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Analysis of an Abstract Delayed Fractional Integro-Differential System via the α-Resolvent Operator

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Abstract: This paper explores the mild solutions of partial impulsive fractional integro-differential systems of order $1 < \alpha < 2$ in a Banach space. We derive the solution of the system under the assumption that the homogeneous part of the system admits an α-resolvent operator. Krasnoselskii's fixed point theorem is used for the existence of solution, while uniqueness is ensured using Banach's fixed point theorem. The stability of the system is analyzed through the framework of Hyers–Ulam stability using Lipschitz conditions. Finally, examples are presented to illustrate the applicability of the theoretical results.

Keywords: integro-differential system; fractional; impulse; delay; stability in terms of Ulam; α-resolvent operator

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1. Introduction

We investigate the system:

$$\begin{cases} {}^cD_0^{\alpha}u(s) &= \mathcal{A}u(s) + \int_0^s E(s-t)u(t)dt + w(s, u_s, \int_0^s e(s,t,u_t)dt), \ \ s \in [0,T] = I, \ s \neq s_j, \\ u(0) &= \chi + \mu(u) \in \mathcal{B}, \ \ u'(0) = 0, \\ \Delta u(s_j) &= J_j(u_{s_j}), \ \ j \in \{0,1,\dots,m\}, \end{cases} \tag{1}$$

where ${}^cD_0^{\alpha}$ denotes the Caputo derivative with $1 < \alpha < 2$, $\mathcal{A}$ is a closed linear operator with domain $D(\mathcal{A})$ that satisfies the Hille–Yosida axiom, $\overline{D(\mathcal{A})} = \mathcal{U}$ is a Banach space ($\mathcal{BS}$), and E represents the set of operators mapped from $D(\mathcal{A})$ to $\mathcal{U}$ that are linear and bounded. The mappings $u_s : (-\infty, T] \to \mathcal{U}$ defined by $u_s(\theta) = u(s+\theta)$ are elements of an abstract space $\mathcal{B}$ defined axiomatically. Consider the sequence $0 = s_0 < s_1 < \cdots < s_{m+1} = T$ of specified values and $w : I \times \mathcal{B} \times \mathcal{U} \to \mathcal{U}$, $e : I \times I \times \mathcal{B} \to \mathcal{U}$ suitable mappings. The jump $\Delta u(s)$ for any function u at a specific point s is defined as $\Delta u(s) = u(s^+) - u(s^-)$.

In applied mathematics, fractional calculus is an intransitive area that works with integrals and derivatives of real-number or complex-number powers. Developing calculus for differential and integral operators of such powers generalizes classical calculus. Due to its multiple uses in viscoelasticity, biology, control hypotheses, information processing system, and image processing [1–5], fractional calculus has garnered significant relevance and appreciation. Fractional differential equations are also utilized to examine processes such as the rate of substrate dimerization during electrochemical reduction and the analysis of ground water flow problems.

The existence and uniqueness of solutions are key requirements for boundary value problems involving fractional differential equations, revealing the specific behavior of the solution. The existence of mild solutions for fractional integro-differential equations is established through the use of fixed point $(\mathcal{FP})$ theory in Banach spaces $(\mathcal{BS}s)$ using the Caputo fractional derivative in [6]. Shu et al. [7] investigated mild solutions for impulsive fractional evolution equations of order $0 < \alpha < 1$. Building on analytic results using the Mittag-Leffler function proposed a new and more suitable definition of mild solutions for these equations. In recent years, much work has been done on different classes of fractional and integro-differential equations using approaches such as semigroup theory [8,9], resolvent operator theory [10–12] and α-resolvent operator theory [13–18].

It has also be shown that stability analysis is a key aspect of the qualitative research on fractional differential equations [19]. This approach, which emphasizes the stability of differential equations rather than seeking explicit solutions, is recognized for its effectiveness in producing solutions that closely approximate the exact ones. Recently, the authors of [20] examined the Hyers–Ulam stability $(\mathcal{HUS})$ of a coupled system involving Ψ-Caputo fractional derivatives with multipoint–multistrip integral-type boundary conditions, while Sene et al. [6] demonstrated the $\mathcal{HUS}$ of the mild solution for fractional integro-differential equations. For recent literature on stability analysis of differential equations, interested readers can see [21–26] and the references therein.

There has been a surge of research focused on solving systems using the α-resolvent operator. Many studies have built on the related findings, exploring various aspects and applications of the α-resolvent operator in solving fractional integro-differential systems. In 2012, Agarwal et al. [13] investigated the qualitative properties and the existence of an α-resolvent operator for the system

$$\begin{cases} {}^cD_0^\alpha u(s) & = \mathcal{A}u(s) + \int_0^s E(s-\varepsilon)u(\varepsilon)d\varepsilon, \quad s \in I, \\ u(0) & = u_0,\ u'(0) = u_1, \end{cases} \tag{2}$$

where $1 < \alpha < 2$ and where $\mathcal{A}$, $\{E(s)\}_{s\geq 0}$ are closed linear operators defined on a domain which is dense in $\mathcal{U}$. They further examined the existence and regularity of solutions for the nonhomogeneous system

$$\begin{cases} {}^cD_0^\alpha u(s) & = \mathcal{A}u(s) + \int_0^s E(s-\varepsilon)u(\varepsilon)d\varepsilon + \mathcal{Z}(s), \quad s \in I, \\ u(0) & = u_0,\ u'(0) = u_1, \end{cases} \tag{3}$$

where $\mathcal{Z} \in L^1([0,T],\mathcal{U})$. They assumed that the α-resolvent operator is exponentially bounded, meaning that there exists some $w > 0$ such that $\|\tau_\alpha(s)\| \leq Me^{ws}$. Additionally, they demonstrated that the mild and classical solutions of (3) coincide when $\mathcal{Z} \in L^1([0,T],D(\mathcal{A}))$.

Following the work of Agarwal et al., Santos et al. [27] established the existence of mild solutions of fractional integro-differential equations with state-dependent delay using the α-resolvent operator and $\mathcal{FP}$ theory. Vijayakumar et al. [18] examined the fractional integro-differential inclusions in $\mathcal{BS}$s via the resolvent operator and provided the sufficient conditions for controllability by employing Bohnenblust–Karlin's $\mathcal{FP}$ theorem. Similarly, the authors of [17] investigated the approximate controllability of fractional semilinear integro-differential equations using α-resolvent operators, offering two alternative sets of necessary conditions for the problem. The first set employs functional analysis theories and the compactness of the associated resolvent operator, while the second utilizes Gronwall's inequality. For insights into existence results, controllability, and approximate controllability of fractional integro-differential equations, interested readers may see [15,17,20,28–30] and the references therein.

Impulsive delayed fractional integro-differential equations are commonly encountered as models in various applications, which has led to significant attention on the study of such equations in recent years. The literature on this topic primarily focuses on first-order impulsive differential equations with delays; for examples, see [11,14,31,32]. The problem of solution existence for partial fractional differential equations using the α-resolvent operator and $\mathcal{FP}$ theory has been explored in a number of recent works [6,7,15,27,30]. In our current work, we extend and build upon previous research in the area of fractional integro-differential equations with delay. In [33], Agarwal et al. provided the sufficient conditions for the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delays. Here, we further extend this to impulsive systems, which introduces additional complexity. Furthermore, while studies such as [34,35] have investigated the existence and uniqueness of solutions for fractional differential equations with delays, our study introduces the use of the α-resolvent operator for partial impulsive fractional integro-differential systems with $1 < \alpha < 2$, offering a more generalized framework for analyzing these systems. In previous research, studies have either focused on analyzing partial fractional integro-differential systems without impulses, or have considered partial impulsive integro-differential systems while employing different fixed point techniques. The main novelty of this work lies in the methodological approach employed to investigate the studied instantaneous impulsive fractional integro-differential system containing delay and incorporating the Caputo derivative with order $1 < \alpha < 2$. To the best of our knowledge, this is the first study to examine the existence of a mild solution for System (1) using the α-resolvent operator and Krasnoselskii's $\mathcal{FP}$ approach. Furthermore, this study establishes the $\mathcal{HUS}$ for System (1), which has yet to be studied in the literature by using the α-resolvent approach for fractional systems.

The rest of this paper is organized as follows: in Section 2, we provide the essential definitions and results that are employed in the subsequent sections; in Section 3, we provide the variation of constants formula for System (1); in Section 4, we outline the sufficient conditions for the existence and uniqueness of the mild solution and establish the $\mathcal{HUS}$; finally, examples are provided in the concluding section.

