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Krein–Sobolev Orthogonal Polynomials II

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Abstract: In a recent paper, Littlejohn and Quintero studied the orthogonal polynomials $\{K_n\}_{n=0}^{\infty}$—which they named *Krein–Sobolev polynomials*—that are orthogonal in the classical Sobolev space $H^1[-1,1]$ with respect to the (positive-definite) inner product $(f,g)_{1,c} := -\frac{(f(1)-f(-1))(\overline{g}(1)-\overline{g}(-1))}{2} + \int_{-1}^{1}(f'(x)\overline{g}'(x) + cf(x)\overline{g}(x))dx$, where c is a fixed, positive constant. These polynomials generalize the Althammer (or Legendre–Sobolev) polynomials first studied by Althammer and Schäfke. The Krein–Sobolev polynomials were found as a result of a left-definite spectral study of the self-adjoint Krein Laplacian operator $\mathcal{K}_c$ $(c > 0)$ in $L^2(-1,1)$. Other than K_0 and K_1, these polynomials are not eigenfunctions of $\mathcal{K}_c$. As shown by Littlejohn and Quintero, the sequence $\{K_n\}_{n=0}^{\infty}$ forms a complete orthogonal set in the first left-definite space $(H^1[-1,1],(\cdot,\cdot)_{1,c})$ associated with $(\mathcal{K}_c, L^2(-1,1))$. Furthermore, they show that, for $n \geq 1$, $K_n(x)$ has n distinct zeros in $(-1,1)$. In this note, we find an explicit formula for Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$.

Keywords: one-dimensional Krein Laplacian self-adjoint operator; left-definite spectral theory; Althammer polynomials; Krein–Sobolev polynomials

MSC: 05C38, 15A15; 05A15; 15A18

1. Introduction

In a recent paper, Littlejohn and Quintero [1] studied the *Krein–Sobolev polynomials* $\{K_n\}_{n=0}^{\infty}$, which are orthogonal in the classical Sobolev space:

$$H^1[-1,1] = \{f : [-1,1] \to \mathbb{C} \mid f \in AC[-1,1]; f' \in L^2(-1,1)\}, \tag{1}$$

which is endowed with the inner product

$$(f,g)_{1,c} = -\frac{(f(1)-f(-1))(\overline{g}(1)-\overline{g}(-1))}{2} + \int_{-1}^{1}(f'(x)\overline{g}'(x) + cf(x)\overline{g}(x))dx. \tag{2}$$

Throughout this paper, unless otherwise specified, we assume c is a fixed, positive constant. This is not immediately obvious, especially since the discrete term in (2) has a minus sign, that $(\cdot,\cdot)_{1,c}$ defines a (positive-definite) inner product. These Krein–Sobolev polynomials naturally arise in the spectral analysis of the shifted one-dimensional Krein Laplacian self-adjoint operator $\mathcal{K}_c : \mathcal{D}(\mathcal{K}_c) \subset L^2(-1,1) \to L^2(-1,1)$ defined by

$$\mathcal{K}_c f = -f'' + cf \tag{3}$$

$$f \in \mathcal{D}(\mathcal{K}_c) := \left\{ f \in \Delta \mid f'(1) = f'(-1) = \frac{f(1)-f(-1)}{2} \right\}; \tag{4}$$

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here, Δ is the maximal domain associated with the expression $m[y] = -y'' + cy$ in $L^2(-1,1)$, given by

$$\Delta := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC[-1,1]; f'' \in L^2(-1,1)\}.$$

We note that, when $c = 0$, the bilinear form $(\cdot,\cdot)_{1,0}$ is only a pseudo inner product so, unless otherwise indicated, we assume $c > 0$. We note that the subscript 1 in $(\cdot,\cdot)_{1,c}$ refers to the fact that $(H^1[-1,1], (\cdot,\cdot)_1)$ is called the *first* left-definite space (see Sections 2 and 3) associated with $(\mathcal{K}_c, L^2(-1,1)$.

The Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$ are, in a sense, a generalization of *Althammer's polynomials* $\{A_n\}_{n=0}^{\infty}$, first studied by Althammer [2] in 1962, and later by Schäfke [3] in 1972. Althammer showed the sequence$\{A_n\}_{n=0}^{\infty}$ is orthogonal with respect to the Sobolev inner product

$$\langle f, g\rangle_{\lambda} = \int_{-1}^{1} (f(x)\overline{g}(x) + \lambda f'(x)\overline{g}'(x))dx \tag{5}$$

(where λ is a fixed, positive parameter) in $H^1[-1,1]$. Notice that when $\lambda = 1/c$ and $f \in H^1[-1,1]$ is an even function, then

$$(f,g)_{1,c} = c\langle f, g\rangle_{1/c} \quad (g \in H^1[-1,1]) \tag{6}$$

(later, we see that both $A_{2n}(x)$ and $K_{2n}(x)$ are even; in fact, we show that $A_{2n}(x) = K_{2n}(x)$ for all $n \in \mathbb{N}_0$).

In the literature, the sequence $\{A_n\}_{n=0}^{\infty}$ is known as *Althammer's polynomials* or *Sobolev–Legendre polynomials*, which has the latter name because the initial construction of $\{A_n\}_{n=0}^{\infty}$ involves the difference of two Legendre polynomials. In [2] and in a later 1972 contribution by Schäfke [3], several explicit properties of these polynomials were established, including an exact formula for $A_n(x)$.

Althammer's work is considered to be one of the first publications on the subject of Sobolev orthogonal polynomials, which is an area that has seen massive growth since the 1990s. For informative and detailed accounts of the history of the study of Sobolev orthogonal polynomials, we recommend the sources [4,5]. Sobolev orthogonal polynomials have applications in various areas, particularly in numerical analysis and solving boundary value problems for differential equations, where their ability to provide smooth approximations with good convergence properties is valuable.

Littlejohn and Quintero [1] found several properties of the Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$. However, they were not able to find an explicit formula for K_n for all $n \in \mathbb{N}_0$. It is the purpose of this note to give an explicit representation of these polynomials.

The contents of this paper are as follows. In Section 2, we recall properties of the classical self-adjoint Krein Laplacian operator $\mathcal{K}_0$ and the shifted Krein Laplacian operator $\mathcal{K}_c$ when $c > 0$. The operator $\mathcal{K}_c$ $(c > 0)$ is a positive, self-adjoint operator in $L^2[-1,1]$. Consequently, by applying a general left-definite theory of positive self-adjoint operators developed by Littlejohn and Wellman in [6,7], there is a continuum of Hilbert spaces $\{H_r\}_{r>0}$, the so-called *left-definite spaces*, generated by positive powers $\mathcal{K}_c^r$ of $\mathcal{K}_c$ $(r > 0)$. In general, we call H_1 the first left-definite space. In the case of the shifted Krein Laplacian $\mathcal{K}_c$, it is the case that the first left-definite space is $(H^1[-1,1], (\cdot,\cdot)_{1,c})$. Section 3 motivates left-definite theory and discusses the main results of abstract left-definite theory needed for this paper. In Section 4, we review properties of the Krein–Sobolev polynomials in the first left-definite space $(H^1[-1,1], (\cdot,\cdot)_{1,c})$ for the shifted Krein Laplacian operator $\mathcal{K}_c$ in $L^2(-1,1)$. The construction of $\{K_n\}_{n=0}^{\infty}$ is intricate and similar to the construction of the Althammer polynomials developed in [2,3]. In Section 5, we discuss these Althammer polynomials and present our new result, namely, an explicit formula of each Krein–Sobolev polynomial $K_n(x)$. Lastly, in Section 6, we summarize our main result and suggest an open problem on density of polynomials in certain constrained Hilbert spaces.

2. The Krein Laplacian Self-Adjoint Operator

In $L^2[-1,1]$, with standard inner product

$$(f,g)_{L^2[-1,1]} = \int_{-1}^{1} f(x)\overline{g}(x)dx,$$

there are uncountably many unbounded, self-adjoint operators T generated by the second-order differential expression

$$m[y] := -y''.$$

In this continuum, one such self-adjoint operator is the Krein Laplacian operator $\mathcal{K}_0$ defined in (3) and (4) when $c = 0$. Complete details on an analytic study of this operator in $L^2[-1,1]$ can be found in [8] in Section 10. In $L^2[-1,1]$, the Krein Laplacian operator $\mathcal{K}_0$ has a discrete spectrum given by

$$\sigma(\mathcal{K}_0) = \{0\} \cup \{(n\pi)^2\}_{n=1}^{\infty} \cup \{\mu_n^2\}_{n=1}^{\infty}, \tag{7}$$

and (complete) eigenfunctions

$$\{1, x\} \cup \{\cos n\pi x\}_{n=1}^{\infty} \cup \{\sin \mu_n x\}_{n=1}^{\infty}, \tag{8}$$

where $\mu_n > 0$ is a solution of the transcendental equation

$$\tan \mu = \mu \quad (n \in \mathbb{N}).$$

More specifically, the eigenspace of $\lambda = 0$, equivalently the null space of $\mathcal{K}_0$, is two-dimensional and spanned by $\{1, x\}$. The eigenspace for the eigenvalue $\lambda = (n\pi)^2$ $(n \in \mathbb{N})$ is one-dimensional and spanned by the eigenfunction $\cos n\pi x$, while the eigenspace for each $\lambda = \mu_n^2$ is one-dimensional and spanned by $\sin \mu_n x$.

