 Information Extraction from Objects (IEO)

Inputs:

$(L, \check{X}, \check{Y}, \check{R})$: Aggregated L-fuzzy context.
T: Set of labels defined from truncated symmetric trapezoidal fuzzy numbers.
$\mathcal{U}_{\check{X}} = \{(\check{x}_i, l_{\check{x}_i}), \check{x}_i \in \check{X}, l_{\check{x}_i} \in T\}$: Set of pairs representing the situation to analyze.

Output: $P \in L^{\check{Y}}$: Attributes obtained from initial requirements.

Steps:

for all $(\check{x}_i, l_{\check{x}_i}) \in \mathcal{U}_{\check{X}}$ **do**
  for all $\check{x} \in \check{X}$ **do**
    if $\check{x} = \check{x}_i$ **then**
      $E_{\check{x}_i}(\check{x}) \leftarrow 1$ ▷ Departure situation
    else
      $E_{\check{x}_i}(\check{x}) \leftarrow 0$
    end if
  end for
  for all $(\check{x}, \check{y}) \in \check{X} \times \check{Y}$ **do**
    $\check{R}_{l_{\check{x}_i}}(\check{x}, \check{y}) = \check{R}(\check{x}, \check{y})_{l_{\check{x}_i}}$ ▷ Relation of the labeled L-fuzzy context
  end for
  while $E_{\check{x}_i} \neq \varphi(E_{\check{x}_i})$ **do**
    $E_{\check{x}_i} \leftarrow \varphi(E_{\check{x}_i})$ ▷ Calculate the associated concept
  end while
  for all $\check{y} \in \check{Y}$ **do**

$E_{\check{x}_i 1}(\check{y}) \leftarrow (E_{\check{x}_i})_1(\check{y})$ ▷ Intension of the concept
end for
end for
$P \leftarrow \bigcap_{(\check{x}_i, l_{\check{x}_i}) \in \mathcal{U}_{\check{X}}} E_{\check{x}_i 1}$ ▷ Intersection of the intensions

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Algorithm A2 Information Extraction from Attributes (IEA)

Inputs:

$(L, \check{X}, \check{Y}, \check{R})$: Aggregated L-fuzzy context.
T: Set of labels defined from truncated symmetric trapezoidal fuzzy numbers.
$\mathcal{U}_{\check{Y}} = \{(\check{y}_j, l_{\check{y}_j}), \check{y}_j \in \check{Y}, l_{\check{y}_j} \in T\}$: Set of pairs representing the situation to analyze.

Output: $E \in L^{\check{X}}$: Objects obtained from departure set.

Steps:

for all $(\check{y}_j, l_{\check{y}_j}) \in \mathcal{U}_{\check{Y}}$ **do**
for all $\check{y} \in \check{Y}$ **do**
if $\check{y} = \check{y}_j$ **then**
$P_{\check{y}_j}(\check{y}) \leftarrow 1$ ▷ Departure situation
else
$P_{\check{y}_j}(\check{y}) \leftarrow 0$
end if
end for
for all $(\check{x}, \check{y}) \in \check{X} \times \check{Y}$ **do**
$\check{R}_{l_{\check{y}_j}}(\check{x}, \check{y}) = \check{R}(\check{x}, \check{y})_{l_{\check{y}_j}}$ ▷ Relation of the labeled L-fuzzy context
end for
while $P_{\check{y}_j} \neq \phi(P_{\check{y}_j})$ **do**
$P_{\check{y}_j} \leftarrow \phi(P_{\check{y}_j})$ ▷ Calculate the associated concept
end while
for all $\check{x} \in \check{X}$ **do**
$P_{\check{y}_j 2}(\check{x}) \leftarrow (P_{\check{y}_j})_2(\check{x})$ ▷ Extension of the concept
end for
end for
$E \leftarrow \bigcap_{(\check{y}_j, l_{\check{y}_j}) \in \mathcal{U}_{\check{Y}}} P_{\check{y}_j 2}$ ▷ Intersection of the extensions

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