2. Basics

Let $\mathcal{U} = (\mathcal{U}, \|.\|)$, $U = (D(\mathcal{A}), \|.\|_U)$ be $\mathcal{BS}$s, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{U} \to \mathcal{U}$ be a closed linear operator, and $\|u\|_U = \|\mathcal{A}u\| + \|u\| \ \ \forall\, u \in D(\mathcal{A})$. $L(\mathcal{U}; U)$ denote the $\mathcal{BS}$ endowed with the uniform operator topology and consisting of operators from $\mathcal{U}$ to U which is linear and bounded. When $\mathcal{U} = U$, then $L(\mathcal{U}; U)$ is written as $L(\mathcal{U})$, with $(D(\mathcal{A}))$ denoting the domain of $\mathcal{A}$ endowed with the graph norm. Let $0 \in \rho(\mathcal{A})$ be the resolvent set of $\mathcal{A}$; then, $\mathcal{A}^{-1}$ exists. Additionally, if $\mathcal{A}$ meets the Hille–Yosida condition [36], then we have s_1 and s_2 in $\mathbb{R}$ such that

$$
\begin{aligned}
(s_1, \infty) &\subset \rho(\mathcal{A}), \text{ and} \\
\|(\lambda - \mathcal{A})^{-n}\| &\leq \frac{s_2}{(\lambda - s_1)^n}, \quad n = 1, 2, \ldots, \quad \lambda > s_1.
\end{aligned}
$$

For impulsive conditions, we introduce additional terms and notation. We define $PC(I, \mathcal{U})$ as the space consisting of functions $u : [0, T] \to \mathcal{U}$ such that u is continuous at $s \neq s_j$, $u(s_j^-) = u(s_j)$ and $u(s_j^+)$ exists for each $j = 1, 2, \ldots, m$. In this paper, $PC(I, \mathcal{U})$ is equipped with the norm $\|u\|_{PC} = \sup_{s \in [0,T]} |u(s)|$. Evidently, $(PC(I, \mathcal{U}), \|\cdot\|_{PC})$ is a $\mathcal{BS}$.

For convenience, let $s_0 = 0$ and $T = s_{m+1}$. For every $u \in PC(I, \mathcal{U})$, we have $\bar{u}_j$, which is defined for $j = 0, 1, \ldots, m$, belongs to $C([s_j, s_{j+1}]; \mathcal{U})$, and is expressed as follows:

$$\bar{u}_j = \begin{cases} u(s), & s \in (s_j, s_{j+1}], \\ u(s_j^+), & s = s_j. \end{cases}$$

In addition, if $\mathcal{B}$ is a subset of $PC(I, \mathcal{U})$, we define $\bar{\mathcal{B}}_j$ as $\bar{\mathcal{B}}_j = \{\bar{u}_j : u \in \mathcal{B}\}$. For a foundational study on differential equations with impulses, see [37,38].

Lemma 1. *[31] A subset $\mathcal{B}$ of $PC(I, \mathcal{U})$ is relatively compact in $PC(I, \mathcal{U})$ if and only if $\bar{\mathcal{B}}_j$ are relatively compact in $C([s_j, s_{j+1}]; \mathcal{U})$ for every $j = 0, 1, 2, \ldots, m$,.*

For the phase space $\mathcal{B}$, we use an axiomatic definition following a framework akin to that in [32]. A linear space $\mathcal{B}$ comprises functions that map from $(-\infty, 0]$ to $\mathcal{U}$ and is equipped with seminorm $\|.\|_{\mathcal{B}}$. Moreover, $\mathcal{B}$ meets certain axiomatic criteria.

- If $u : (-\infty, \xi + T] \to \mathcal{U}$, $T \in (0, \infty)$ and $u_\xi \in \mathcal{B}$, and if $u|_{[\xi, \xi+T]} \in PC([\xi, \xi + a]; \mathcal{U})$, then the following is true for $\xi \le s \le \xi + T$:
  (i) u_s is in $\mathcal{B}$; (ii) $\|u(s)\| \le H\|u_s\|_{\mathcal{B}}$, $H > 0$; (iii) $\|u_s\|_{\mathcal{B}} \le F(s - \xi)\sup\{\|u(t)\| : \xi \le t \le s\} + G(s - \xi)\|u_\xi\|_{\mathcal{B}}$, F and G are functions from $[0, \infty)$ to $[1, \infty)$, F is continuous, G is locally bounded, and H, F, and G do not depend on $u(\cdot)$.
- The space $\mathcal{B}$ is complete.

Definition 1. *[9] The fractional integral I_0^α with 0 as a lower limit and order α for u is*

$$I_0^\alpha u(s) = \frac{1}{\Gamma(\alpha)} \int_0^s \frac{u(r)}{(s - r)^{1-\alpha}} dr, \ s > 0, \ \alpha > 0,$$

where $\Gamma(.)$ is a gamma function. This definition is valid as long as the right-hand side is defined pointwise for $0 \le s < \infty$.

Definition 2. *[9] The $\mathcal{RL}$ derivative ${}^R D_0^\alpha$ with a lower limit 0 and order α for u is defined as*

$${}^R D_0^\alpha u(s) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{ds^n} \int_0^s \frac{u(r)}{(s - r)^{1-n+\alpha}} dr, \ s > 0, \ n - 1 < \alpha < n,$$

which is valid as long as the right-hand side is defined pointwise for $s \in (o, \infty)$.

Definition 3. *[9] The Caputo derivative ${}^c D_0^\alpha$ with a lower limit 0 and order α for u is defined as*

$${}^c D_0^\alpha u(s) = {}^R D_0^\alpha \left(u(s) - \sum_{k=0}^{n-1} \frac{s^k}{k!} u^k(0) \right), \ s > 0, \ n - 1 < \alpha < n.$$

We address the solution of System (1) by employing the α-resolvent operator, which is defined below.

Definition 4. *[36] A set of operators $(\tau_\alpha(s))_{s \ge 0}$ such that $\tau_\alpha(s) \in \mathcal{L}(\mathcal{U})$ is an α-resolvent operator for System (2) when the following conditions are fulfilled:*
(T_1) $\tau_\alpha(.) : [0, \infty) \to \mathcal{L}(\mathcal{U})$ is strongly continuous and

$$\tau_\alpha(0)u = 0, \ \forall u \text{ in } \mathcal{U}, 1 < \alpha < 2.$$

(T_2) *For* u *in* $\mathcal{U}, \tau_\alpha(.)u \in C([0,\infty), D(\mathcal{U}) \cap C^1([0,\infty), \mathcal{U})$,

$$^{c}D_0^\alpha \tau_\alpha(s)u = \mathcal{A}\tau_\alpha(s)u + \int_0^s E(s-r)\tau_\alpha(r)udr, \tag{4}$$

$$^{c}D_0^\alpha \tau_\alpha(s)u = \tau_\alpha(\varepsilon)\mathcal{A}u + \int_0^s \tau_\alpha(s-r)E(r)udr, \ \forall u \in \mathcal{U}, \ s \geq 0. \tag{5}$$

In this paper, we adopt the following conditions:

(P_1) Suppose that $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{U} \to \mathcal{U}$ is a closed linear operator with $\overline{(D(\mathcal{A}))} = \mathcal{U}$. Let $\alpha \in (1,2)$. For some $\Phi_0 \in (0, \frac{\pi}{2}]$, for each $\Phi < \Phi_0$ let there be $0 < L_0 = L_0(\Phi)$ such that $\Lambda \in \rho(\mathcal{A})$ for each $\Lambda \in \Pi_{0,\alpha\varphi} = \{\Lambda \in \mathbb{C} : |arg(\Lambda)| < \Lambda\varphi, \Lambda \neq 0\}$. Finally, $\|R(\Lambda, \mathcal{A})\| \leq \frac{L_0}{|\Lambda|}$, $\forall \Lambda \in \Pi_{0,\alpha\varphi}$, where $\varphi = \Phi + \frac{\pi}{2}$.

(P_2) $E(s) : D(E) \subseteq \mathcal{U} \to \mathcal{U}$ is closed linear operator for all $s > 0$ with $D(\mathcal{A}) \subseteq D(E(s))$, and $E(s)u$ is strongly measurable for $s \in (0,\infty)$ and each $u \in D(\mathcal{A})$. We have $q(.) \in L^1_{loc}(\mathbb{R}^+)$ such that $\tilde{q}(\Lambda)$ exists for the real part of Λ greater than 0 and $\|E(s)u\| \leq q(s)\|u\|_1, \forall s > 0, \ u \in D(\mathcal{A})$. Furthermore, $\tilde{E} : \Pi_{0,\frac{\pi}{2}} \to L(D(\mathcal{A}), \mathcal{U})$ has an analytic extension to $\Pi_{0,\varphi}$ (also represented by $\tilde{E}$), such that

$$\|\tilde{E}(\Lambda)u\| \leq \|\tilde{E}(\Lambda)\|\|u\|_1, \ \forall u \in D(\mathcal{A})$$

and

$$\|\tilde{E}(\lambda)\| \leq O(\frac{1}{|\lambda|}), \text{ as } \lambda \to \infty.$$

(P_3) There is a subspace $D \subseteq D(\mathcal{A})$ such that $\overline{D} = (D(\mathcal{A}))$ and $L_1 > 0$ such that $\mathcal{A}(D) \subseteq D(\mathcal{A})$, $\tilde{E}(\lambda)(D) \subseteq D(\mathcal{A})$, $\|\mathcal{A}\tilde{E}(\lambda)u\| \leq L_1\|u\|$ for each $u \in D$ and every $\lambda \in \Pi_{0,\varphi}$.