An application of the general left-definite theory developed by Littlejohn and Wellman (see [6,7]) to the Krein Laplacian $\mathcal{K}_0$ produces a first left-definite space that is only a *pseudo* inner product space, since the two-dimensional eigenspace $W = \text{span}\{1, x\}$ acts like the *zero* element for this inner product. To avoid the awkwardness of considering equivalence classes of functions, we shift the spectrum (7) by $c > 0$. This allows for a left-definite analysis in a Hilbert function space with a positive-definite inner product. Consequently, from hereon, we study the *shifted* Krein Laplacian $\mathcal{K}_c$ in $L^2[-1,1]$ when $c > 0$ is fixed. In this case, $\mathcal{K}_c$ is self-adjoint and bounded below by cI in $L^2(-1,1)$; that is,

$$(\mathcal{K}_c f, f)_{L^2[-1,1]} \geq c(f,f)_{L^2[-1,1]} \quad (f \in \mathcal{D}(\mathcal{K}_c)); \tag{9}$$

moreover, the spectrum of $\mathcal{K}_c$ is given by

$$\sigma(\mathcal{K}_c) = \{\lambda + c \mid \lambda \in \sigma(\mathcal{K}_0)\} = \sigma(\mathcal{K}_0) + c,$$

where $\sigma(\mathcal{K}_0)$ is given in (7). The eigenfunctions of $\mathcal{K}_c$ are, of course, the same as for $\mathcal{K}_0$. The first left-definite space was computed in [1] to be the Sobolev space $(H^1[-1,1], (\cdot,\cdot)_{1,c})$.

As mentioned in the introduction, it is not immediately clear that $(\cdot,\cdot)_{1,c}$ is a positive-definite inner product on $H^1[-1,1] \times H^1[-1,1]$. Even though it necessarily is an inner product from left-definite theory, we give an elementary proof. For $f \in H^1[-1,1]$, we see from the Cauchy–Schwarz inequality that

$$|f(1)-f(-1)| = \left|\int_{-1}^{1} f'(x)dx\right| \leq \int_{-1}^{1} |f'(x)|dx \leq \left(\int_{-1}^{1} |f'(x)|^2 dx\right)^{1/2} \left(\int_{-1}^{1} dx\right)^{1/2}$$
$$= \sqrt{2}\left(\int_{-1}^{1} |f'(x)|^2 dx\right)^{1/2}.$$

It follows that

$$\frac{|f(1)-f(-1)|^2}{2} \leq \int_{-1}^{1} |f'(x)|^2 dx \leq \int_{-1}^{1} \left(|f'(x)|^2 + c|f(x)|^2\right) dx, \tag{10}$$

and, consequently,

$$(f,f)_{1,c} = -\frac{|f(1)-f(-1)|^2}{2} + \int_{-1}^{1} \left(|f'(x)|^2 + c|f(x)|^2\right) dx \geq 0.$$

Furthermore, if $(f,f)_{1,c} = 0$, we see from (10) that

$$\int_{-1}^{1} |f(x)|^2 dx = 0,$$

implying that $f = 0$ almost everywhere for $x \in (-1,1)$. However, since $f \in AC[-1,1]$, we must have $f \equiv 0$ on $[-1,1]$. It now follows that $(\cdot,\cdot)_{1,c}$ is a positive-definite inner product on $H^1[-1,1] \times H^1[-1,1]$.

