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The Existence and Stability of a Periodic Solution of a Nonautonomous Delayed Reaction–Diffusion Predator–Prey Model

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Abstract: In this study, we research a nonautonomous, three-species, delayed reaction–diffusion predator–prey model (RDPPM). Firstly, we derive sufficient conditions to guarantee the existence of a strictly positive, spatially homogeneous periodic solution (SHPS) for the delayed, nonautonomous RDPPM. These conditions are obtained using the comparison theorem for delayed differential equations and the fixed point theorem. Secondly, we present sufficient conditions to ensure the global asymptotic stability of the SHPS for the delayed, nonautonomous RDPPM. These conditions are established through the application of the upper and lower solution method (UALSM) for delayed parabolic partial differential equations (PDEs), along with Lyapunov stability theory. Finally, to demonstrate the practical application of our results, we numerically validate the proposed conditions using a 2-periodic, delayed, nonautonomous RDPPM.

Keywords: RDPPM; nonautonomous; periodic solution; global stability; UALSM

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1. Introduction

In this study, our focus is on the following nonautonomous, three-species, delayed Lotka–Volterra RDPPM, formulated as follows

$$\begin{cases} \dfrac{\partial s_1(x,t)}{\partial t} - d_1(t)\Delta s_1(x,t) = s_1(x,t)[r_1(t) - a_{11}(t)s_1(x,t-\tau_1) - a_{12}(t)s_2(x,t-\tau_2) \\ \qquad\qquad -a_{13}(t)s_3(x,t-\tau_3)], \\ \dfrac{\partial s_2(x,t)}{\partial t} - d_2(t)\Delta s_2(x,t) = s_2(x,t)[-r_2(t) - a_{22}(t)s_2(x,t-\tau_2) + a_{21}(t)s_1(x,t-\tau_1) \\ \qquad\qquad -a_{23}(t)s_3(x,t-\tau_3)], \\ \dfrac{\partial s_3(x,t)}{\partial t} - d_3(t)\Delta s_3(x,t) = s_3(x,t)[-r_3(t) - a_{33}(t)s_3(x,t-\tau_3) + a_{31}(t)s_1(x,t-\tau_1) \\ \qquad\qquad +a_{32}(t)s_2(x,t-\tau_2)], \end{cases} \tag{1}$$

with Neumann boundary conditions and positive initial conditions

$$\begin{cases} \partial s_i(x,t)/\partial n = 0, (x,t) \in \partial\Omega \times R^+, i = 1,2,3, \\ s_i(x,t) = \eta_{i0}(x,t) > 0, x \in \Omega, t \in [-\tau, 0], \tau = \max\{\tau_1, \tau_2, \tau_3\}, \end{cases} \tag{2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ with boundary $\partial\Omega$; Δ is a Laplace operator on Ω; $\partial/\partial n$ stands for the outward normal derivation on $\partial\Omega$; $s_i(x,t)$ represent the density of *i*-th species at point $x = (x_1, x_2, \cdots, x_n)$ and the time of t; τ_1, τ_2 and τ_3 are bounded constants representing delays and $\tau = \max\{\tau_1, \tau_2, \tau_3\}$; $d_1(t)$, $d_2(t)$ and $d_3(t)$ denote the diffusion rates of the prey, the middle predator, and the top predator species at time t, respectively; $a_{ii}(t)$ $(i = 1, 2, 3)$ represents interaction within i-th species; $a_{12}(t)$, $a_{13}(t)$, and $a_{23}(t)$ denote the capturing rates of the middle predator and the top predator, respectively; $a_{21}(t)$, $a_{31}(t)$, and $a_{32}(t)$ denote the conversion rates of the middle predator and the top predator; $r_1(t)$, $r_2(t)$, and $r_3(t)$ are the reproduction rates of prey (in the absence of a predator) and the natural death rates of the middle and top predators, respectively. All coefficients of the models (1.1) and (1.2) are positive and continuous ω- periodic functions. Models (1) and (2) describe the predator–prey relationship of three species, where 1-th species s_1 is the prey that is preyed upon by both 2-th species s_2 and 3-th species s_3, 2-th species s_2 is an middle predator that preys on 1-th species s_1 and is preyed upon by 3-th species s_3, and 3-th species s_3 is the top predator that preys on both 1-th species s_1 and 2-th species s_2. Models (1) and (2) represent a significant model in biomathematics. Their simplified versions have been extensively studied since the 1920s when Lotka and Volterra introduced the classic Lotka–Volterra model. In particular, the stability of positive equilibrium solutions and periodic solutions for predator–prey models and the persistence of systems have always been a hot topic of concern for many scientific and technological workers. In 2006, Zhang and Teng [1] studied a two-species Lotka–Volterra PPM with periodic coefficients and gained very simple criteria to ensure the existence and stability of a positive periodic solution for the model. In 2008, Shi and Chen [2] constructed and investigated a PPM with a stage structure and obtained some sufficient conditions to guarantee the globally asymptotical stability for the nontrivial periodic solution. In 2010, Wang [3] studied the permanence of a periodic PPM where the prey disperses in a patchy environment with two patches and provided criteria to ensure the permanence of the model. In 2013, Kim and Baek [4] studied an impulsively controlled PPM and provided very simple criteria to ensure the permanence of the model and the existence of a nontrivial periodic solution. In 2016, Zhang and Teng [5] investigated a periodic PPM with Gompertz growth function and obtained sufficient criteria to ensure the permanence of the grey population and the global attractivity of predator-extinction periodic solutions. In 2019, Deng et al. [6] investigated a Lotka–Volterra PPM incorporating predator cannibalism and provided some sufficient conditions to ensure the existence and global stability of the possible equilibria for the model. In 2020, Zhang et al. [7] researched a Lotka–Volterra PPM with non-selective harvesting and obtained sufficient conditions for the permanence of two populations and the globally asymptotical stability for the positive equilibrium point with the help of a suitable Lyapunov function. In 2021, Kaushik and Banerjee [8] studied a PPM with a stage structure on predator and counterattacking behavior such that the prey can attack a juvenile predator and gain the permanence of two populations and the stability of the positive equilibrium point. In 2022, He and Li [9] studied a fear effect PPM with mutual interference or group defense and obtained some conditions to ensure the global stability of the interior equilibrium point. In 2023, Quan et al. [10] studied a PPM with impulsive diffusion and transient (or nontransient) impulsive harvesting and obtained criteria to ensure the globally asymptotic stability of the trivial solution and the periodic solution for the model. In 2024, Mishra et al. [11] studied a PPM with role reversal and proved that the role reversal mechanism in predator–prey interactions can prevent the cyclic dynamics of the population. Since animals naturally tend to congregate around water and food sources, incorporating diffusion terms into the aforementioned system yields a new model that more precisely captures the underlying dynamics of population interactions. However, the methodologies outlined in the previous literature cannot be readily applied to analyze such reaction–diffusion predator–prey models.

In recent years, with the continuous improvement of reaction–diffusion equation theory, research on the properties of the reaction–diffusion predator–prey model (RDPPM) has attracted increasing attention from scholars [12,13]. Especially, research on the stability of periodic solutions and the permanence of RDPPM has achieved more and more excellent results. For example, in 2013, Ko and Ahn [14] investigated an RDPPM with one prey and two competing predators and obtained some judgment criteria to ensure the persistence and attractiveness of the solution of the model. In 2015, Moussaoui [15] researched an RDPPM with homogeneous Neumann boundary conditions and achieved some conditions to guarantee the existence and global stability of the positive periodic solution of the model. In 2017, Wang [16] studied an RDPPM with two predators and one prey and obtained sufficient conditions to guarantee the existence of time-periodic solutions for the model. In 2020, Wu and Zhao [17] investigated an RDPPM with the Allee effect and threshold hunting and provided some conditions to ensure the asymptotic stability of the equilibrium point of the model by constructing a Jacobian matrix. In 2021, Montano and Lisena [18] studied an RDPPM with a Holling-type II functional response and obtained some sufficient conditions ensuring the extinction of one predator and the stable coexistence of the surviving predator and its prey. In 2022, Yan and Zhang [19] investigated an RDPPM with a B-D function response and obtained stability and instability criteria for the positive constant equilibrium point of the model. In the same year, Gnanasekaran et al. [20] studied the following RDPPM with the chemotaxis term

$$\begin{cases} u_{1t} = \mu_1 \Delta u_1 + \chi\, \nabla \cdot (u_1^m \nabla v) + \sigma_1\, u_1 (1 - u_1 - a_1 u_2), & x \in \Omega,\ t > 0, \\ u_{2t} = \mu_2 \Delta u_2 + \xi\, \nabla \cdot (u_2^m \nabla v) + \sigma_2\, u_2 (1 + a_2 u_1 - u_2), & x \in \Omega,\ t > 0, \\ v_t = \mu_3 \Delta v - \gamma v + g(u_1, u_2), & x \in \Omega,\ t > 0, \\ \partial u_1 / \partial n = \partial u_2 / \partial n = \partial v / \partial n = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(\cdot, 0) = u_{10},\, u_2(\cdot, 0) = u_{20},\, v(\cdot, 0) = v_0, & x \in \Omega, \end{cases} \tag{3}$$

and studied the global existence and boundedness of the classical solution by using certain useful inequalities and a crucial lemma. In 2023, Meng and Feng [21] studied an RDPPM with prey refuge and hunting cooperation and provided some conditions to ensure the non-existence and existence of the non-constant positive equilibrium solution for the model. In 2024, Gnanasekaran et al. [22] researched the global existence and asymptotic behavior of the following RDPPM with the chemotaxis term