In the sequelae, for $r \in (0,\infty)$, $\phi \in (\frac{\pi}{2}, \varphi)$, $\Pi_{r,\phi} = \{\Lambda \in \mathbb{C} : \Lambda \neq 0, |\Lambda| > r, |arg(\Lambda)| < \phi\}$. By $\gamma_{r,\phi}, \gamma^i_{r,\phi}$, $i = 1,2,3$ we denotes the paths $\gamma^1_{r,\phi} = \{se^{i\phi} : s \geq r\}$, $\gamma^2_{r,\phi} = \{se^{i\varepsilon} : -\phi \leq \varepsilon \leq \phi\}$, $\gamma^3_{r,\phi} = \{se^{-i\phi} : s \geq r\}$ and $\gamma_{r,\phi} = \cup_{i=1}^3 \gamma^i_{r,\phi}$ oriented counterclockwise. Moreover, $\rho_\alpha(G_\alpha) = \Big\{\Lambda \in \mathbb{C} : G_\alpha(\Lambda) = \Lambda^{\alpha-1}(\Lambda^\alpha I_d - \mathcal{A} - \tilde{E}(\Lambda))^{-1} \in L(\mathcal{U})\Big\}$.

We describe $(\tau_\alpha(s))_{s\geq 0}$ by

$$\tau_\alpha(s) = \frac{1}{2\pi i}\int_{\gamma_{r,\theta}} e^{\Lambda s}G_\alpha(\Lambda)d\Lambda, \ \ s > 0 \ \text{ and } \tau_\alpha(s) = I_d, \text{ when } s = 0.$$

Theorem 1 ([17]). *Assuming that conditions* $(P_1) - (P_3)$ *are met, then System (2) possesses an* α*-resolvent operator.*

Theorem 2 ([13]). *The mapping* τ_α *from* $(0,\infty)$ *to* $L(\mathcal{U})$ *is uniformly continuous, while the mapping* τ_α *from* $[0,\infty)$ *to* $L(\mathcal{U})$ *is strongly continuous.*

In our study, we assume that conditions $(P_1) - (P_3)$ are satisfied.

Definition 5 ([17], Definition 2.5). *Let* α *be in the interval* $(1,2)$. *We introduce the operator* $\mathcal{S}_\alpha$*, defined by*

$$\mathcal{S}_\alpha(s)u = \int_0^s g_{\alpha-1}(s-r)\tau_\alpha(r)uds, \text{ for every } s \geq 0, \ \text{ where } \ g_\alpha(r) = \frac{r^{\alpha-1}}{\Gamma(\alpha)}, \ \ r > 0, \ \alpha > 0.$$

Lemma 2 ([17], Lemma 2.6). $\tau_\alpha(.)$ *exhibits exponential boundedness within* $L(\mathcal{U})$.

Lemma 3 ([17], Lemma 2.7)**.** *If $\tau_\alpha(.)$ is bounded by an exponential function in $L(\mathcal{U})$, then $\mathcal{S}_\alpha(.)$ is also bounded by an exponential function in $L(\mathcal{U})$.*

Lemma 4 ([17], Lemma 2.8)**.** *If $\tau_\alpha(.)$ is bounded by an exponential function in $L(D(\mathcal{A}))$, then $\mathcal{S}_\alpha(.)$ is also bounded by an exponential function in $L(D(\mathcal{A}))$.*

Lemma 5 ([17], Lemma 2.18)**.** *Let $E \subseteq \mathcal{U}$ such that E is non-empty, closed, bounded, and convex. Consider an upper semicontinuous mapping $H : E \to 2^{\mathcal{U}} \backslash \emptyset$ that takes closed and convex values such that $H(E)$ is compact and $H(E) \subseteq E$. Then, H has an $\mathcal{FP}$.*

Theorem 3. *Suppose that $\mathcal{U}$ is a $\mathcal{BS}$ and that $B \subseteq \mathcal{U}$ such that it is bounded, closed, and convex. Let T, R be maps from B into $\mathcal{U}$ so that $Tu + Rw \in B$for all $u, w \in B$. If T is a contraction and R is completely continuous, then $Tu + Ru = u$ has a solution in B.*

3. Representation of the Solution

For System (1), we utilize the α-resolvent operator to obtain the variation of constants formula.

Theorem 4. *Suppose that $w : I \times \mathcal{B} \times \mathcal{U} \to \mathcal{U}$, $e : I \times I \times \mathcal{B} \to \mathcal{U}$ are continuous functions, $E(s)$ is an operator which is bounded and linear, and $u(0) = \chi + \mu(u) \in \mathcal{B}$. If $u(.)$ is a classical solution to (1) within I, then*

$$u(s) = \tau_\alpha(s)(\chi(0) + \mu(u(0))) + \sum_{0<s_j<s} \tau_\alpha(s - s_j)J_j(u(s_j)) + \int_0^s \mathcal{S}_\alpha(s - r)w(r, u_r, \int_0^r e(r, \varepsilon, u_\varepsilon)d\varepsilon)dr \text{ for } s \in I. \tag{6}$$

Proof. Applying I_0^α to each side of (1) for s in the interval $[0, s_1]$ provides us with

$$\begin{aligned} u(s) &= \chi(0) + \mu(u)(0) + \int_0^s g_\alpha(s - r)\mathcal{A}u(r)dr + \int_0^s g_\alpha(s - r) \int_0^r E(r - \varepsilon)u(\varepsilon)d\varepsilon dr \\ &\quad + \int_0^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr. \end{aligned} \tag{7}$$

Letting $s \in (s_1, s_2]$ and applying I_0^α on both sides of (1), we obtain

$$\begin{aligned} u(s) &= u(s_1^+) + \int_{s_1}^s g_\alpha(s - r)\mathcal{A}u(r)dr + \int_{s_1}^s g_\alpha(s - r) \int_0^r E(r - \varepsilon)u(\varepsilon)d\varepsilon dr \\ &\quad + \int_{s_1}^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr. \end{aligned}$$

Now, putting values

$$\begin{aligned} u(s) &= u(s_1) + J_1(u(s_1)) + \int_o^s g_\alpha(s - r)\mathcal{A}u(r)dr - \int_o^{s_1} g_\alpha(s_1 - r)\mathcal{A}u(r)dr + \int_{s_1}^s g_\alpha(s - r) \\ &\quad \int_0^r E(r - \varepsilon)u(\varepsilon)d\varepsilon dr + \int_{s_1}^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr, \end{aligned} \tag{8}$$

we can substitute $s = s_1$ in (7) and use this result in (8) to obtain

$$
\begin{aligned}
u(s) &= \chi(0) + \mu(u)(0) + J_1(u(s_1)) + \int_0^{s_1} g_\alpha(s_1 - r)\mathcal{A}u(r)dr + \int_0^{s_1} g_\alpha(s_1 - r)\int_0^r E(r-\varepsilon)u(\varepsilon)d\varepsilon dr \\
&+ \int_0^{s_1} g_\alpha(s_1 - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr + \int_o^s g_\alpha(s - r)\mathcal{A}u(r)dr - \int_o^{s_1} g_\alpha(s_1 - r)\mathcal{A}u(r)dr \\
&+ \int_{s_1}^s g_\alpha(s - r)\int_0^r E(r-\varepsilon)u(\varepsilon)d\varepsilon dr + \int_{s_1}^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr \\
&= \chi(0) + \mu(u)(0) + J_1(u(s_1)) + \int_o^s g_\alpha(s - r)\mathcal{A}u(r)dr + \int_0^s g_\alpha(s - r)\int_0^r E(r-\varepsilon)u(\varepsilon)d\varepsilon dr \\
&+ \int_0^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr.
\end{aligned}
$$

Similarly, for $s \in (s_2, s_3]$ we can take the integral from s_1^+ to s:

$$
\begin{aligned}
u(s) &= \chi(0) + \mu(u)(0) + J_1(u(s_1)) + J_2(u(s_2)) + \int_o^s g_\alpha(s - r)\mathcal{A}u(r)dr + \int_0^s g_\alpha(s - r) \\
&\int_0^r E(r-\varepsilon)u(\varepsilon)d\varepsilon dr + \int_0^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr.
\end{aligned}
$$