3. A Glimpse of Left-Definite Operator Theory

Left-definite spectral theory has its origins in the seminal work of H. Weyl [9] in his analytical study of second-order Sturm–Liouville differential equations. Indeed, consider the differential expression

$$m[y](x) = -(p(x)y'(x))' + q(x)y(x) \quad (x \in I = (a,b)),$$

where, for simplicity sake, we assume $p, q \in C(I)$ are both positive functions; here, $-\infty \leq a < b \leq \infty$. There are two 'natural' Hilbert space settings to study symmetric and self-adjoint operators generated by $m[\cdot]$. Indeed, consider the spectral equation

$$m[y](x) = \lambda w(x)y(x) \quad (x \in I), \tag{11}$$

where $w \in L^1(a,b)$ is positive almost everywhere on (a,b). The first setting is the Hilbert space $L^2((a,b);w)$; since the weight function w appears on the right-hand side of (11), we call $L^2((a,b);w)$ the *right-definite setting*. The *left-definite setting* is a certain Sobolev space S with inner product

$$(f,g)_{\mathrm{Sob}} = \int_a^b \left(p(x)f'(x)\overline{g}'(x) + q(x)f(x)\overline{g}(x)\right) dx. \tag{12}$$

The inner product in (12) is formally obtained from the classic Dirichlet identity

$$\int_a^b \left(p(x)f'(x)\overline{g}'(x) + q(x)f(x)\overline{g}(x)\right) dx = \int_a^b m[f](x)\overline{g}(x)dx - \{f,g\}(x)\,|_a^b, \tag{13}$$

where $\{f,g\}(x) = -p(x)f'(x)g(x)$ is the Dirichlet form associated with $m[\cdot]$. The term '*left-definite*' arises from the fact that the left-hand side of (11) generates the inner product in (12). The mathematical literature, especially during the period 1970–2005, contains numerous articles on the left-definite study of second-order Sturm–Liouville equations; for example, see [10–13]. A further discussion of left-definite theory applied to second-order

Sturm–Liouville equations can be found in the texts [14] , Chapter 5, and [15], Chapters 5 and 12.

In [6,7] (see also [16]), the authors generalize the notion of left-definite theory to *arbitrary* self-adjoint operators A in a Hilbert space $(H, (\cdot, \cdot)_H)$, which are bounded below by a positive constant k in H; that is, A satisfies the inequality

$$(Af, f)_H \geq k(f, f)_H \quad (f \in \mathcal{D}(A)), \tag{14}$$

for some $k > 0$. Indeed, as shown in [6,7], Littlejohn and Wellman show that such an operator A produces a continuum of left-definite Hilbert spaces $\{H_r, (\cdot, \cdot)_r\}_{r>0}$. The space H_r is called the r^{th} left-definite space associated with (H, A). Left-definite spaces are special cases of Hilbert scales, as described in [17,18].

We briefly describe the salient results of this left-definite operator theory for the purposes of this paper. Definition 1 below, taken from Definitions 2.1 and 3.1 in [6], is motivated by five common features observed in examples listed in the above-mentioned papers.

Suppose V is a real or complex vector space and $(\cdot, \cdot)$ is an inner product on $V \times V$ such that $H = (V, (\cdot, \cdot))$ is a separable Hilbert space. In addition, suppose $A : \mathcal{D}(A) \subset H \to H$ is self-adjoint and bounded below by kI for some positive constant k; that is, the inequality in (14) is satisfied. From the Hilbert space spectral theorem, for each $r > 0$, A^r is self-adjoint and bounded below by $k^r I$.

Definition 1. *Let $r > 0$; suppose V_r is a subspace of V, and $(\cdot, \cdot)_r$ is an inner product on $V_r \times V_r$. Let $H_r = (V_r, (\cdot, \cdot)_r)$. We say that H_r is an r^{th} left-definite space associated with (H, A) if each of the following properties are satisfied:*

(1) *H_r is a Hilbert space;*
(2) *$\mathcal{D}(A^r)$ is a subspace of V_r;*
(3) *$\mathcal{D}(A^r)$ is dense in H_r;*
(4) *$(x, x)_r \geq k^r(x, x)$ for $x \in V_r$;*
(5) *$(x, y)_r = (A^r x, y)$ for $x \in \mathcal{D}(A^r)$ and $y \in V_r$.*

The following theorem clarifies the existence, and uniqueness, of each H_r. The proof of this theorem can be found in [6] in Theorems 3.1, 3.4, and Theorem 3.7.