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi\,\nabla\cdot(u_1\nabla v_1) + u_1(\sigma_1 - a_1u_1 + e_1u_2), & x\in\Omega,\ t>0, \\ u_{2t} = d_2\Delta u_2 + \xi\,\nabla\cdot(u_2\nabla v_2) + u_2(\sigma_2 - a_2u_2 - e_2u_1), & x\in\Omega,\ t>0, \\ v_{1t} = d_3\Delta v_1 + \alpha_1u_2 - \beta_1v_1, & x\in\Omega,\ t>0, \\ v_{2t} = d_4\Delta v_2 + \alpha_2u_1 - \beta_2v_2, & x\in\Omega,\ t>0, \\ \partial u_1/\partial n = \partial u_2/\partial n = \partial v_1/\partial n = \partial v_2/\partial n = 0, & x\in\partial\Omega,\ t>0, \\ u_1(x,0)=u_{10},\, u_2(x,0)=u_{20},\, v_1(x,0)=v_{10},\, v_2(x,0)=v_{20}, & x\in\Omega, \end{cases} \tag{4}$$

and demonstrated the uniqueness and boundedness of the classical solution for the system. Furthermore, the convergence of the solution was established by constructing an appropriate Lyapunov functional. It is worth mentioning that the above models are autonomous RDPPMs. Given that birth rates, death rates, and population interactions are not constant, nonautonomous RDPPMs provide a more accurate simulation of population dynamics in predator–prey systems. However, the methods used in the above literature make it difficult to study nonautonomous RDPPMs. Inspired using the above works, with help of the Lyapunov stability theory and the upper and lower solutions method for PDEs, we recently studied a three-species nonautonomous RDPPMs [23]

$$\begin{cases} \partial u_1(x,t)/\partial t - d_1(t)\Delta u_1(x,t) = u_1(x,t)[r_1(t) - a_{11}(t)u_1(x,t) - \dfrac{a_{12}(t)u_2(x,t)}{b_{12}(t)u_2(x,t)+u_1(x,t)} - \dfrac{a_{13}(t)u_3(x,t)}{b_{13}(t)u_3(x,t)+u_1(x,t)}], \\ \partial u_2(x,t)/\partial t - d_2(t)\Delta u_2(x,t) = u_2(x,t)[-r_2(t) + \dfrac{a_{21}(t)u_1(x,t)}{b_{12}(t)u_2(x,t)+u_1(x,t)} - a_{23}(t)u_3(x,t)], \\ \partial u_3(x,t)/\partial t - d_3(t)\Delta u_3(x,t) = u_3(x,t)[-r_3(t) + \dfrac{a_{31}(t)u_1(x,t)}{b_{13}(t)u_3(x,t)+u_1(x,t)} - a_{32}(t)u_2(x,t)], \\ u_i(x,t)/\partial n = 0,\, u_i(x,0) = u_{i0}(x) \ge 0, \quad x\in\Omega,\ t>0, \end{cases} \tag{5}$$

and derived some sufficient conditions that ensure the global asymptotic stability of strictly positive SHPS for the model.

On the other hand, the dynamic behavior of the RDPPM usually depends on the system states of past time, which induces time delay in the equations of the system. Using delayed reaction–diffusion equations (DRDEs), many real natural phenomena are described and explained well. In recent years, research on DRDEs has attracted more and more attention from scholars. Early research on DRDEs was mostly included in academic works [24,25]. In recent years, some excellent achievements have been obtained in the study of periodic solutions and equilibrium points to the DRDPPM. For example, in 2016, Li [26] considered a DRDPPM with hyperbolic mortality and analyzed the impact of time delay on the stability of the equilibrium solution for the model. In 2017, Zhang and Li [27] studied a DRDPPM with nonlinear prey harvesting and hyperbolic mortality and obtained the globally stable conditions for the unique constant positive equilibrium of the model. In 2018, Ma et al. [28] studied a DRDPPM with mutual interference among the predators and discussed the spatiotemporal dynamics induced by delay and diffusion in the model. In 2019, Chen et al. [29] studied a ratio-dependent DRDPPM with the Neumann boundary conditions and analyzed the global stability of the SHPS for the model. In 2020, Jiang et al. [30] investigated a DRDPPM with a ratio-dependent function and derived stability criteria for the positive equilibrium of the model. In 2021, Djilali and Bentout [31] researched a DRDPPM with the prey social behavior and predator rivalry

$$\begin{cases} \frac{du}{dt} = u(1-u) - \frac{\sqrt{u}v}{1+b\sqrt{u}} + d_1\Delta u, \\ \frac{dv}{dt} = \rho v(-m_1 - m_2 v) + \frac{\sqrt{u_t}}{1+b\sqrt{u_t}} + d_2\Delta v, \\ \partial u_x(0,t) = \partial u_x(\pi,t) = \partial v_x(0,t) = \partial v_x(\pi,t) = 0, \ t \geq 0, \\ u(x,t) = u_0(x,t) \geq 0, v(x,t) = v_0(x,t) \geq 0, (x,t) \in [0,\pi]\times[-\tau,0], \end{cases} \tag{6}$$

and obtained some criteria to ensure the stability of the nonhomogeneous and homogeneous positive periodic solution for the system. In 2022, Xu et al. [32] analyzed a general DRDPPM with predator maturation delay and gained the globally asymptotic stability of the positive equilibrium solution for the model. In 2023, Yuan and Guo [33] studied a DRDPPM with spatial nonlocality and obtained stability criteria of positive steady-state solutions for the model. In 2024, Ma and Meng [34] studied a DRDPPM with a memory-based delay

$$\begin{cases} \frac{\partial u}{\partial t} = u(u-A)(1-u) - m_1 uy + d_1\Delta u, \ x \in \Omega, t > 0, \\ \frac{\partial y}{\partial t} = ry - \alpha y^2 + e_1 uy - \frac{m_2(1-c)yz}{a+(1-c)y} - E_1 y + d_2\Delta y, \ x \in \Omega, t > 0, \\ \frac{\partial z}{\partial t} = \frac{e_2(1-c)yz}{a+(1-c)y} - \beta z^2 - dz - E_2 z \\ \qquad + d_3\Delta z + d_{32}\nabla(z\nabla y(t-\tau)), \ x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} = 0, \ x \in \partial\Omega, t > 0, \\ u(x,0) = y(x,0) = z(x,0) = 0, \ x \in \Omega, \end{cases} \tag{7}$$

and obtained sufficient conditions to ensure the globally asymptotic stability for the positive constant equilibrium solution. It is worth mentioning that the issues studied in the above literature are the stability of constant equilibrium solutions or steady-state solutions of autonomous DRDPPMs. Moreover, the research methods in the previous paper, such as eigenvalues, cannot be used to study nonautonomous DRDPPMs. As far as we know, the results of the stability of periodic solutions to nonautonomous DRDPPMs are rarely published.

Due to the involvement of multiple factors such as time, space, time delay, and diffusion in nonautonomous DRDPPMs, it is more difficult to study the stability of its periodic solution and the persistence of the system. As we already know, the stability of the time-periodic solution for nonautonomous DRDPPMs has not been investigated before. Inspired by the above literature, in this paper, we will study a nonautonomous DRDPPMs (1) and (2). The rest of this paper is organized as follows: The innovations and achievements of this article are presented at the end of the Introduction. In Section 2, we will investigate the existence of the time-periodic solution of the nonautonomous DRDPPM by using the delay differential inequalities and fixed point theory. In Section 3, we will give very simple criteria to ensure the globally asymptotical stability of the time-periodic solution for the model with the help of the upper and lower solution methods and Lyapunov stability theory. In Section 4, we will conduct numerical simulations to validate our theoretical findings. Finally, we will present our conclusions.

Remark 1. *The innovations and accomplishments of this article are outlined as follows: (1) By incorporating time delays and variable coefficients into existing population models, we propose a novel nonautonomous DRDPPM that more accurately captures the interactions among populations; (2) leveraging the upper and lower solution methods, Lyapunov stability theory, and fixed point theory, we have innovatively developed new analytical approaches. These methods have allowed us to derive sufficient conditions for the existence and global stability of the positive time-periodic solution of the new model; (3) the technique of constructing Lyapunov functions for delayed differential equations step by step can be used to investigate the related problems, which will provide an effective method for studying the stability of periodic solutions to delayed PDEs; and (4) compared with the existing results, the stable solution obtained in this article is a time-periodic solution rather than a constant periodic solution or a solution for a steady-state system, which will be more in line with the objective law of seasonal cyclical changes in population density. To our knowledge, this is the first attempt to study a nonautonomous DRDPPM by using the above method.*

2. Existence of the Strictly Positive SHPS for the Nonautonomous DRDPPM

Set $\varphi(t)$ as the ω- periodic function in $\mathbb{R}_+$, we denote

$$\varphi^m = \sup\{\varphi(t), t \in \mathbb{R}_+\}, \quad \varphi^l = \inf\{\varphi(t), t \in \mathbb{R}_+\}$$

Next, we investigate the following functional differential equations (FDEs) corresponding to model (1)

$$\begin{cases} \dfrac{ds_1(t)}{dt} = s_1(t)[r_1(t) - a_{11}(t)s_1(t-\tau_1) - a_{12}(t)s_2(t-\tau_2) - a_{13}(t)s_3(t-\tau_3)], \\ \dfrac{ds_2(t)}{dt} = s_2(t)[-r_2(t) - a_{22}(t)s_2(t-\tau_2) + a_{21}(t)s_1(t-\tau_1) - a_{23}(t)s_3(t-\tau_3)], \\ \dfrac{ds_3(t)}{dt} = s_3(t)[-r_3(t) - a_{33}(t)s_3(t-\tau_3) + a_{31}(t)s_1(t-\tau_1) + a_{32}(t)s_2(t-\tau_2)]. \end{cases} \tag{8}$$

For model (8), set

$$M_1 = \frac{r_1^m}{a_{11}^l}\exp(r_1^m\tau_1), M_2 = \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l}\exp((a_{21}^m M_1 - r_2^l)\tau_2),$$

$$M_3 = \frac{(a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l}{a_{33}^l}\exp(((a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l)\tau_3),$$

$$m_1 = \frac{r_1^l - a_{12}^m M_2 - a_{13}^m M_3}{a_{11}^m}\exp((r_1^l - a_{12}^m M_2 - a_{13}^m M_3 - a_{11}^m M_1)\tau_1),$$

$$m_2 = \frac{a_{21}^l m_1 - a_{23}^m M_3 - r_2^m}{a_{22}^m}\exp((a_{21}^l m_1 - a_{23}^m M_3 - r_2^m - a_{22}^m M_2)\tau_2),$$

$$m_3 = \frac{a_{31}^l m_1 + a_{32}^l m_2 - r_3^m}{a_{33}^m}\exp((a_{31}^l m_1 + a_{32}^l m_2 - r_3^m - a_{33}^m M_3)\tau_3),$$

then we have the following result.

Theorem 1. *Suppose that*

$$(H_1)\ a_{21}^m M_1 > r_2^l,\quad (H_2)\ (a_{31}^m + a_{32}^m)\max\{M_1, M_2\} > r_3^l,$$
$$(H_3)\ r_1^l > a_{12}^m M_2 + a_{13}^m M_3,\quad (H_4)\ a_{21}^l m_1 - a_{23}^m M_3 > r_2^m,\quad (H_5)\ a_{31}^l m_1 + a_{32}^l m_2 > r_3^m.$$

Then model (8) is permanent.