Proceeding in a similar fashion,

$$
\begin{aligned}
u(s) &= \chi(0) + \mu(u)(0) + \sum_{0<s_j<s} J_j(u(s_j)) + \int_o^s g_\alpha(s - r)\mathcal{A}u(r)dr + \int_0^s g_\alpha(s - r)\int_0^r E(r-\varepsilon)u(\varepsilon)d\varepsilon dr \\
&+ \int_0^s g_\alpha(s - r)w(r, u_r, \int_0^r e(r, \eta, u_\eta)d\eta))dr. \qquad (9)
\end{aligned}
$$

Now, considering (5), we obtain

$$
\tau_\alpha(s)u = u + \int_o^s g_\alpha(s - r)\tau_\alpha(r)\mathcal{A}udr + \int_0^s g_\alpha(s - r)\int_0^r \tau_\alpha(r-\varepsilon)E(r)ud\varepsilon dr. \qquad (10)
$$

From (10), we have

$$
I_d = \tau_\alpha - g_\alpha * \tau_\alpha\mathcal{A} - g_\alpha * \tau_\alpha * E,
$$

which implies that

$$
\begin{aligned}
I_d * u &= (\tau_\alpha - g_\alpha * \tau_\alpha\mathcal{A} - g_\alpha * \tau_\alpha * E) * u \\
&= \tau_\alpha * (u - g_\alpha\mathcal{A}u - g_\alpha * E * u) \\
&= \tau_\alpha * (\chi(0) + \mu(u)(0) + \sum_{0<s_j<s} J_j(u(s_j)) + g_\alpha * w) \\
&= \tau_\alpha * (\chi(0) + \mu(u)(0)) + \sum_{0<s_j<s} \tau_\alpha * J_j(u(s_j)) + g_1 * g_{\alpha-1} * \tau_\alpha * w). \qquad (11)
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\int_0^r u(r)dr &= \int_0^s \tau_\alpha(r)(\chi(0)+\mu(u)(0)dr + \int_0^s \sum_{0<r_j<r} \tau_\alpha(r-r_j)J_j(u(r_j))dr + \int_0^s \int_0^r g_{\alpha-1}(r-\varepsilon) \\
&\int_0^\varepsilon \tau_\alpha(\varepsilon - t)w(t,u_t,\int_0^u e(t,\eta,u_\eta)d\eta)dtd\varepsilon dr \\
&= \int_0^s \tau_\alpha(r)(\chi(0)+\mu(u)(0)dr + \sum_{0<r_j<r}\int_0^s \tau_\alpha(r-r_j)J_j(u(r_j))dr + \int_0^r\int_0^r\int_u^r g_{\alpha-1}(r-\varepsilon) \\
&\tau_\alpha(\varepsilon - t)w(t,u_t,\int_0^u e(t,\eta,u_\eta)d\eta)d\varepsilon dtdr \\
&= \int_0^s \tau_\alpha(r)(\chi(0)+\mu(u)(0)dr + \sum_{0<r_j<r}\int_0^s \tau_\alpha(r-r_j)J_j(u(r_j))dr + \int_0^s\int_0^r\int_0^{r-t} g_{\alpha-1}(r-t-\varepsilon) \\
&\tau_\alpha(\varepsilon)w(t,u_t,\int_0^t e(t,\eta,u_\eta)d\eta)d\varepsilon dtdr \\
&= \int_0^s \tau_\alpha(r)(\chi(0)+\mu(u)(0)dr + \sum_{0<r_j<r}\int_0^s \tau_\alpha(r-r_j)J_j(u(r_j))dr + \int_0^s\int_0^r \mathcal{S}_\alpha(r-t) \\
&w(t,u_t,\int_0^t e(t,\eta,u_\eta)d\eta)dtdr.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
u(s) &= \tau_\alpha(s)(\chi(0)+\mu(u)(0)) + \sum_{0<s_j<s}\tau_\alpha(s-s_j)J_j(u(s_j)) + \int_0^s \mathcal{S}_\alpha(s-u) \\
&w(t,u_t,\int_0^t e(t,\eta,u_\eta)d\eta)dt. \qquad (12)
\end{aligned}$$

Definition 6. *A mild solution of System (1) is a function $u : (-\infty, T] \to \mathcal{U}$ if the following conditions are fulfilled:*

- $u_0 = \chi(0) + \mu(u)(0)$,
- $u(s) \in PC(I;\mathcal{U})$, $J_j(u_{s_j}) = u(s_j^+) - u(s_j)$, *where* $j = 1, 2, \ldots, m$,
- *and*

$$\begin{aligned}
u(s) &= \tau_\alpha(s)(\chi(0)+\mu(u)(0)) + \sum_{0<s_j<s}\tau_\alpha(s-s_j)J_j(u(s_j)) + \int_0^s \mathcal{S}_\alpha(s-t) \\
&w(t,u_u,\int_0^t e(t,\eta,u_\eta)d\eta)dt, \quad \textit{where } s \in I.
\end{aligned}$$

Remark 1. *The mild solution $u(s)$ is unique if for any two mild solutions $u_1(s)$ and $u_2(s)$ we have $u_1(s) = u_2(s)$ for all $s \in [0, T]$.*

4. Main Results

We explore the existence, uniqueness, and $\mathcal{HUS}$ for the problem in (1) and outline several assumptions that are used in our analysis.

(G_0) There exists a constant M_τ and $M_{\mathcal{S}}$ such that $\|\tau_\alpha(s)\| \le M_\tau$ and $\|\mathcal{S}_\alpha(s)\| \le M_{\mathcal{S}}$, $\forall\, s \in I$.

(G_1) (1) $w : I \times \mathcal{B} \times \mathcal{U} \to \mathcal{U}$ satisfies the subsequent axioms:
(i) $w(s, ., .) : \mathcal{B} \times \mathcal{U} \to \mathcal{U}$ is continuous for almost every $s \in I$.
(ii) $s \to w(s, \phi, u)$ is measurable for each $(\phi, u) \in \mathcal{B} \times \mathcal{U}$.

(iii) Given a mapping m_w from I to $[0, \infty)$ and a continuous mapping u_w which is non-decreasing from $[0, \infty)$ to $[0, \infty)$, the following inequality holds for all $s \in I$ and any $(\phi, u) \in \mathcal{B} \times \mathcal{U}$:

$$\|w(s, \phi, u)\| \leq m_w(s) u_w(\|\phi\| + \|u\|).$$

(2) $w : I \times \mathcal{B} \times \mathcal{U} \to D(\mathcal{A})$ exhibits Lipschitz continuity. There exists a constant $L_w > 0$ for which the following inequality holds for each $\phi_1, \ \phi_2 \in \mathcal{B}$ and $u_1, \ u_2$ in $\mathcal{U}$:

$$\|w(s_1, \phi_1, u_1) - w(s_2, \phi_2, u_2)\| \leq L_w(|s_1 - s_2| + \|\phi_1 - \phi_2\| + \|u_1 - u_2\|).$$

(G_2) (1) $e : I \times I \times \mathcal{B} \to \mathcal{U}$ satisfies the following axioms:

(i) $e(s, t, u) : \mathcal{B} \to \mathcal{U}$ is continuous for each s, t in I.

(ii) $e(., ., u) : I \times I \to \mathcal{U}$ is measurable for any given u in $\mathcal{U}$.

(iii) Given m_e from I to $[0, +\infty)$ and a function u_e that is non-decreasing from $[0, \infty)$ to $[0, \infty)$, the following inequality holds for each $t, \ s$ in I and every u in $\mathcal{U}$:

$$\|e(s, t, u)\| \leq c_e m_e(s) u_e(\|u\|_{\mathcal{B}}),$$

where $c_e > 0$.

(2) $e : I \times I \times \mathcal{B} \to \mathcal{U}$ is Lipschitz continuous. Provided that $L_e > 0$, the following inequality holds for each t, s belonging to I and for all $u_1, \ u_2$ belonging to $\mathcal{B}$:

$$\|e(s, t, u_1) - e(s, t, u_2)\| \leq L_e(|u_1 - u_2|).$$

(G_3) (1) If μ is a continuous mapping from $C(I; \mathcal{U})$ to $C([-r, 0], \mathcal{U})$, it follows that for each u in $C(I; \mathcal{U})$,

$$\|\mu(u)\| \leq c_\mu \|u\| + d.$$

(2) $\mu : C(I; \mathcal{U}) \to \overline{D(\mathcal{A})}$ is Lipschitz continuous. Given $L_\mu > 0$, the following inequality holds for each u and ϕ in $C(I; \mathcal{U})$:

$$\|\mu(u) - \mu(\phi)\| \leq L_\mu \|u - \phi\|.$$

(G_4) (1) $J_j : \mathcal{B} \to \mathcal{U}$ are completely continuous and there exist mappings x_j from $[0, +\infty)$ to $(0, +\infty)$, which is non-decreasing; thus, for any u belonging to $\mathcal{B}$ and $j \in \{1, 2, \ldots, m\}$,

$$\|J_j(u)\| \leq x_j(\|u\|_{\mathcal{B}}), \quad \liminf_{\xi \to +\infty} \frac{x_j(\xi)}{\xi} = \varphi_j < +\infty.$$