Theorem 1. *Suppose A is a self-adjoint operator in the separable Hilbert space $H = (V, (\cdot, \cdot))$, which is bounded below by kI for some $k > 0$. Let $r > 0$, and define*

$$\begin{aligned} V_r &:= \mathcal{D}(A^{r/2}) \\ (x, y)_r &:= (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r) \\ H_r &:= (V_r, (\cdot, \cdot)_r). \end{aligned}$$

Then, H_r is the unique r^{th} left-definite space associated with (H, A). Moreover, we have the following:

(i) *If A is bounded, then $V = V_r$ and the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are equivalent for all $r > 0$;*
(ii) *If A is unbounded, then V_r is a proper subspace of V, and for $0 < r < s$, V_s is a proper subspace of V_r; moreover, none of the inner products $(\cdot, \cdot)$, $(\cdot, \cdot)_r$, or $(\cdot, \cdot)_s$ are equivalent;*
(iii) *If $\{\phi_n\}$ is a (complete) set of orthogonal eigenfunctions of A in H, then they are also a (complete) orthogonal set in each left-definite space H_r.*

From Theorem 1, we see that $H^1[-1, 1]$ is the *form domain* of $\mathcal{K}_c$; that is to say, $H^1[-1, 1] = \mathcal{D}(\mathcal{K}_c^{1/2})$.

4. The Sobolev Space $H^1[-1,1]$ and the Krein–Sobolev Orthogonal Polynomials

From Theorem 1(iii), the eigenfunctions of the shifted Krein Laplacian operator $\mathcal{K}_c$, given in (8), form a complete orthogonal set in $(H^1[-1,1],(\cdot,\cdot)_{1,c})$. The main purpose of the Littlejohn–Quintero paper [1] was to construct a *different* complete orthogonal set in $H^1[-1,1]$, namely, the Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$. We remark, however, that the polynomial $K_n(x)$ is *not* an eigenfunction of $\mathcal{K}_c$ for any $c \geq 0$ when $n \geq 2$.

The construction of the Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$ is similar to Althammer's construction of the polynomials $\{A_n\}_{n=0}^{\infty}$, which we discuss in Section 5. Both polynomial sets $\{A_n\}_{n=0}^{\infty}$ and $\{K_n\}_{n=0}^{\infty}$ are even or odd polynomials, and both are normalized so that $A_n(1) = K_n(1) = 1$. Furthermore, as we will see in Section 5, when n is even and $\lambda = 1/c$ (see (5)), $A_n(x) = K_n(x)$. However, the Althammer and Krein–Sobolev polynomials of odd degrees > 3 are different.

For $n \in \mathbb{N}_0$, define

$$S_n(x) := P_n(x) - P_{n-2}(x) \quad (P_{-2}(x) = P_{-1}(x) = 0), \tag{15}$$

where $\{P_n\}_{n=0}^{\infty}$ is the sequence of classical Legendre polynomials normalized by $P_n(1) = 1$ for each $n \in \mathbb{N}_0$. For various properties of the Legendre polynomials, and orthogonal polynomials in general, see the classic text [19] , Chapter IV, of Szegö.

Calculations show that $S_0(x) = 1$, $S_1(x) = x$, $S_2(x) = \frac{3}{2}(x^2-1)$, etc.; moreover, $S_n(1) = 0$ for $n \geq 2$.

In [1], the following technical result was established for $\{S_n\}_{n=0}^{\infty}$.

Proposition 1. *The following properties for the polynomials $\{S_n\}_{n=0}^{\infty}$ hold:*

(i) *For $n \geq 2$, $S_n(1) = 0$;*

(ii)

$$S_n(-x) = (-1)^n S_n(x);$$

(iii)

$$S_n(1) - S_n(-1) = \begin{cases} 2 & \textit{if } n = 1 \\ 0 & \textit{if } n \neq 1 \end{cases};$$

(iv)

$$(S_n, S_n)_{1,c} = \begin{cases} 2c & \textit{if } n = 0 \\ \frac{2}{3}c & \textit{if } n = 1 \\ 2c\left(\dfrac{1}{2n+1} + \dfrac{1}{2n-3}\right) + 2(2n-1) & \textit{if } n > 1; \end{cases}$$

(v)

$$(S_n, S_m)_{1,c} = 0 \textit{ for } |n-m| \neq 0, 2;$$

(vi)

$$(S_n, S_{n-2})_{1,c} = -\frac{2c}{2n-3}. \tag{16}$$

We are now in position to define the polynomials $\{K_n\}_{n=0}^{\infty}$ from $\{S_n\}_{n=0}^{\infty}$; as shown in [1], the sequence $\{K_n\}_{n=0}^{\infty}$ forms a complete orthogonal set (see Theorem 3) with respect to the inner product $(\cdot,\cdot)_{1,c}$.