Proof. According to the first equation of model (8), one has

$$\begin{aligned}\frac{ds_1(t)}{dt} &= s_1(t)[r_1(t) - a_{11}(t)s_1(t-\tau_1) - a_{12}(t)s_2(t-\tau_2) - a_{13}(t)s_3(t-\tau_3)] \\ &\le s_1(t)[r_1^m - a_{11}^l s_1(t-\tau_1)].\end{aligned} \tag{9}$$

From the Lemma 2.2 in [35], it follows that

$$\limsup_{t\to+\infty} s_1(t) \le \frac{r_1^m}{a_{11}^l}\exp(r_1^m \tau_1) = M_1. \tag{10}$$

Moreover, based on the second equation of model (8), it can be deduced that

$$\begin{aligned}\frac{ds_2(t)}{dt} &\le s_2(t)[-r_2(t) - a_{22}(t)s_2(t-\tau_2) + a_{21}(t)s_1(t-\tau_1)] \\ &\le s_2(t)[-r_2^l - a_{22}^l s_2(t-\tau_2) + a_{21}^m M_1].\end{aligned}$$

By (H_1), we have $a_{21}^m M_1 - r_2^l > 0$. Thus, by the Lemma 2.2 in [35],

$$\limsup_{t\to+\infty} s_2(t) \le \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l}\exp((a_{21}^m M_1 - r_2^l)\tau_2) = M_2. \tag{11}$$

By the third equation of model (8), one has

$$\begin{aligned}\frac{ds_3(t)}{dt} &= s_3(t)[-r_3(t) - a_{33}(t)s_3(t-\tau_3) + a_{31}(t)s_1(t-\tau_1) + a_{32}(t)s_2(t-\tau_2)] \\ &\le s_3(t)[-r_3^l + a_{31}^m M_1 + a_{32}^m M_2 - a_{33}^l s_3(t-\tau_3)] \\ &\le s_3(t)[-r_3^l + (a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - a_{33}^l s_3(t-\tau_3)].\end{aligned}$$

By (H_2), we have $(a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l > 0$. Thus, from the Lemma 2.2 in [33], one has

$$\limsup_{t\to+\infty} s_3(t) \le \frac{(a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l}{a_{33}^l}\exp(((a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l)\tau_3) = M_3. \tag{12}$$

On the other hand, based on the first equation of model (8),

$$\frac{ds_1(t)}{dt} \ge s_1(t)[r_1^l - a_{11}^m s_1(t-\tau_1) - a_{12}^m M_2 - a_{13}^m M_3] = s_1(t)[r_1^l - a_{12}^m M_2 - a_{13}^m M_3 - a_{11}^m s_1(t-\tau_1)].$$

By (H_3), we have $r_1^l - a_{12}^m M_2 - a_{13}^m M > 0$. Thus, form Lemma 2.3 in [35],

$$\liminf_{t\to+\infty} s_1(t) \ge \frac{r_1^l - a_{12}^m M_2 - a_{13}^m M_3}{a_{11}^m}\exp((r_1^l - a_{12}^m M_2 - a_{13}^m M_3 - a_{11}^m M_1)\tau_1) = m_1. \tag{13}$$

Similarly, by the second and third equations in model (8), one has

$$\frac{ds_2(t)}{dt} = s_2(t)[-r_2(t) - a_{22}(t)u_2(t-\tau_2) + a_{21}(t)s_1(t-\tau_1) - a_{23}(t)s_3(t-\tau_3)]$$
$$\geq s_2(t)[-r_2^m - a_{22}^m s_2(t-\tau_2) + a_{21}^l m_1 - a_{23}^m M_3]$$
$$\geq s_2(t)[a_{21}^l m_1 - a_{23}^m M_3 - r_2^m - a_{22}^m s_2(t-\tau_2)],$$

and

$$\frac{ds_3(t)}{dt} = s_3(t)[-r_3(t) - a_{33}(t)s_3(t-\tau_3) + a_{31}(t)s_1(t-\tau_1) + a_{32}(t)s_2(t-\tau_2)]$$
$$\geq s_3(t)[-r_3^m - a_{33}^m u_3(t-\tau_3) + a_{31}^l m_1 + a_{32}^l m_2]$$
$$\geq s_3(t)[a_{31}^l m_1 + a_{32}^l m_2 - r_3^m - a_{33}^m u_3(t-\tau_3)].$$

From $(H_4), (H_5)$ and the Lemma 2.3 in [35], we have

$$\liminf_{t\to+\infty} s_2(t) \geq \frac{a_{21}^l m_1 - a_{23}^m M_3 - r_2^m}{a_{22}^m} \exp((a_{21}^l m_1 - a_{23}^m M_3 - r_2^m - a_{22}^m M_2)\tau_2) = m_2. \quad (14)$$

and

$$\liminf_{t\to+\infty} s_3(t) \geq \frac{a_{31}^l m_1 + a_{32}^l m_2 - r_3^m}{a_{33}^m} \exp((a_{31}^l m_1 + a_{32}^l m_2 - r_3^m - a_{33}^m M_3)\tau_3) = m_3. \quad (15)$$

By (10)–(15), we see that model (8) is permanent. □

Theorem 2 *Suppose that* $(H_1)-(H_5)$ *holds. Then there exists a strictly positive spatial homogeneity* ω*- periodic solution of (1) and (2).*

Proof. Based on the existence and uniqueness theorem of solutions of FDEs (see Theorem 2.3, [36]), we define a Poincaré mapping $\psi : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ in the following form

$$\psi(S_0) = S(t, \omega, S_0)$$

where $S(t,\omega,S_0) = (s_1(t), s_2(t), s_3(t))$ is a positive solution of FDEs (8) subject to the IC $S_0 = (\eta_{10}(t), \eta_{20}(t), \eta_{30}(t)),\ t \in [-\tau, 0].$ We define

$$Z = \left\{(\eta_{10}, \eta_{20}, \eta_{30}) \in \mathbb{R}_+^3 \,\middle|\, m_i \leq \eta_{i0} \leq M_i, i = 1, 2, 3\right\},$$

then $Z \subset R_+^3$ is a convex and compact set. By Theorem 1 and continuity of solution of FDEs (8) with regard to the IC, see (Theorem 4.1, [36]]), ψ is a completely continuous mapping from Z to Z. By Lemma 2.4 in [36], (8) has a positive ω- periodic solution $(s_1^*(t), s_2^*(t), s_3^*(t))$ which is the spatial homogeneity ω- periodic solution for model (1) (see Definition 2.2, [37]). □

3. Stability of the Strictly Positive SHPS for the Nonautonomous DRDPPM

In this section, our primary focus is on the nonautonomous DRDPPM, as defined by Equations (1) and (2). Utilizing the method of upper and lower solutions for delayed parabolic PDEs, along with Lyapunov stability theory, we establish sufficient conditions to guarantee the global asymptotic stability of the strictly positive SHPS for (1) and (2).

Theorem 3. *Suppose that* $(H_1)-(H_5)$ *and the following assumptions hold.*

(H_6) $A_1 = a_{11}^l - a_{11}^m \tau_1 [r_1^m + a_{11}^m M_1 + a_{12}^m M_2 + a_{13}^m M_3] - M_1 (a_{11}^m)^2 \tau_1 - (1 + M_2 a_{22}^m \tau_2) a_{21}^m - (1 + M_3 a_{33}^m \tau_3) a_{31}^m > 0,$

(H_7) $A_2 = a_{22}^l - a_{22}^m \tau_2 [r_2^m + a_{22}^m M_2 + a_{21}^m M_1 + a_{23}^m M_3] - M_2 (a_{22}^m)^2 \tau_2 - (1 + M_1 a_{11}^m \tau_1) a_{12}^m - (1 + M_3 a_{33}^m \tau_3) a_{32}^m > 0,$

(H_8) $A_3 = a_{33}^l - a_{33}^m \tau_3 [r_3^m + a_{33}^m M_3 + a_{31}^m M_1 + a_{32}^m M_2] - M_3 (a_{33}^m)^2 \tau_3 - (1 + M_1 a_{11}^m \tau_1) a_{13}^m - (1 + M_2 a_{22}^m \tau_2) a_{23}^m > 0.$

Then (1) and (2) have a spatial homogeneity strictly positive and globally asymptotical stable ω*- periodic solution* $(s_1^*(t), s_2^*(t), s_3^*(t))$*, that is, the solution* $(s_1(x,t), s_2(x,t), s_3(x,t))$ *of (1) and (2) with any IC fulfills*

$$\lim_{t\to\infty}\left(s_i(x,t) - s_i^*(t)\right) = 0, \text{ uniformly for } x \in \bar{\Omega},\ i = 1,2,3. \tag{16}$$

Proof. By Theorem 2, (1) and (2) have a spatial homogeneity strictly positive ω- periodic solution. We prove the stability of the solution. Let $l_i = \min_{x\in\bar{\Omega},\, t\in[-\tau,0]} \eta_{i0}(x,t),$ $r_i = \max_{x\in\bar{\Omega},\, t\in[-\tau,0]} \eta_{i0}(x,t).$ Then $0 < l_i \le \eta_{i0}(x,t) \le r_i.$ Let $(\tilde{s}_1(t), \tilde{s}_2(t), \tilde{s}_3(t))$ and $(\hat{s}_1(t), \hat{s}_2(t), \hat{s}_3(t))$ be the solutions of (8) subject to initial values $(\eta_{10}(t), \eta_{20}(t), \eta_{30}(t)) = (r_1, r_2, r_3)$ and $(\eta_{10}(t), \eta_{20}(t), \eta_{30}(t)) = (l_1, l_2, l_3)$, then there exist upper and lower solutions $(\tilde{s}_1(t), \tilde{s}_2(t), \tilde{s}_3(t))$ and $(\hat{s}_1(t), \hat{s}_2(t), \hat{s}_3(t))$ of (1) and (2). By Theorem 2.1 in [38], (1) and (2) have a unique solution $(s_1(x,t), s_2(x,t), s_3(x,t)), (x,t) \in \bar{\Omega}\times(-\tau,+\infty)$, which satisfies

$$(\hat{s}_1(t), \hat{s}_2(t), \hat{s}_3(t)) \le (s_1(x,t), s_2(x,t), s_3(x,t)) \le (\tilde{s}_1(t), \tilde{s}_2(t), \tilde{s}_3(t)).$$

If we can prove

$$\lim_{t\to\infty} \tilde{s}_i(t) - s_i^*(t) = \lim_{t\to\infty} \hat{s}_i(t) - s_i^*(t) = 0, (i = 1,2,3), \tag{17}$$

then (16) is established. Consequently, to achieve (17), we must demonstrate that the solution $(s_1(t), s_2(t), s_3(t))$ of FDEs (8), with any positive initial condition $(\eta_{10}(t), \eta_{20}(t), \eta_{30}(t))$, satisfies

$$\lim_{t\to\infty}\left(s_i(t) - s_i^*(t)\right) = 0, i = 1,2,3. \tag{18}$$

By Theorem 1, we have

$$m_i \le s_i(t) \le M_i, i = 1,2,3, \text{ when } t > T,$$

where M_i, m_i and T are positive numbers.