(2) $J_j : \mathcal{B} \to \mathcal{U}$ are continuous and there exists $L_{J_j} > 0$, for which the subsequent inequality holds for $j = 1, 2, \ldots, m$ and for all $u_1, \ u_2$ belonging to $\mathcal{B}$:

$$\|J_j(u_1) - J_j(u_2)\| \leq L_{J_j} \|u_1 - u_2\|_{\mathcal{B}}.$$

Theorem 5. *If u_0 belongs to $\overline{D(\mathcal{A})}$, then 0 is in $\rho(\mathcal{A})$ and the conditions $(G_0) - (G_4)$ are satisfied; then, the mild solution to System (1) is unique in $[-r, T]$ given that*

$$q_0 = M_\tau L_\mu + M_\tau \sum_{j=1}^{m} L_{J_j} + T M_{\mathcal{S}} L_w (1 + T) < 1. \tag{13}$$

Proof. Suppose $u : [-r,k] \to \mathcal{U}$ to be a function such that its restriction $u_{[0,T]}$ to the interval $[0,T]$ belongs to $PC(I,\mathcal{U})$. Consider the operator P defined on $PC(I,\mathcal{U})$ by

$$(Pu)(s) = \begin{cases} \chi(s) + \mu(u)(s), \quad s \in [-r,0] \\ \tau_\alpha(s)(\chi(0) + \mu(u(0))) + \sum_{0<s_j<s} \tau_\alpha(s-s_j) J_j(u(s_j)) \\ + \int_0^s \mathcal{S}_\alpha(s-r) w(r, u_r, \int_0^r e(r,\varepsilon,u_\varepsilon) d\varepsilon) dr, \ \text{ for } s \in I. \end{cases} \tag{14}$$

For $0 \le s_1 < s_2 \le T$, consider

$$\begin{aligned} \|(Pu)(s_2) - (Pu)(s_1)\| \le\ & \|\tau_\alpha(s_2) - \tau_\alpha(s_1)\| \|(\chi(0) + \mu(u(0)))\| + \sum_{0<s_j<s} \|\tau_\alpha(s_2 - s_j) - \tau_\alpha(s_1 - s_j)\| \\ & \|J_j(u(s_j))\| + \int_0^{s_1-\epsilon} \|(\mathcal{S}_\alpha(s_2-r) - \mathcal{S}_\alpha(s_1-r)) w(r,u_r, \int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\| dr \\ & + \int_{s_1-\epsilon}^{s_1} \|(\mathcal{S}_\alpha(s_2-r) - \mathcal{S}_\alpha(s_1-r)) w(r,u_r, \int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\| dr \\ & + \int_{s_1}^{s_2} \|\mathcal{S}_\alpha(s_2-r) w(r,u_r, \int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\| dr \\ \le\ & \|\tau_\alpha(s_2) - \tau_\alpha(s_1)\| (\|\chi\| + c_\mu \|u\| + d) + \sum_{0<s_j<s} \|\tau_\alpha(s_2 - s_j) - \tau_\alpha(s_1 - s_j)\| \\ & x_j(\|u\|_{\mathcal{B}}) + \int_0^{s_1-\epsilon} \|\mathcal{S}_\alpha(s_2-r) - \mathcal{S}_\alpha(s_1-r)\| m_w(r) u_w(\|u_r\|_{\mathcal{B}} + \| \int_0^r e(r,\varepsilon, \\ & u_\varepsilon)d\varepsilon\|) dr + M_{\mathcal{S}} \int_{s_1}^{s_2} m_w(r) u_w(\|u_r\|_{\mathcal{B}} + \|\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon\|) dr \\ \le\ & \|\tau_\alpha(s_2) - \tau_\alpha(s_1)\| (\|\chi\| + c_\mu \|u\| + d) + \sum_{0<s_j<s} \|\tau_\alpha(s_2 - s_j) - \tau_\alpha(s_1 - s_j)\| \\ & x_j(\|u\|_{\mathcal{B}}) + \int_0^{s_1-\epsilon} \|\mathcal{S}_\alpha(s_2-r) - \mathcal{S}_\alpha(s_1-r)\| m_w(r) u_w(\|u_r\|_{\mathcal{B}} + T c_e m_e(r) \\ & u_e(\|u_\varepsilon\|)) dr + M_{\mathcal{S}} \int_{s_1}^{s_2} m_w(r) u_w(\|u_r\|_{\mathcal{B}} + T c_e m_e(r) u_e(\|u_\varepsilon\|) dr. \end{aligned}$$

The right side approaches 0 as $s_1 \to s_2$ for sufficiently small ϵ, as the compactness of $\mathcal{S}_\alpha(s)$ ensures continuity in the uniform operator topology; therefore, $Pu \in PC(I,\mathcal{U})$. For $s \in (s_j, s_{j+1}]$, $j \in \{1,2,\ldots,m\}$ and $u, w \in PC(I,\mathcal{U})$, we have

$$\begin{aligned} \|(Pu)(s) - (Pw)(s)\| =\ & \|\tau_\alpha(s)\| \|\mu(u(0)) - \mu(w(0))\| + \sum_{j=1}^m \|\tau_\alpha(s-s_j)\| \|J_j(u(s_j)) - J_j(w(s_j))\| \\ & + \int_0^s \|\mathcal{S}_\alpha(s) \left(w(r,u_r, \int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon) - w(r,w_r, \int_0^r e(r,\varepsilon,w_\varepsilon)d\varepsilon) \right) \| dr \\ \le\ & M_\tau L_\mu \|u - w\|_{PC} + M_\tau \sum_{j=1}^m L_{J_j} \|u-w\|_{PC} + T M_{\mathcal{S}} L_w (1+T) \|u-w\|_{PC} \\ =\ & \left(M_\tau L_\mu + M_\tau \sum_{j=1}^m L_{J_j} + T M_{\mathcal{S}} L_w (1+T) \right) \|u-w\|_{PC}, \end{aligned}$$

as $\left(M_\tau L_\mu + M_\tau \sum_{j=1}^m L_{J_j} + T M_{\mathcal{S}} L_w(1+T) \right) < 1$; therefore, P is a contraction. By applying the Banach $\mathcal{FP}$ theorem, P possesses an $\mathcal{FP}$. This $\mathcal{FP}$ corresponds to the unique mild solution of System (1). □

Theorem 6. *If the conditions (G_0)–(G_4) are met, then there is at least one mild solution for System (1) if*

$$M_\tau(\|\chi\| + d + x_j(r)) + M_{\mathcal{S}} \int_0^s m_w(r)u_w(r + Tc_e m_e(r)u_e(r))dr + M_\tau c_\mu r - r < 1 \quad (15)$$
$$M_\tau L_\mu < 1. \quad (16)$$

Proof. Consider the operator P on $PC(I, \mathcal{U})$ by

$$(Pu)(s) = \begin{cases} \chi(s) + \mu(u)(s), \quad s \in [-r, 0] \\ \tau_\alpha(s)(\chi(0) + \mu(u(0))) + \sum_{0<s_j<s} \tau_\alpha(s - s_j)J_j(u(s_j)) \\ + \int_0^s \mathcal{S}_\alpha(s - r)w(r, u_r, \int_0^r e(r, \varepsilon, u_\varepsilon)d\varepsilon)dr, \quad \text{for } s \in I. \end{cases} \quad (17)$$

We employ Theorem (3) in our proof, which is outlined in the subsequent steps.

Step 1. First, we demonstrate the continuity of P over the interval $(s_i, s_{i+1}]$. For this, let $u^n, u \in PC(I, \mathcal{U})$ such that $\|u^n - u\|_{PC} \to 0$ as $n \to \infty$. Consider

$$\begin{aligned} \|(Pu^n)(s) - (Pu)(s)\|_{PC} &= \|\tau_\alpha(s)\|\|\mu(u^n(0)) - \mu(u(0))\| + \sum_{j=1}^m \|\tau_\alpha(s - s_j)\|\|J_j(u^n(s_j)) - J_j(u(s_j))\| \\ &+ \int_0^s \|\mathcal{S}_\alpha(s)\|\|(w(r, u_r^n, \int_0^r e(r, \varepsilon, u_\varepsilon^n)d\varepsilon) - w(r, u_r, \int_0^r e(r, \varepsilon, u_\varepsilon)d\varepsilon))\|dr \\ &\leq M_\tau L_\mu \|u^n - u\|_{PC} + M_\tau \sum_{j=1}^m L_{J_j}\|u^n - u\|_{PC} + TM_{\mathcal{S}}L_w(1 + T)\|u^n - u\|_{PC}. \end{aligned}$$

Applying the limit $n \to \infty$, we obtain $\lim_{n\to\infty} \|(Pu^n)(s) - (Pu)(s)\|_{PC} = 0$.

Step 2. We establish that bounded sets are mapped to bounded sets within $PC(I, \mathcal{U})$ by P. For this, let $u \in B_r = \{u \in PC(I, \mathcal{U}) : \|u\|_{PC} \leq r\}$,

$$\begin{aligned} \|Pu(s)\| &\leq \|\tau_\alpha(s)(\chi(0) + \mu(u(0)))\| + \sum_{0<s_j<s} \|\tau_\alpha(s - s_j)J_j(u(s_j))\| + \int_0^s \|\mathcal{S}_\alpha(s - r)w(r, u_r, \\ &\int_0^r e(r, \varepsilon, u_\varepsilon)d\varepsilon)\|dr \ \text{ for } s \in I \\ &\leq M_\tau\Big(\|\chi\| + c_\mu\|u\| + d + x_j(\|u\|_{\mathcal{B}})\Big) + M_{\mathcal{S}} \int_0^s m_w(r)u_w\Big(\|u_r\|_{\mathcal{B}} + \|\int_0^r e(r, \varepsilon, u_\varepsilon)d\varepsilon)\|\Big)dr \\ &\leq M_\tau\Big(\|\chi\| + c_\mu r + d + x_j(r)\Big) + M_{\mathcal{S}} \int_0^s m_w(r)u_w\Big(r + Tc_h m_e(r)u_e(r)\Big)dr. \end{aligned} \quad (18)$$

Using (26), we obtain

$$\|Pu(s)\| \leq r - M_\tau c_\mu r = r_1,$$

implying that bounded sets are mapped into bounded sets in $PC(I, \mathcal{U})$.

Step 3. We show that bounded sets are mapped into equicontinuous sets of functions on $(s_i, s_{i+1}]$ by P. Letting $u \in B_r$ and $s', s'' \in (s_i, s_{i+1}]$ such that $s_i < s' < s'' \leq s_{i+1}$, we obtain

$$
\begin{aligned}
&\|(Pu)(s'')-(Pu)(s')\|_{PC}\\
\leq\ & \|(\tau_\alpha(s'')-\tau_\alpha(s'))\|\|(\chi(0)+\mu(u(0)))\|+\sum_{j=1}^{m}\|\tau_\alpha(s''-s_j)-\tau_\alpha(s'-s_j)\|\|J_j(u(s_j))\|\\
&+\int_0^{s'-\epsilon}\|(\mathcal{S}_\alpha(s''-r)-\mathcal{S}_\alpha(s'-r))\|\|w(r,u_r,\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\|dr+\int_{s'-\epsilon}^{s'}\|(\mathcal{S}_\alpha(s''-r)\\
&-\mathcal{S}_\alpha(s'-r))\|\|w(r,u_r,\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\|dr+\int_{s'}^{s''}\|\mathcal{S}_\alpha(s''-r)\|\|w(r,u_r,\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)\|dr\\
\leq\ & \|\tau_\alpha(s'')-\tau_\alpha(s')\|(\|\chi\|+c_\mu\|u\|+d)+\sum_{j=1}^{m}\|\tau_\alpha(s''-s_j)-\tau_\alpha(s'-s_j)\|x_j(\|u(s_j)\|_{\mathcal{B}})+\int_0^{s'-\epsilon}\\
&\|(\mathcal{S}_\alpha(s''-r)-\mathcal{S}_\alpha(s'-r))\|m_w(r)u_w(r+Tc_hm_e(r)u_e(r))dr+\int_{s'-\epsilon}^{s'}\|\mathcal{S}_\alpha(s''-r)-\mathcal{S}_\alpha(s'-r)\|\\
&m_w(r)u_w(r+Tc_hm_e(r)u_e(r))dr+\int_{s'}^{s''}\|\mathcal{S}_\alpha(s''-r)\|m_w(r)u_w(r+Tc_hm_e(r)u_e(r))dr.
\end{aligned}
$$

As $s'\to s''$ for sufficiently small ϵ, the right hand side approaches 0 regardless of $u\in B_r$. This follows from Theorem (2) and the compactness of $\mathcal{S}_\alpha(s)$, which ensures continuity in the uniform operator topology.

Step 4. Consider the operator $(Pu)(s)=(Tu)(s)+(R_iu)(s)$, where

$$(Tu)(s)=\tau_\alpha(s)(\chi(0)+\mu(u(0)))\quad\text{and}$$

$$(R_iu)(s)=\sum_{j=1}^{i}\tau_\alpha(s-s_j)J_j(u(s_j))+\int_0^s\mathcal{S}_\alpha(s-r)w(r,u_r,\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)dr.$$

We prove that R_i maps B_r into a precompact set in $\mathcal{U}$. We have to show that the set $\phi(s)=\{(R_iu)(s):u\in B_r\}$ is precompact in $\mathcal{U}$. Suppose that $s\in[0,T]$ and ϵ belongs to the set of real numbers such that $0<\epsilon<s$. We define an operator $R_{i,\epsilon}$ for $u\in B_r$ by

$$R_{i,\epsilon}=\sum_{j=1}^{i}\tau_\alpha(s-s_j)J_j(u(s_j))+\int_0^{s-\epsilon}\mathcal{S}_\alpha(s-r)w(r,u_r,\int_0^r e(r,\varepsilon,u_\varepsilon)d\varepsilon)dr.$$

The set $\phi_\epsilon(s)=\{(R_{i,\epsilon}u)(s):u\in B_r\}$ is precompact in $\mathcal{U}$ for each ϵ, where $0<\epsilon<s$, as $\mathcal{S}_\alpha(s)$ is a compact operator. In addition, for $0<\epsilon<s$ we have

$$|(R_iu)(s)-(R_{i,\epsilon}u)(s)|\leq M_{\mathcal{S}}\int_{s-\epsilon}^{s}m_w(r)u_w(r+Tc_em_e(r)u_e(r))dr.$$

Hence, there are precompact sets that can be made arbitrarily close to the set $\phi(s)$, which is precompact in $\mathcal{U}$, implying that $(R_iB_r)(s)$ are relatively compact in $\mathcal{U}$.

Step 5. We prove that $(R_iu)(s)$ is completely continuous and that $(Tu)(s)+(R_iw)(s)\in B_r$ for $i=1,2,\ldots,m$. The definitions of T and R_i are provided in the previous step.

By following Steps 1 through 4 for $(R_iu)(s)$, it is straightforward to show that $(R_iu)(s)$ is completely continuous.

From (18), we have

$$\|(Pu)(s)\|\leq M_\tau(\|\chi\|+d+x_j(r))+M_{\mathcal{S}}\int_0^s m_w(r)u_w(r+Tc_em_e(r)u_e(r))dr+M_\tau c_\mu r.$$

Because

$$M_\tau(\|\chi\|+d+x_j(r))+M_{\mathcal{S}}\int_0^s m_w(r)u_w(r+Tc_em_e(r)u_e(r))dr+M_\tau c_\mu r-r<1,$$

we have $\|(Pu)(s)\| \leq r$. Accordingly if $u,\ w \in B_r$, then $Tu + R_i w \in B_r$.

Step 6. Here, we show that T is a contraction for $s \in (s_i, s_{i+1}]$ and $u, w \in PC(I, \mathcal{U})$. Consider

$$\begin{aligned} \|(Tu)(s) - (Tw)(s)\| &\leq \|\tau_\alpha\| \|\mu(u)(0) - \mu(w)(0)\| \\ &\leq M_\tau L_\mu \|u - w\|_{PC}. \end{aligned}$$

From (27), we have $M_\tau L_\mu < 1$; hence, T is a contraction. From Theorem 3, we can deduce that System (1) has at least one mild solution over I. □

To prove $\mathcal{HUS}$ for the given system, let us define the $\mathcal{HUS}$ first.

Definition 7. *System (1) is said to have $\mathcal{HUS}$ if there are positive constants ϵ and $\mathcal{C}$ in such a way that for any $\tilde{u}$ mild solution in $PC(I, \mathcal{U})$ satisfying*

$$\begin{cases} |{}^c D_0^\alpha u(s) - \mathcal{A}u(s) - \int_0^s E(s-t)u(\varepsilon)dt - w(s, u_s, \int_0^s e(s,t,u_t)dt)| < \epsilon_a \\ |\Delta u(s_j) - J_j(u_{s_j})| < \epsilon_j \end{cases} \tag{19}$$

there exists a mild solution $u \in PC(I, \mathcal{U})$ to (1) for which $\|\tilde{u}(s) - u(s)\| \leq \mathcal{C}\epsilon$.

Remark 2. *Every $\tilde{u} \in PC(I, \mathcal{U})$ is regarded as a mild solution of (19) if and only if the following conditions are met:*

(i) There exist ϕ belonging to PC and a sequence of functions ϕ_j for which $|\phi(s)| \leq \epsilon_a$ and $|\phi_j(s)| \leq \epsilon_j$ for $s \in (s_i, s_{i+1}]$ with $i = 0, 1, \ldots, m$ and $j = 1, 2, \ldots, m$.