Let

$$V_{11}(t) = \left|\ln s_1(t) - \ln s_1^*(t)\right|.$$

We denote by $D^+V_{11}(t)$ the right-side derivative of $V_{11}(t)$, then

$$
\begin{aligned}
D^{+}V_{11}(t) &= \operatorname{sgn}\,(s_1(t)-s_1^{*}(t))[-a_{11}(t)(s_1(t-\tau_1)-s_1^{*}(t-\tau_1))-a_{12}(t)(s_2(t-\tau_2)-s_2^{*}(t-\tau_2)) \\
&\quad -a_{13}(t)(s_3(t-\tau_3)-s_3^{*}(t-\tau_3))] \\
&= \operatorname{sgn}\,(s_1(t)-s_1^{*}(t))[-a_{11}(t)(s_1(t)-s_1^{*}(t))-a_{12}(t)(s_2(t-\tau_2)-s_2^{*}(t-\tau_2)) \\
&\quad -a_{13}(t)(s_3(t-\tau_3)-s_3^{*}(t-\tau_3))+a_{11}(t)\int_{t-\tau_1}^{t}(\dot{s}_1(\theta)-\dot{s}_1^{*}(\theta))d\theta] \\
&= \operatorname{sgn}\,(s_1(t)-s_1^{*}(t))[-a_{11}(t)(s_1(t)-s_1^{*}(t))-a_{12}(t)(s_2(t-\tau_2)-s_2^{*}(t-\tau_2)) \\
&\quad -a_{13}(t)(s_3(t-\tau_3)-s_3^{*}(t-\tau_3))+a_{11}(t)\int_{t-\tau_1}^{t}(s_1(\theta)[r_1(\theta)-a_{11}(\theta)s_1(\theta-\tau_1) \\
&\quad -a_{12}(\theta)s_2(\theta-\tau_2)-a_{13}(\theta)s_3(\theta-\tau_3)] \\
&\quad -s_1^{*}(\theta)[r_1(\theta)-a_{11}(\theta)s_1^{*}(\theta-\tau_1)-a_{12}(\theta)s_2^{*}(\theta-\tau_2)-a_{13}(\theta)s_3^{*}(\theta-\tau_3)])d\theta] \\
&= \operatorname{sgn}\,(s_1(t)-s_1^{*}(t))[-a_{11}(t)(s_1(t)-s_1^{*}(t))-a_{12}(t)(s_2(t-\tau_2)-s_2^{*}(t-\tau_2)) \\
&\quad -a_{13}(t)(s_3(t-\tau_3)-s_3^{*}(t-\tau_3))+a_{11}(t)\int_{t-\tau_1}^{t}((s_1(\theta)-s_1^{*}(\theta))[r_1(\theta) \\
&\quad -a_{11}(\theta)s_1^{*}(\theta-\tau_1)-a_{12}(\theta)s_2^{*}(\theta-\tau_2)-a_{13}(\theta)s_3^{*}(\theta-\tau_3)] \\
&\quad -s_1(\theta)[a_{11}(\theta)(s_1(\theta-\tau_1)-s_1^{*}(\theta-\tau_1))+a_{12}(\theta)(s_2(\theta-\tau_2)-s_2^{*}(\theta-\tau_2)) \\
&\quad +a_{13}(\theta)(s_3(\theta-\tau_3)-s_3^{*}(\theta-\tau_3))])d\theta] \\
&\le -a_{11}(t)\left|s_1(t)-s_1^{*}(t)\right|+a_{12}(t)\left|s_2(t-\tau_2)-s_2^{*}(t-\tau_2)\right|+a_{13}(t)\left|s_3(t-\tau_3)-s_3^{*}(t-\tau_3)\right| \\
&\quad +a_{11}(t)\int_{t-\tau_1}^{t}([r_1(\theta)+a_{11}(\theta)s_1^{*}(\theta-\tau_1)+a_{12}(\theta)u_2^{*}(\theta-\tau_2) \\
&\quad +a_{13}(\theta)s_3^{*}(\theta-\tau_3)]\left|s_1(\theta)-s_1^{*}(\theta)\right|+s_1(\theta)[a_{11}(\theta)\left|s_1(\theta-\tau_1)-s_1^{*}(\theta-\tau_1)\right| \\
&\quad +a_{12}(\theta)\left|s_2(t-\tau_2)-s_2^{*}(\theta-\tau_2)\right|+a_{13}(\theta)\left|s_3(\theta-\tau_3)-s_3^{*}(\theta-\tau_3)\right|])d\theta.
\end{aligned}
\tag{19}
$$

Let

$$
\begin{aligned}
V_{12}(t) &= \int_{t-\tau_1}^{t}\int_{\mu}^{t}a_{11}(\mu+\tau_1)([r_1(\theta)+a_{11}(\theta)s_1^{*}(\theta-\tau_1)+a_{12}(\theta)s_2^{*}(\theta-\tau_2) \\
&\quad +a_{13}(\theta)s_3^{*}(\theta-\tau_3)]\left|s_1(\theta)-s_1^{*}(\theta)\right|+s_1(\theta)[a_{11}(\theta)\left|s_1(\theta-\tau_1)-s_1^{*}(\theta-\tau_1)\right| \\
&\quad +a_{12}(\theta)\left|s_2(t-\tau_2)-s_2^{*}(\theta-\tau_2)\right|+a_{13}(\theta)\left|s_3(\theta-\tau_3)-s_3^{*}(\theta-\tau_3)\right|])d\theta d\mu.
\end{aligned}
\tag{20}
$$

By (19) and (20),

$$\begin{aligned}
D^{+}\sum_{i=1}^{2}V_{1i}(t) &\le -a_{11}(t)\left|s_1(t)-s_1^*(t)\right| + a_{12}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| + a_{13}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&\quad + \int_{t-\tau_1}^{t} a_{11}(\mu+\tau_1)d\mu([r_1(t)+a_{11}(t)s_1^*(t-\tau_1)+a_{12}(t)s_2^*(t-\tau_2) \\
&\quad + a_{13}(t)s_3^*(t-\tau_3)]\left|s_1(t)-s_1^*(t)\right| + s_1(t)[a_{11}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| \\
&\quad + a_{12}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| + a_{13}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|]) \\
&\le -a_{11}(t)\left|s_1(t)-s_1^*(t)\right| + a_{12}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| + a_{13}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&\quad + \int_{t-\tau_1}^{t} a_{11}(\mu+\tau_1)d\mu([r_1(t)+a_{11}(t)M_1+a_{12}(t)M_2+a_{13}(t)M_3]\left|s_1(t)-s_1^*(t)\right| \\
&\quad + M_1\int_{t-\tau_1}^{t} a_{11}(s+\tau_1)d\mu[a_{11}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| + a_{12}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| \\
&\quad + a_{13}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|]) \\
&\le (-a_{11}^l + a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2 + a_{13}^m M_3])\left|s_1(t)-s_1^*(t)\right| \\
&\quad + M_1(a_{11}^m)^2\tau_1\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| + (1+M_1 a_{11}^m\tau_1)a_{12}^m\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| \\
&\quad + (1+M_1 a_{11}^m\tau_1)a_{13}^m\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|.
\end{aligned} \tag{21}$$

Let

$$\begin{aligned}
V_{13}(t) &= M_1(a_{11}^m)^2\tau_1\int_{t-\tau_1}^{t}\left|(s_1(w)-s_1^*(w)\right|dw + (1+M_1 a_{11}^m\tau_1)a_{12}^m\int_{t-\tau_2}^{t}\left|(s_2(w)-s_2^*(w)\right|dw \\
&\quad + (1+M_1 a_{11}^m\tau_1)a_{13}^m\int_{t-\tau_3}^{t}\left|(s_3(w)-s_3^*(w)\right|dw,
\end{aligned} \tag{22}$$

and

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t). \tag{23}$$

By (21) and (22), we have

$$\begin{aligned}
D^{+}V_1(t) &\le (-a_{11}^l + a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2 + a_{13}^m M_3] + M_1(a_{11}^m)^2\tau_1)\left|s_1(t)-s_1^*(t)\right| \\
&\quad + (1+M_1 a_{11}^m\tau_1)a_{12}^m\left|s_2(t)-s_2^*(t)\right| + (1+M_1 a_{11}^m\tau_1)a_{13}^m\left|s_3(t)-s_3^*(t)\right|.
\end{aligned} \tag{24}$$

Similarly, we define

$$V_{21}(t) = \left|\ln s_2(t) - \ln s_2^*(t)\right|,$$

then we have

$$
\begin{aligned}
D^{+}V_{21}(t) &= \operatorname{sgn}\,(s_2(t)-s_2^*(t))[-a_{22}(t)(s_2(t-\tau_2)-s_2^*(t-\tau_2))+a_{21}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{23}(t)(s_3(t-\tau_3)-s_3^*(t-\tau_3))] \\
&= \operatorname{sgn}\,(s_2(t)-s_2^*(t))[-a_{22}(t)(s_2(t)-s_2^*(t))+a_{21}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{23}(t)(s_3(t-\tau_3)-s_3^*(t-\tau_3))+a_{22}(t)\int_{t-\tau_2}^{t}(\dot{s}_2(\theta)-\dot{s}_2^*(\theta))d\theta] \\
&= \operatorname{sgn}\,(s_2(t)-s_2^*(t))[-a_{22}(t)(s_2(t)-s_2^*(t))+a_{21}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{23}(t)(s_3(t-\tau_3)-s_3^*(t-\tau_3))+a_{22}(t)\int_{t-\tau_2}^{t}(s_2(\theta)[-r_2(\theta)-a_{22}(\theta)s_2(\theta-\tau_2) \\
&\quad +a_{21}(\theta)s_1(\theta-\tau_1)-a_{23}(\theta)s_3(\theta-\tau_3)] \\
&\quad -s_2^*(\theta)[-r_2(\theta)-a_{22}(\theta)s_2^*(\theta-\tau_2)+a_{21}(\theta)s_1^*(\theta-\tau_1)-a_{23}(\theta)s_3^*(\theta-\tau_3)])d\theta \\
&= \operatorname{sgn}\,(s_2(t)-s_2^*(t))[-a_{22}(t)(s_2(t)-s_2^*(t))+a_{21}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{23}(t)(s_3(t-\tau_3)-s_3^*(t-\tau_3))+a_{22}(t)\int_{t-\tau_2}^{t}((s_2(\theta)-s_2^*(\theta))[-r_2(\theta)-a_{22}(\theta)s_2^*(\theta-\tau_2) \\
&\quad +a_{21}(\theta)s_1^*(\theta-\tau_1)-a_{23}(\theta)s_3^*(\theta-\tau_3)]-s_2(\theta)[a_{22}(\theta)(s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2)) \\
&\quad -a_{21}(\theta)(s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1))+a_{23}(\theta)(s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3))])d\theta \\
&\le -a_{22}(t)\left|s_2(t)-s_2^*(t)\right|+a_{21}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right|+a_{23}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&\quad +a_{22}(t)\int_{t-\tau_2}^{t}([r_2(\theta)+a_{22}(\theta)s_2^*(\theta-\tau_2)+a_{21}(\theta)s_1^*(\theta-\tau_1)+a_{23}(\theta)s_3^*(\theta-\tau_3)]\left|s_2(\theta)-s_2^*(\theta)\right| \\
&\quad +s_2(\theta)[a_{22}(\theta)\left|s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2)\right|+a_{21}(\theta)\left|s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1)\right| \\
&\quad +a_{23}(\theta)\left|s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3)\right|])d\theta.
\end{aligned}
\tag{25}
$$

Let

$$
\begin{aligned}
V_{22}(t) &= \int_{t-\tau_2}^{t}\int_{\mu}^{t}a_{22}(\mu+\tau_2)([r_2(\theta)+a_{22}(\theta)s_2^*(\theta-\tau_2)+a_{21}(\theta)s_1^*(\theta-\tau_1) \\
&\quad +a_{23}(\theta)s_3^*(\theta-\tau_3)]\left|s_2(\theta)-s_2^*(\theta)\right|+s_2(\theta)[a_{22}(\theta)\left|s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2)\right| \\
&\quad +a_{21}(\theta)\left|s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1)\right|+a_{23}(\theta)\left|s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3)\right|])d\theta d\mu.
\end{aligned}
\tag{26}
$$

By (25) and (26), we have

$$
\begin{aligned}
D^{+}\sum_{i=1}^{2}V_{2i}(t) \le{}& -a_{22}(t)\left|s_2(t)-s_2^*(t)\right| + a_{21}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right| + a_{23}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&+\int_{t-\tau_2}^{t} a_{22}(\mu+\tau_2)d\mu\,([r_2(t)+a_{22}(t)s_2^*(t-\tau_2)+a_{21}(t)s_1^*(t-\tau_1) \\
&+a_{23}(t)s_3^*(t-\tau_3)]\left|s_2(t)-s_2^*(t)\right| + s_2(t)[a_{22}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| \\
&+a_{21}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| + a_{23}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|]) \\
\le{}& -a_{22}(t)\left|s_2(t)-s_2^*(t)\right| + a_{21}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right| + a_{23}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&+\int_{t-\tau_2}^{t} a_{22}(\mu+\tau_2)d\mu\,([r_2(t)+a_{22}(t)M_2+a_{21}(t)M_1+a_{23}(t)M_3]\left|s_2(t)-s_2^*(t)\right| \\
&+M_2\int_{t-\tau_2}^{t} a_{22}(\mu+\tau_2)d\mu\,[a_{22}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| + a_{21}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| \\
&+a_{23}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|]) \\
\le{}& (-a_{22}^l + a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1 + a_{23}^m M_3])\left|s_2(t)-s_2^*(t)\right| \\
&+(1+M_2a_{22}^m\tau_2)a_{21}^m\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| \\
&+M_2(a_{22}^m)^2\tau_2\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| + (1+M_2a_{22}^m\tau_2)a_{23}^m\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right|.
\end{aligned} \tag{27}
$$

Let

$$
\begin{aligned}
V_{23}(t) ={}& M_2(a_{22}^m)^2\tau_2\int_{t-\tau_2}^{t}\left|(s_2(w)-s_2^*(w)\right|dw + (1+M_2a_{22}^m\tau_2)a_{21}^m\int_{t-\tau_1}^{t}\left|(s_1(w)-s_1^*(w)\right|dw \\
&+(1+M_2a_{22}^m\tau_2)a_{23}^m\int_{t-\tau_3}^{t}\left|(s_3(w)-s_3^*(w)\right|dw,
\end{aligned} \tag{28}
$$

and

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t). \tag{29}$$

By (27) and (28),

$$
\begin{aligned}
D^{+}V_2(t) \le{}& (-a_{22}^l + a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1 + a_{23}^m M_3] + M_2(a_{22}^m)^2\tau_2)\left|s_2(t)-s_2^*(t)\right| \\
&+(1+M_2a_{22}^m\tau_2)a_{21}^m\left|s_1(t)-s_1^*(t)\right| + (1+M_2a_{22}^m\tau_2)a_{23}^m\left|s_3(t)-s_3^*(t)\right|.
\end{aligned} \tag{30}
$$

Let $V_{31}(t) = \left|\ln s_3(t) - \ln s_3^*(t)\right|$, one has

$$\begin{aligned}
D^{+}V_{31}(t) &= \operatorname{sgn}(s_3(t)-s_3^*(t))[-a_{33}(t)(s_3(t-\tau_3)-s_3^*(t-\tau_3))+a_{31}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{32}(t)(s_2(t-\tau_2)-s_2^*(t-\tau_2))] \\
&= \operatorname{sgn}(s_3(t)-s_3^*(t))[-a_{33}(t)(s_3(t)-s_3^*(t))+a_{31}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{32}(t)(s_2(t-\tau_2)-s_2^*(t-\tau_2))+a_{33}(t)\int_{t-\tau_3}^{t}(\dot{s}_3(\theta)-\dot{s}_3^*(\theta))d\theta] \\
&= \operatorname{sgn}(s_3(t)-s_3^*(t))[-a_{33}(t)(s_3(t)-s_3^*(t))+a_{31}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{32}(t)(s_2(t-\tau_2)-s_2^*(t-\tau_2))+a_{33}(t)\int_{t-\tau_3}^{t}(s_3(\theta)[-r_3(\theta)-a_{33}(\theta)s_3(\theta-\tau_3) \\
&\quad +a_{31}(\theta)s_1(\theta-\tau_1)+a_{32}(\theta)s_2(\theta-\tau_2)] \\
&\quad -s_3^*(\theta)[-r_3(\theta)-a_{33}(\theta)s_3^*(\theta-\tau_3)+a_{31}(\theta)s_1^*(\theta-\tau_1)+a_{32}(\theta)s_2^*(\theta-\tau_2)])d\theta \\
&= \operatorname{sgn}(s_3(t)-s_3^*(t))[-a_{33}(t)(s_3(t)-s_3^*(t))+a_{31}(t)(s_1(t-\tau_1)-s_1^*(t-\tau_1)) \\
&\quad -a_{32}(t)(s_2(t-\tau_2)-s_2^*(t-\tau_2))+a_{33}(t)\int_{t-\tau_3}^{t}(s_3(\theta)-s_3^*(\theta))[-r_3(\theta) \\
&\quad -a_{33}(\theta)s_3^*(\theta-\tau_3)+a_{31}(\theta)s_1^*(\theta-\tau_1)+a_{32}(\theta)s_2^*(\theta-\tau_2)] \\
&\quad -s_3(\theta)[a_{33}(\theta)(s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3))-a_{31}(\theta)(s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1)) \\
&\quad -a_{32}(\theta)(s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2))])d\theta] \\
&\le -a_{33}(t)\left|s_3(t)-s_3^*(t)\right|+a_{31}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right|+a_{32}(t)\left|s_s(t-\tau_2)-s_s^*(t-\tau_2)\right| \\
&\quad +a_{33}(t)\int_{t-\tau_3}^{t}([r_3(\theta)+a_{33}(\theta)s_3^*(\theta-\tau_2)+a_{31}(\theta)s_1^*(\theta-\tau_1)+a_{32}(\theta)s_2^*(\theta-\tau_2)]\left|s_3(\theta)-s_3^*(\theta)\right| \\
&\quad +s_3(\theta)[a_{33}(\theta)\left|s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3)\right|+a_{31}(\theta)\left|s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1)\right| \\
&\quad +a_{32}(\theta)\left|s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2)\right|])d\theta.
\end{aligned} \tag{31}$$

Let

$$\begin{aligned}
V_{32}(t) = \int_{t-\tau_3}^{t}\int_{\mu}^{t} & a_{33}(\mu+\tau_3)([r_3(\theta)+a_{33}(\theta)s_3^*(\theta-\tau_3)+a_{31}(\theta)s_1^*(\theta-\tau_1) \\
&+a_{32}(\theta)s_2^*(\theta-\tau_2)]\left|s_3(\theta)-s_3^*(\theta)\right|+s_3(\theta)[a_{33}(\theta)\left|s_3(\theta-\tau_3)-s_3^*(\theta-\tau_3)\right| \\
&+a_{31}(\theta)\left|s_1(\theta-\tau_1)-s_1^*(\theta-\tau_1)\right|+a_{32}(\theta)\left|s_2(\theta-\tau_2)-s_2^*(\theta-\tau_2)\right|])d\theta d\mu.
\end{aligned} \tag{32}$$