(ii) The following equations hold:

$$\begin{cases} {}^c D_0^\alpha u(s) &= \mathcal{A}u(s) + \int_0^s E(s-t)u(t)dr + w(s, u_s, \int_0^s e(s,t,u_t)dt) \\ &\quad +\phi(s), \quad s \in (s_i, s_{i+1}],\ i = 0, 1, \ldots, m, \\ \Delta u(s_j) &= J_j(u_{s_j}) + \phi_j(s), \quad j = 1, 2, \ldots, m. \end{cases} \tag{20}$$

Theorem 7. *Assuming that (G_0), $(G_1)(2)$, $(G_2)(2)$, $(G_3)(2)$, and $(G_4)(2)$ hold, System (1) exhibits $\mathcal{HUS}$.*

Proof. Let $\tilde{u}$ satisfy the inequality

$$\begin{cases} |{}^c D_0^\alpha u(s) - \mathcal{A}u(s) - \int_0^s E(s-t)u(t)dt - w(s, u_t, \int_0^s e(s,t,u_t)dr)| < \epsilon_a \\ |\Delta u(s_j) - J_j(u_{s_j})| < \epsilon_j. \end{cases} \tag{21}$$

From (21),

$$\begin{cases} {}^c D_0^\alpha \tilde{u}(s) = \mathcal{A}\tilde{u}(s) + \int_0^s E(s-t)\tilde{u}(t)dr + w(s, \tilde{u}_t, \int_0^s e(s,t,\tilde{u}_t)dt) + \phi(s) \\ \Delta \tilde{u}(s_j) = J_j(\tilde{u}_{s_j}) + \phi_j(s), \quad j = 1, 2, \ldots, m \end{cases}$$

such that $|\phi(s)| < \epsilon_a$ and $|\phi_j(s)| < \epsilon_j$.

Upon solving, we obtain

$$\begin{aligned} \tilde{u}(s) \quad = \quad & \tau_\alpha(s)(\chi(0) + \mu(\tilde{u}(0))) + \sum_{0<s_j<s} \tau_\alpha(s-s_j)J_j(\tilde{u}(s_j)) + \int_0^t \mathcal{S}_\alpha(s-s_j)(w(r, \tilde{u}_r, \\ & \int_0^r e(r,t,\tilde{u}_t)dt))ds + \sum_{j=1}^m \tau_\alpha(s-s_j)\phi_j(s) + \int_0^s \mathcal{S}_\alpha(s-r)\phi(r)dr, \text{ when } s \in I. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\|\tilde{u}(s)-u(s)\| &= \|\tau_\alpha(s)(\chi(0)+\mu(\tilde{u}(0)))+\sum_{0<s_j<s}\tau_\alpha(s-s_j)J_j(\tilde{u}(s_j))+\int_0^s \mathcal{S}_\alpha(s-s_j)(w(r,\tilde{u}_r,\int_0^r \\
&\quad e(r,t,\tilde{u}_t)dt))ds+\sum_{j=1}^{m}\tau_\alpha(s-s_j)\phi_j(s)+\int_0^s \mathcal{S}_\alpha(s-r)\phi(r)dr-\{\tau_\alpha(s)(\chi(0)+\mu(u(0))) \\
&\quad +\sum_{0<s_j<s}\tau_\alpha(s-s_j)J_j(u(s_j))+\int_0^s \mathcal{S}_\alpha(s-s_j)(w(r,u_r,\int_0^r e(r,t,u_t)dt))d\varepsilon\}\| \\
&\leq \|\tau_\alpha(s)\|\|\mu(\tilde{u}(0))-\mu(u(0))\|+\sum_{j=1}^{m}\|\tau_\alpha(s-s_j)\|\|J_j(\tilde{u}(s_j))-J_j(u(s_j))\|+\|\int_0^s \mathcal{S}_\alpha(s-s_j)\| \\
&\quad \|(w(r,\tilde{u}_r,\int_0^r e(r,t,\tilde{u}_t)dt))ds-(w(r,u_r,\int_0^r e(r,t,u_t)dt))ds\|+\|\sum_{j=1}^{m}\tau_\alpha(s-s_j)\varphi_j(s)\| \\
&\quad +\|\int_0^s \mathcal{S}_\alpha(s-r)\varphi(r)dr\| \\
&\leq M_\tau L_\mu\|\tilde{u}(0)-u(0)\|+M_\tau\sum_{j=1}^{m}L_{J_j}\|\tilde{u}(s_j)-u(s_j)\|+(TM_{\mathcal{S}}L_w) \\
&\quad (\|\tilde{u}_r-u_r\|+L_eT\|\tilde{u}_t-u_t)\|)+m\epsilon_jM_\tau+T\epsilon_aM_{\mathcal{S}} \\
&\leq \left(M_\tau L_\mu+M_\tau\sum_{j=1}^{m}L_{J_j}+(TM_{\mathcal{S}}L_w)(1+L_eT)\right)\|\tilde{u}(s)-u(s)\|+m\epsilon_jM_\tau+T\epsilon_aM_{\mathcal{S}} \\
&\leq \frac{m\epsilon_jM_\tau+T\epsilon_aM_{\mathcal{S}}}{\left(1-M_\tau L_\mu-M_\tau\sum_{j=1}^{m}L_{J_j}-(TM_{\mathcal{S}}L_w)(1+L_eT)\right)},
\end{aligned}
$$

which implies that

$$\|\tilde{u}(s)-u(s)\| \leq \mathcal{C}\epsilon,$$

where

$$\mathcal{C}=\frac{mM_\tau+TM_{\mathcal{S}}}{\left(1-M_\tau L_\mu-M_\tau\sum_{j=1}^{m}L_{J_j}-(TM_{\mathcal{S}}L_w)(1+L_eT)\right)} \text{ and } \epsilon=\max\{\epsilon_a,\epsilon_j\},\ j=1,\ldots,m.$$

Thus, System (1) exhibits $\mathcal{HUS}$. □

Remark 3. *The Hyers–Ulam stability of the impulsive fractional integro-differential system is established using the Caputo fractional derivative and Lipschitz-type assumptions. The Caputo derivative is suitable for fractional-order dynamics and captures the nonlocal nature of fractional derivatives. Formulating these assumptions for the system's components was challenging due to the impulsive and fractional characteristics. This approach provides a novel way to study stability in such systems, where conventional methods may not apply.*

5. Example

Example 1.

$$
\begin{cases}
\frac{\partial^\alpha}{\partial r^\alpha}v(r,s) &= \frac{\partial^2}{\partial s^2}v(r,s)+\int_0^s(s-r)^\sigma e^{-\varepsilon(s-r)}\frac{\partial^2}{\partial\eta^2}v(s,\eta)d\eta+\mathcal{D}(r,v(r-\epsilon,s), \\
&\quad \int_0^r\theta(r,v(s,s-\phi))dr) \text{ for } r\in[0,r_1)\cup(r_1,1], s\in[0,\pi],\ (r,s)\neq(r_j,s), \\
v(r,0) &= v(r,\pi)=0,\ r\in I, \\
v(0,s) &= v_0(s)+q(v(\epsilon,s)), \epsilon\in[0,1], s\in[0,\pi], \\
\Delta v(r_1,s) &= J_1(v(r_1,s)), r_1\in[0,1], s\in[0,\pi].
\end{cases}
\tag{22}
$$

In this system, we have $\frac{\partial^\alpha}{\partial r^\alpha} = {}^cD_0^\alpha$*, where* $\alpha \in (1,2)$*,* σ, ε *are constants. Assuming that* $\mathcal{V} = L^2([0,\pi])$ *is a complete normed space and the space* $\mathcal{B} = C_0 \times L^2(g,\mathcal{V})$*.* $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{V} \to \mathcal{V}$ *defined by*

$$\mathcal{A}v = v'' \text{ with domain } D(\mathcal{A}) = \{v \in \mathcal{V} : v'' \in \mathcal{V}, v(0) = v(\pi) = 0\}$$

is the infinitesimal generator of an analytic semigroup $(\tau(t))_{t\geq 0}$ *on* $\mathcal{V}$ *implies that* $\mathcal{A}$ *is sectorial of type and that* (P_1) *is satisfied. We consider an operator* $E(r) : D(\mathcal{A}) \subseteq \mathcal{V} \to \mathcal{V}$ *defined as* $E(r)v = r^\sigma e^{\varepsilon r} v''$*,* $v \in D(\mathcal{A})$*,* $r \geq 0$*. It is also clear that* (P_2) *and* (P_3) *are satisfied with* $b(r) = r^\sigma e^{\varepsilon r}$ *and* $D = C_0^\infty([0,\pi])$*. The space of infinitely differentiable function vanishes at* $s = 0$ *and* $s = \pi$*. Because* (P_1)*–*(P_3) *are satisfied, an α-resolvent operator exists for System (1) when* $\mathcal{D} = 0$*. Let us choose* $f = e^{2r}$*, where* $-s < r < 0$ *for the phase space* $\mathcal{B}$*, and let* $\int_{-s}^0 f(r)dr = \frac{1}{2}$ *for* $-s < r < 0$ *and*

$$\|\varphi\|_{\mathcal{B}} = \int_{-s}^0 f(r) \sup_{\zeta\in[r,0]} \|\varphi(\zeta)\|_{L^2} dr.$$

Now, let $\theta : [0,1] \times [0,1] \times \mathcal{B} \to L^2([0,\pi]), \mathcal{D} : [0,1] \times \mathcal{B} \times L^2([0,\pi]) \to L^2([0,\pi])$ *be defined as*

$$\theta(r,s,m) = \mathcal{C}_4 e^{\mathcal{C}_5 r} s + \mathcal{C}_6 m,$$

$$\mathcal{D}(r,m,n) = (\mathcal{C}_1(r) + \mathcal{C}_2)(m) + \mathcal{C}_3 n,$$

$$J_J(r) = \mathcal{C}_j r^2,$$

$$q(v(r,s)) = \sum_{j=1}^{2} \epsilon_j |v(t_j,s)|, \;\; 0 < t_1 < t_2 < 1, \; t_1, t_2 \neq r_1.$$

Upon verification, the Lipschitz constants are $L_\theta = \mathcal{C}_4 e^{\mathcal{C}_5}$*,* $L_{\mathcal{D}} = max(|\mathcal{C}_2|, |T\mathcal{C}_1 + \mathcal{C}_2|)$*,* $L_J = 2Ba_j$*, where B is the bound of input values of* J_j*,* $L_q = \sum_{j=1}^2 \epsilon_j$*, where* $\mathcal{C}_1, \ldots, \mathcal{C}_6$ *and* a_j *are constants. By comparison, we can see that* $\mathcal{D}$ *satisfies* $(G_1)(1)(iii)$*,* θ *satisfies* $(G_2)(1)(iii)$*, q satisfies* $(G_3)(1)$*, and* J_J *satisfies* $(G_4)(1)$*.*

After comparing, System (1) is an abstract structure of System (22).

Corollary 1. *System (1) possesses a unique mild solution provided that*

$$M_\tau \sum_{j=1}^{2} \epsilon_j + 2M_\tau \sum_{j=1}^{2} B\mathcal{C}_j + 2M_{\mathcal{S}} L_{\mathcal{D}} < 1. \tag{23}$$

Example 2.

$$\begin{cases} {}^cD_0^\alpha u(s,x) &= a(x)\Delta u(s,x) + \int_0^s e^{-b(s-t)} u(t,x)\,dt + \gamma(s)\|u_s\|_{H^1}^2 + \kappa(s)\int_0^s f(t,u_t)\,dt, \\ & \quad s \in [0,T],\; s \neq s_j, \\ u(0,x) &= \chi(x) + \lambda\|u\|_{L^2}^2, \quad u_s'(0,x) = 0, \\ \Delta u(s_j,x) &= \delta_j \int_\Omega u_{s_j}^2\,dx, \quad j \in \{1,2,\ldots,m\}. \end{cases} \tag{24}$$

Here, $\mathcal{U} = L^2(\Omega)$*,* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with a* C^2*-boundary,* $\mathcal{B} = C([-\infty,T], L^2(\Omega))$*, a space of continuous* L^2*-valued functions.* $\mathcal{A}u(s,x) = a(x)\Delta u(s,x)$*,* $\quad D(\mathcal{A}) = \{u \in H^2(\Omega) : u = 0 \text{ on } \partial\Omega\}$*, and* $D(\bar{\mathcal{A}}) = L^2(\Omega)$*.* $E(s)v = e^{-bs}v$*,* $\quad b > 0$*,* $\|E(s)v\|_{L^2} \leq e^{-bs}\|v\|_{L^2}$*, bounded and linear on* $L^2(\Omega)$*.*

$\mathcal{A}$ *is a closed linear operator, as* $\mathcal{A} = a(x)\Delta$*, and* Δ *is closed on* $D(\mathcal{A})$*. In addition,* $\mathcal{A}$ *is linear, as* $a(x)$ *is a scalar function,* $a(x)$ *is positive,* $\mathcal{A}$ *is sectorial and its spectrum lies in the left complex half-plane, and* $\|R(\lambda,\mathcal{A})\| \leq \frac{L_0}{|\lambda|}$*,* $\quad R(\lambda,\mathcal{A}) = (\lambda I_d - \mathcal{A})^{-1}$*,* $\lambda \in \Pi_{0,\alpha\varphi}$*. Hence,* P_1 *is satisfied.*

$\|E(s)v\|_{L^2} \leq e^{-bs}\|v\|_{L^2}$, *for* $v \in L^2(\Omega)$. *The Laplace transform of* $E(s)$ *is provided by* $\tilde{E}(\lambda) = \int_0^\infty e^{-\lambda s}E(s)\,ds$ *analytic in* λ, *and* $D = H_0^1(\Omega)$ D *satisfies* $\overline{D} = D(\mathcal{A})$, $\mathcal{A}\tilde{E}(\lambda)(D) \subseteq D(\mathcal{A})$, $\|\mathcal{A}\tilde{E}(\Lambda)u\| \leq \|a(x)\|_\infty\|\Delta\tilde{E}(\Lambda)u\| \leq L_1\|u\|$. *Hence,* P_2 *and* P_3 *are satisfied, implying that* $\mathcal{A}$ *generates an* α*-resolvent operator for System (24) when* $w = 0$.

Now, consider

$$w(s, u_s, \int_0^s e(s,t,u_t)\,dt) = \gamma(s)\|u_s\|^2_{H^1} + \kappa(s)\int_0^s f(t,u_t)\,dt,$$

$$J_j(u_{s_j}) = \delta_j \int_\Omega u^2_{s_j}\,dx, \quad \delta_j > 0,$$

$$\mu(u) = \lambda\|u\|^2_{L^2}, \quad \lambda > 0.$$

After calculating the Lipschitz constants for w, J, *and* μ *denoted by* L_w, L_J, *and* L_μ, *respectively, which are provided by*

$$L_w = \sup_{s\in I}\Big(2|\gamma(s)| + |\kappa(s)|L_f\Big),$$

$$L_J = 2\delta_j \sup_{s_j}\|u_{s_j}\| = 2\delta_j M_j, \quad \textit{where } M_j = \sup\|u_{s_j}\|,$$

$$L_\mu = 2\lambda\sup_u\|u\|_{L^2},$$

System (24) can be expressed in abstract form as System (1).

Proposition 1. *If* u_0 *belongs to* $\overline{D(\mathcal{A})}$, *then the mild solution to System (2) is unique in* $[-r, T]$ *given that*

$$2M_\tau\lambda\sup_u\|u\|_{L^2} + 2M_\tau\delta_j\sup_{s_j}\|u_{s_j}\| + TM_{\mathcal{S}}\sup_{s\in I}\Big(2|\gamma(s)| + |\kappa(s)|L_f\Big)(1+T) < 1. \tag{25}$$

Proposition 2. *There is at least one mild solution for System (24) if*

$$M_\tau(\|\chi\| + \lambda\|u\|^2 + \delta_j r^2) + M_{\mathcal{S}}\int_0^s(|\gamma(s)| + |\kappa(r)|)u_w(r + Te^{-br}r)dr - r \quad < \quad 1, \tag{26}$$

$$2M_\tau\lambda\sup_u\|u\|_{L^2} \quad < \quad 1. \tag{27}$$

6. Conclusions

In this article, we have explored the existence and uniqueness of a partial impulsive fractional integro-differential system with finite delay using the concept of the α-resolvent operator. By employing the α-resolvent operator, we derive a solution which serves as the basis for proving both the existence and uniqueness of the system's solution. Using the Banach $\mathcal{FP}$ theorem, we prove the uniqueness of a mild solution, while Krasnoselskii's $\mathcal{FP}$ theorem is employed to demonstrate the existence of a solution under specific conditions. Furthermore, we examine the $\mathcal{HUS}$ of the studied system.

The methodology developed in this work provides a general framework that can be extended to study other fractional integro-differential systems with different types of impulses, delays, and boundary conditions. This approach can also be applied to related problems in Banach spaces, offering a broader perspective on fractional dynamical systems and their stability properties.

Looking ahead, our upcoming research will focus on exploring the controllability of System (1) using the approach from [16–18,39] as well as on examining partial impulsive fractional stochastic and neutral integro-differential systems utilizing the α-resolvent operator [21,40]. Additionally, we will establish other stability concepts, such as Hyers–Ulam–Rassias and Mittag-Leffler–Ulam stability, using the Caputo derivative [24,41].

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