By (31) and (32),

$$\begin{aligned}
D^{+}\sum_{i=1}^{2}V_{3i}(t) \le{}& -a_{33}(t)\left|s_3(t)-s_3^*(t)\right| + a_{31}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right| + a_{32}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| \\
&+\int_{t-\tau_3}^{t} a_{33}(\mu+\tau_3)ds([r_3(t)+a_{33}(t)s_3^*(t-\tau_2)+a_{31}(t)s_1^*(t-\tau_1) \\
&+a_{32}(t)s_2^*(t-\tau_2)]\left|s_3(t)-s_3^*(t)\right| + s_3(t)[a_{33}(t)\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| \\
&+a_{31}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| + a_{32}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right|]) \\
\le{}& -a_{33}(t)\left|s_3(t)-s_3^*(t)\right| + a_{31}(t)\left|(s_1(t-\tau_1)-s_1^*(t-\tau_1))\right| + a_{32}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right| \\
&+\int_{t-\tau_3}^{t} a_{33}(\mu+\tau_3)d\mu([r_3(t)+a_{33}(t)M_3+a_{31}(t)M_1+a_{32}(t)M_2]\left|s_3(t)-s_3^*(t)\right| \\
&+M_3\int_{t-\tau_3}^{t} a_{33}(s+\tau_3)d\mu[a_{33}(t)\left|s_3(t-\tau_2)-s_3^*(t-\tau_2)\right| + a_{31}(t)\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| \\
&+a_{32}(t)\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right|]) \\
\le{}& (-a_{33}^l + a_{33}^m\tau_3[r_3^m + a_{33}^m M_3 + a_{31}^m M_1 + a_{32}^m M_2])\left|s_3(t)-s_3^*(t)\right| \\
&+(1+M_3 a_{33}^m\tau_3)a_{31}^m\left|s_1(t-\tau_1)-s_1^*(t-\tau_1)\right| \\
&+M_3(a_{33}^m)^2\tau_3\left|s_3(t-\tau_3)-s_3^*(t-\tau_3)\right| + (1+M_3 a_{33}^m\tau_3)a_{32}^m\left|s_2(t-\tau_2)-s_2^*(t-\tau_2)\right|.
\end{aligned} \tag{33}$$

Let

$$\begin{aligned}
V_{33}(t) ={}& M_3(a_{33}^m)^2\tau_3\int_{t-\tau_3}^{t}\left|(s_3(w)-s_3^*(w)\right|dw \\
&+(1+M_3 a_{33}^m\tau_3)a_{31}^m\int_{t-\tau_1}^{t}\left|(s_1(w)-s_1^*(w)\right|dw \\
&+(1+M_3 a_{33}^m\tau_3)a_{32}^m\int_{t-\tau_2}^{t}\left|(s_2(w)-s_2^*(w)\right|dw,
\end{aligned} \tag{34}$$

and

$$V_3(t) = V_{31}(t) + V_{32}(t) + V_{33}(t). \tag{35}$$

By (33) and (34),

$$\begin{aligned}
D^{+}V_3(t) \le{}& (-a_{33}^l + a_{33}^m\tau_3[r_3^m + a_{33}^m M_3 + a_{31}^m M_1 + a_{32}^m M_2] \\
&+M_3(a_{33}^m)^2\tau_3)\left|s_3(t)-s_3^*(t)\right| + (1+M_3 a_{33}^m\tau_3)a_{31}^m\left|s_1(t)-s_1^*(t)\right| \\
&+(1+M_3 a_{33}^m\tau_3)a_{32}^m\left|s_2(t)-s_2^*(t)\right|.
\end{aligned} \tag{36}$$

We define a Lyapunov function as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t).$$

By (24), (30), and (36), we obtain

$$D^{+}V(t) \le -A_1\left|s_1(t)-s_1^*(t)\right| - A_2\left|s_2(t)-s_2^*(t)\right| - A_3\left|s_3(t)-s_3^*(t)\right|. \tag{37}$$

Integrating from ω to t on both sides of (37), we have

$$V(t) + \alpha\int_{\omega}^{t}(\sum_{i=1}^{3}\left|s_i(u)-s_i^*(u)\right|)du \le V(\omega) < +\infty, \tag{38}$$

where $\alpha = \max\{A_1, A_2, A_3\} > 0$. Therefore, $V(t)$ is bounded on $[\omega, +\infty)$, and

$$\int_{\omega}^{t} \left(\sum_{i=1}^{3} \left| s_i(u) - s_i^*(u) \right|\right) du \le \frac{V(\omega)}{\alpha} < +\infty. \quad (39)$$

By (39), we have

$$\sum_{i=1}^{3} \left| s_i(t) - s_i^*(t) \right| \in L^1(T, +\infty). \quad (40)$$

By means of Theorem 1, $\sum_{i=1}^{3} \left| s_i(t) - s_i^*(t) \right|$ is uniformity continuous on $[\omega, +\infty)$. With the help of Lemma 8.2 in [39], we can obtain

$$\lim_{t \to +\infty} \left| s_i(t) - s_i^*(t) \right| = 0, \ i = 1, 2, 3.$$

This concludes the proof of Theorem 3. □

Theorem 4. *Suppose that the* ω*-periodic model (1) satisfies assumptions* $(H_1) - (H_8)$*, then model (1) is permanent,* i.e., *the solution* $(s_1(x,t), s_2(x,t), s_3(x,t))$ *of model (1) and (2) with any IC fulfills*

$$m_i \le s_i(x,t) \le M_i, \ i = 1, 2, 3, \textit{ uniformly for } (x,t) \in \bar{\Omega} \times (T, +\infty). \quad (41)$$

Proof. By means of Theorem 2, we have

$$m_i \le s_i^*(t) = s_i^*(t + \omega) \le M_i, t \in [-\tau, +\infty), i = 1, 2, 3, \quad (42)$$

where M_i and m_i are positive numbers.

Moreover, from Theorem 3, one has

$$\lim_{t \to \infty} s_i(x,t) = s_i^*(t), \ i = 1, 2, 3, \text{ uniformly for } x \in \bar{\Omega}. \quad (43)$$

Therefore, by (42) and (43), there exists a positive real number T such that the solution $(s_1(x,t), s_2(x,t), s_3(x,t))$ of models (1) and (2) with any IC fulfills

$$m_i \le s_i(x,t) \le M_i, \text{ uniformly for } (x,t) \in \bar{\Omega} \times (T, +\infty).$$

This completes the proof of Theorem 4. □

4. Numerical Simulations

In this section, we provide a numerical example to prove the correctness of Theorem 3. For the convenience of calculation and numerical simulation, we chose 2-period functions as the coefficients for the nonautonomous periodic DRDPPM (1) and (2).

Example 1. *Consider the following three-species DRDPPM. Based on the assumptions outlined in Theorem 3, and after performing some calculations, we selected specific values for the parameters as demonstrated in models (44) and (45). It is important to note that these chosen parameter values are not unique.*

$$\begin{cases} \dfrac{\partial s_1(x,t)}{\partial t} - \Delta s_1(x,t) = s_1(x,t)[(24+\cos\pi t) - (6+\sin\pi t)s_1(x,t-\tau_1) - (0.75+0.25\sin\pi t)s_2(x,t-\tau_2) \\ \qquad - (0.65+0.35\sin\pi t)s_3(x,t-\tau_3)], \\ \dfrac{\partial s_2(x,t)}{\partial t} - \Delta s_2(x,t) = s_2(x,t)[-(1+\cos\pi t) - (5+\sin\pi t)s_2(x,t-\tau_2) + (1.2+0.2\sin\pi t)s_1(x,t-\tau_1) \\ \qquad - (0.075+0.025\sin\pi t)s_3(x,t-\tau_3)], \\ \dfrac{\partial s_3(x,t)}{\partial t} - \Delta s_3(x,t) = s_3(x,t)[-(1.1+0.1\cos\pi t) - (4+\sin\pi t)s_3(x,t-\tau_3) + (0.85+0.15\sin\pi t)s_1(x,t-\tau_1) \\ \qquad + (0.95+0.05\sin\pi t)s_2(x,t-\tau_2)], \end{cases} \tag{44}$$

with Neumann boundary conditions and positive initial conditions

$$\begin{cases} \partial s_i(0,t)/\partial x = \partial s_i(2\pi,t)/\partial x = 0,\ t \ge 0,\ i = 1,2,3, \\ s_1(x,t) = (4+3t)(1+\cos(x+\pi)), \\ s_2(x,t) = (2+5t)(1-\sin(x+0.5\pi)), \\ s_3(x,t) = (3+2t)(1+\sin(x-0.5\pi)), \\ x \in (0,2\pi),\ t \in [0,\tau],\ \tau = \max\{\tau_1,\tau_2,\tau_3\}. \end{cases} \tag{45}$$

Take $\tau_1 = 0.001, \tau_2 = 0.002, \tau_3 = 0.003,$ we have

$$M_1 = \frac{r_1^m}{a_{11}^l}\exp(r_1^m\tau_1) \approx 5.1266, M_2 = \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l}\exp((a_{21}^m M_1 - r_2^l)\tau_2) \approx 1.8203,$$

$$M_3 = \frac{(a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l}{a_{33}^l}\exp(((a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l)\tau_3) \approx 3.1747,$$

$$m_1 = \frac{r_1^l - a_{12}^m M_2 - a_{13}^m M_3}{a_{11}^m}\exp[(r_1^l - a_{12}^m M_2 - a_{13}^m M_3 - a_{11}^m M_1)\tau_1] \approx 2.5265,$$

$$m_2 = \frac{a_{21}^l m_1 - a_{23}^m M_3 - r_2^m}{a_{22}^m}\exp[(a_{21}^l m_1 - a_{23}^m M_3 - r_2^m - a_{22}^m M_2)\tau_2] \approx 0.3410,$$

$$m_3 = \frac{a_{31}^l m_1 + a_{32}^l m_2 - r_3^m}{a_{33}^m}\exp[(a_{31}^l m_1 + a_{32}^l m_2 - r_3^m - a_{33}^m M_3)\tau_3] \approx 0.1674,$$

$$a_{21}^m M_1 - r_2^l \approx 7.1772 > 0,$$

$$(a_{31}^m + a_{32}^m)\max\{M_1, M_2\} - r_3^l \approx 9.2532 > 0,$$

$$r_1^l - a_{12}^m M_2 + a_{13}^m M_3 \approx 18.005 > 0,$$

$$a_{21}^l m_1 - a_{23}^m M_3 - r_2^m \approx 0.2090 > 0,$$

$$a_{31}^l m_1 + a_{32}^l m_2 - r_3^m \approx 0.8754 > 0,$$

and

$$A_1 = a_{11}^l - a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2 + a_{13}^m M_3] - M_1(a_{11}^m)^2\tau_1$$
$$-(1+M_2 a_{22}^m\tau_2)a_{21}^m - (1+M_3 a_{33}^m\tau_3)a_{31}^m \approx 1.8090 > 0,$$
$$A_2 = a_{22}^l - a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1 + a_{23}^m M_3]$$
$$-M_2(a_{22}^m)^2\tau_2 - (1+M_1 a_{11}^m\tau_1)a_{12}^m - (1+M_3 a_{33}^m\tau_3)a_{32}^m \approx 1.5403 > 0,$$
$$A_3 = a_{33}^l - a_{33}^m\tau_3[r_3^m + a_{33}^m M_3 + a_{31}^m M_1 + a_{32}^m M_2]$$
$$-M_3(a_{33}^m)^2\tau_3 - (1+M_1 a_{11}^m\tau_1)a_{13}^m - (1+M_2 a_{22}^m\tau_2)a_{23}^m \approx 0.3436 > 0.$$

Based on the above calculation results, it is easy to see that that model (44) and (45) satisfy the conditions of Theorem 3. From Theorem 3, model (44) and (45) have a strictly positive spatial homogeneity 2-periodic solution $(s_1^*(t), s_2^*(t), s_3^*(t))$, and for any positive initial conditions the solution $(s_1(x,t), s_2(x,t), s_3(x,t))$ of model (1) and (2) which satisfies

$$\lim_{t\to+\infty}\left|s_i(x,t) - s_i^*(t)\right| = 0,\ i = 1,2,3, \text{ uniformly for } x \in \bar{\Omega}.$$

Utilizing the finite difference method [40] and the MATLAB 7.0 software package, we can derive numerical solutions for model (44) subject to the boundary and initial conditions outlined in (45). These solutions are illustrated in Figures 1–3. From Figures 1–3, it is evident that models (44) and (45) possess a strictly positive, spatially homogeneous, 2-periodic solution. Specifically, in the context of models (44) and (45), the densities of prey and predators exhibit periodic oscillations with a period of 2 and distribute uniformly in space as time progresses sufficiently. In order to verify that the periodic solution of model (44) and (45) is globally asymptotically stable, we selected different initial values and conducted extensive numerical simulations. The results showed that for any positive initial value, the 2-periodic solution of model (44) and (45) is asymptotically stable. Please refer to Figure 4 for details. In order to verify the existence of a globally asymptotically stable periodic solution for models (44) and (45) with different time delays, we conducted numerical simulations for different time delays and found that models (44) and (45) still possess a strictly positive, spatially homogeneous, 2-periodic solution. Figures 5–8 show the results when time delays $\tau_1 = 0.01, \tau_2 = 0.02, \tau_3 = 0.03$.

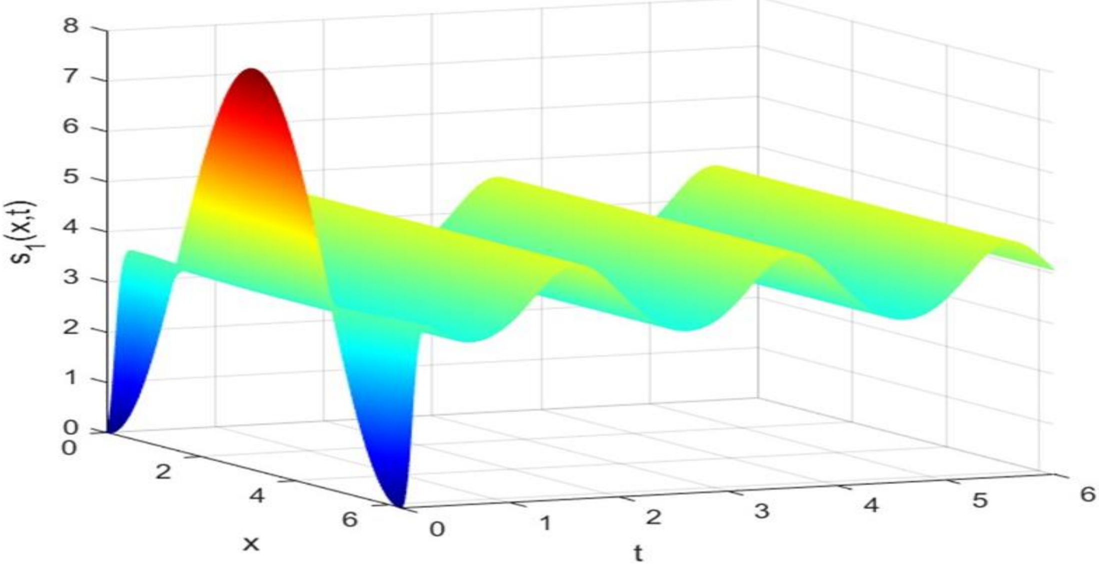

Figure 1. The evolution process of the density for the species $s_1(x,t)$ in models (44) and (45) with $\tau_1 = 0.001, \tau_2 = 0.002, \tau_3 = 0.003.$

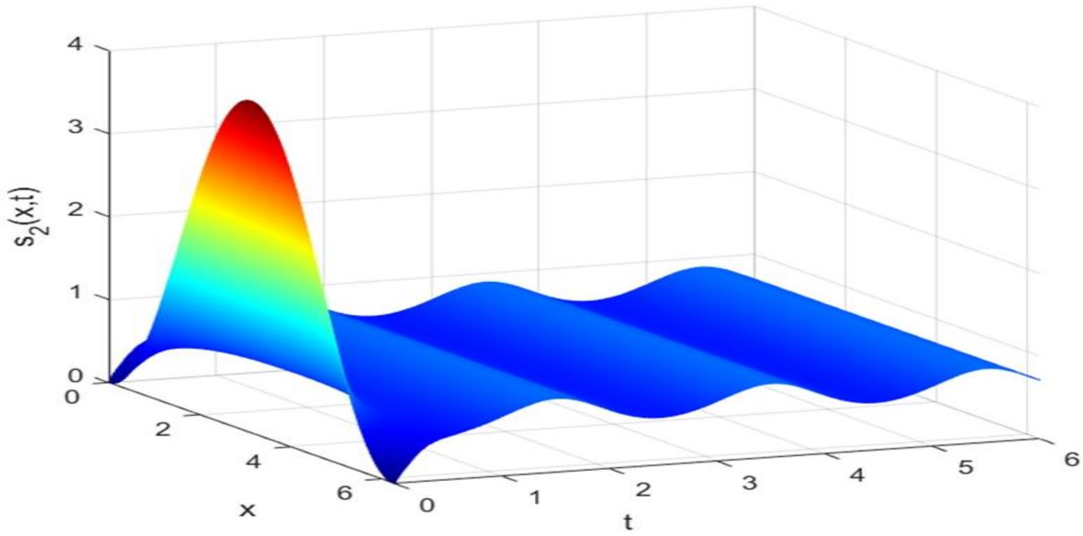

Figure 2. The evolution process of the density for the species $s_2(x,t)$ in models (44) and (45) with $\tau_1 = 0.001, \tau_2 = 0.002, \tau_3 = 0.003.$

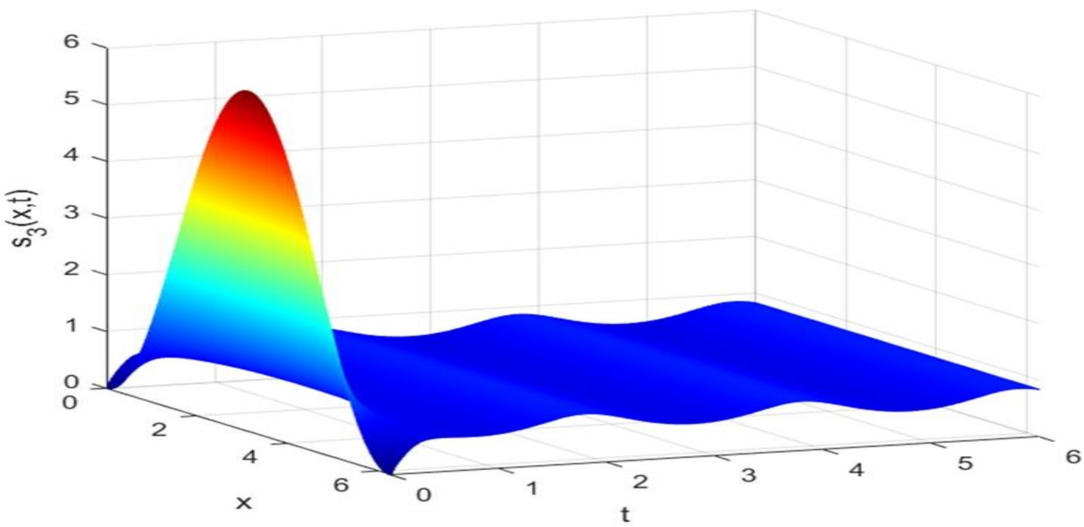

Figure 3. The evolution process of the density for the species $s_3(x,t)$ in models (44) and (45) with $\tau_1 = 0.001, \tau_2 = 0.002, \tau_3 = 0.003.$

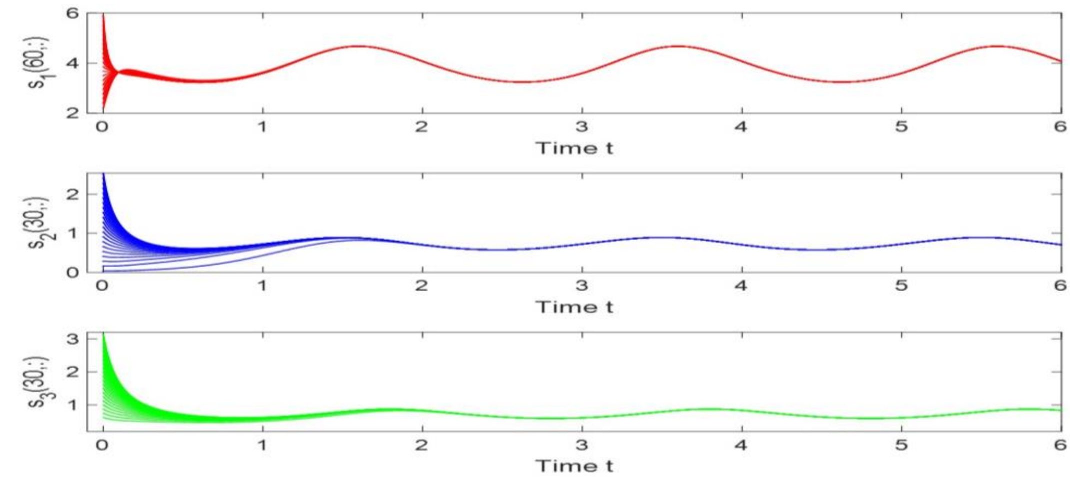

Figure 4. The evolution process of the density for the species $s_1(x,t), s_2(x,t)$ and $s_3(x,t)$ in models (44) and (45) with different initial values and $\tau_1 = 0.001, \tau_2 = 0.002, \tau_3 = 0.003.$

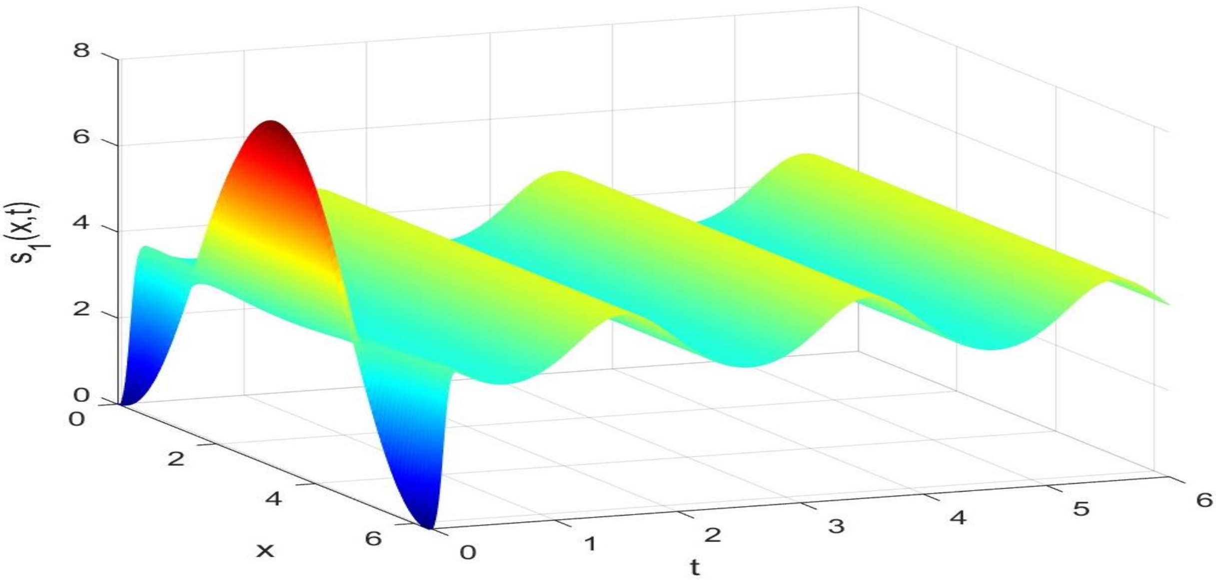

Figure 5. The evolution process of the density for the species $s_1(x,t)$ in models (44) and (45) with $\tau_1 = 0.01, \tau_2 = 0.02, \tau_3 = 0.03.$

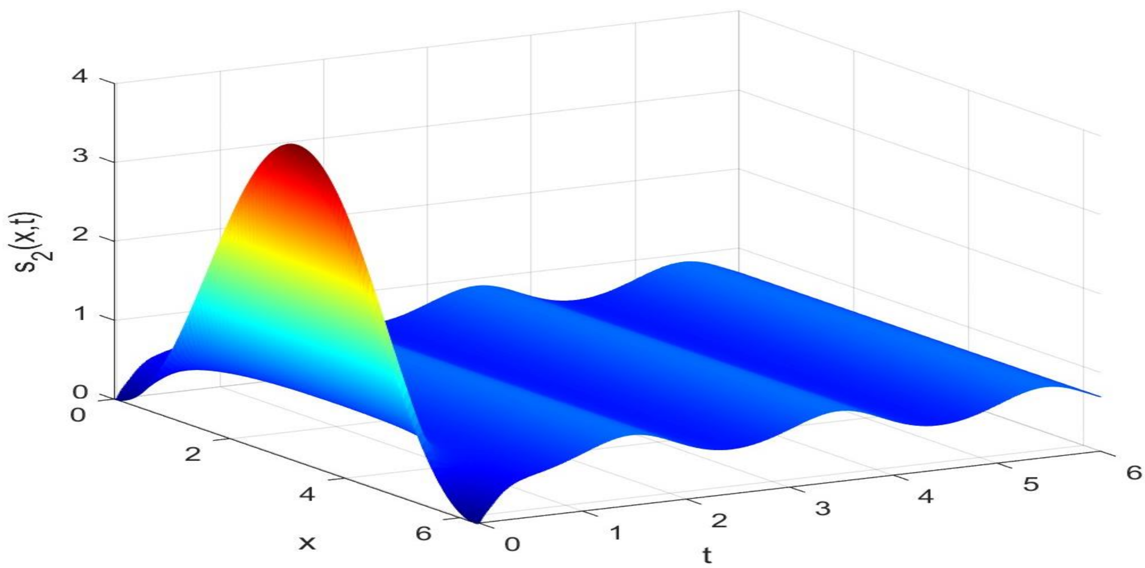

Figure 6. The evolution process of the density for the species $s_2(x,t)$ in models (44) and (45) with $\tau_1 = 0.01, \tau_2 = 0.02, \tau_3 = 0.03$.

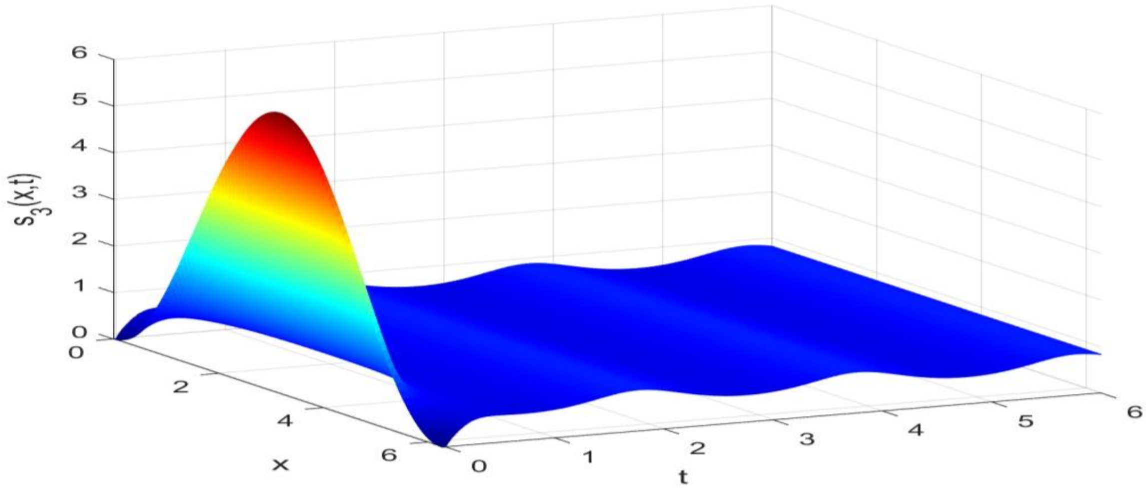

Figure 7. The evolution process of the density for the species $s_3(x,t)$ in models (44) and (45) with $\tau_1 = 0.01, \tau_2 = 0.02, \tau_3 = 0.03$.

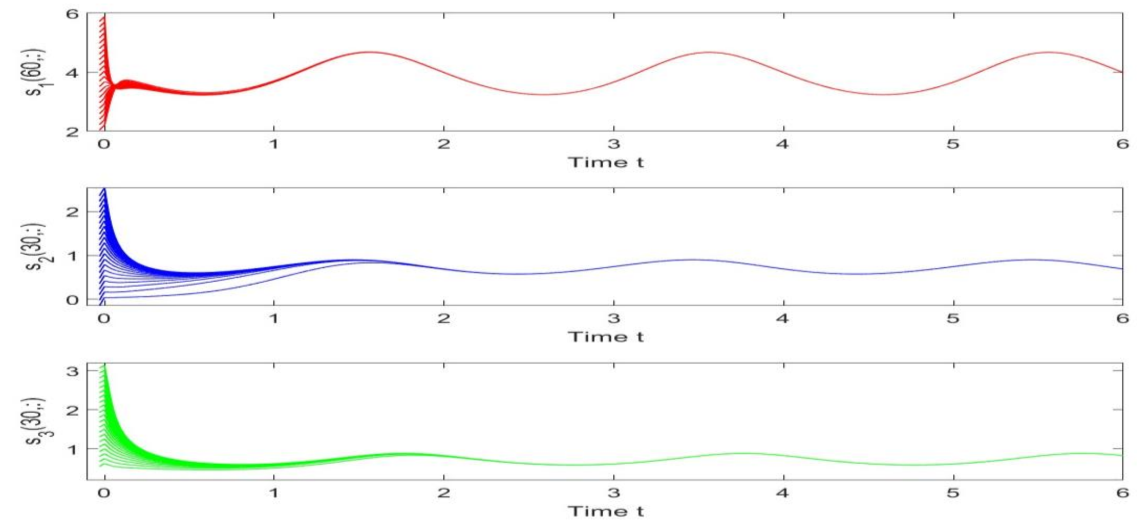

Figure 8. The evolution process of the density for the species $s_1(x,t), s_2(x,t)$ and $s_3(x,t)$ in models (44) and (45) with different initial values and $\tau_1 = 0.01, \tau_2 = 0.02, \tau_3 = 0.03$.

5. Conclusions

This article demonstrates the significant strength of the UALSM in addressing nonlinear nonautonomous reaction–diffusion equations. It has found widespread application in solving problems associated with nonlinear PDEs in various fields such as chemistry, engineering, and mathematical physics. The innovative approach of constructing a Lyapunov function alongside a pair of ordered UALS offers a valuable reference for tackling stability issues in both delay and non-delay nonlinear PDEs.

The periodic solution for a three-species nonautonomous DRDFFM is investigated. The existence and stability of strictly positive SHPS are established for the nonautonomous nonlinear reaction–diffusion equations based on readily verifiable criteria. These criteria improve and generalize certain previous findings. Notably, the sufficient conditions derived in this article are straightforward, rendering the approach highly adaptable for practical applications. It is important to acknowledge that, while this work does not account for functional response in the model, functional response is ubiquitous in ecosystems and can influence system stability. Therefore, our next objective is to explore multi-species non-autonomous DRDPPMs that incorporate functional response. In addition, our method, after being improved, can also be used to study and generalize fractional differential equations [41,42].

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