Definition 2. *For $n \in \mathbb{N}_0$, define*

$$K_n(x) := \sum_{r=0}^{[n/2]} a_{n-2r} S_{n-2r}(x), \tag{17}$$

where $K_{-2}(x) = K_{-1}(x) = 0$, $a_0 = a_1 = 1$, *and*

$$a_n := -\frac{(K_{n-2}, S_{n-2})_{1,c}}{(S_n, S_{n-2})_{1,c}} \quad (n \geq 2). \tag{18}$$

Note that, by (16), each a_n is well defined when $c > 0$. Moreover, calculations show that $K_0(x) = S_0(x) = 1$, and $K_1(x) = S_1(x) = x$. We recall from Section 2 that both K_0 and K_1 are eigenfunctions of the shifted Krein Laplacian operator $\mathcal{K}_c$ (corresponding to the eigenvalue $\lambda = c$). However, $\mathcal{K}_c$ has no other polynomial eigenfunctions.

In [1], the authors proved the following results.

Theorem 2. *The Krein–Sobolev polynomials* $\{K_n\}_{n=0}^{\infty}$ *and the sequence* $\{a_n\}_{n=0}^{\infty}$ *have the following properties:*

(i) $K_n(-x) = (-1)^n K_n(x)$;
(ii) $K_n = a_n S_n + K_{n-2}$;
(iii) *The sequence* $\{a_n\}_{n=0}^{\infty}$ *satisfies* $a_0 = a_1 = a_2 = a_3 = 1$ *and, for* $n \geq 2$,

$$a_{n+2} = a_n\left(1 + \frac{4n^2-1}{c}\right) + \frac{2n+1}{2n-3}(a_n - a_{n-2}); \tag{19}$$

(iv) $a_{n+2} \geq a_n$; *moreover,* $a_n \geq 1$ *for each* $n \in \mathbb{N}_0$;
(v) *For* $n \geq 2$,

$$(a_n S_n + K_{n-2}, S_{n-2})_{1,c} = 0;$$

(vi) K_n *is a polynomial of degree exactly* n, *and* $K_n(1) = 1$.

Theorem 3. *The Krein–Sobolev polynomials* $\{K_n\}_{n=0}^{\infty}$, *defined in (17), form a complete orthogonal set of polynomials in the first left-definite space* $(H^1[-1,1], (\cdot,\cdot)_{1,c})$, *defined in (1) and (2), associated with the shifted Krein Laplacian operator* $\mathcal{K}_c$ *in* $L^2(-1,1)$. *Moreover,*

$$(K_m, K_n)_{1,c} = \frac{2c}{2n+1} a_n a_{n+2} \delta_{n,m} \quad (n, m \in \mathbb{N}_0)$$

where $\delta_{n,m}$ *is the Kronecker delta function.*

Theorem 4. *The roots of the Krein–Sobolev orthogonal polynomials* $\{K_n\}_{n=1}^{\infty}$ *are real, simple, and lie in the interval* $(-1,1)$.

5. An Explicit Representation of the Krein–Sobolev Polynomials

As shown by Schäfke [3], the Althammer polynomials $\{A_n\}_{n=0}^{\infty}$ satisfy the following properties:

(i) $A_0(x) = 1$, $A_1(x) = x$ and, for $n \geq 2$,

$$A_n(x) = \sum_{k=0}^{[n/2]} a'_{n-2k} S_{n-2k},$$

where S_n is defined in (15), and the sequence $\{a'_n\}_{n=0}^{\infty}$ satisfies the recurrence relation $a'_0 = a'_1 = 1$ and

$$a'_{n+2} = a'_n(1 + (4n^2-1)\lambda) + \left(\frac{2n+1}{2n-3}\right)(a'_n - a'_{n-2}) \quad (n \geq 2). \tag{20}$$

Furthermore, each $a'_n > 0$ and is explicitly given by

$$a'_n = \sum_{k=0}^{[(n-1)/2]} \left(\frac{\lambda}{4}\right)^k \frac{1}{(2k)!} \frac{(n+2k-1)!}{(n-2k-1)!} \quad (n \in \mathbb{N}); \tag{21}$$

(ii) For each $n \in \mathbb{N}_0$,

$$A_n(-x) = (-1)^n A_n(x);$$

(iii) For each $n, m \in \mathbb{N}_0$, $\{A_n\}_{n=0}^{\infty}$ satisfy the orthogonality relationship

$$\langle A_n, A_m \rangle_\lambda = \frac{2}{2n+1} a'_n a'_{n+2} \delta_{n,m},$$

where $\langle \cdot, \cdot \rangle_\lambda$ is defined in (5).

From (20), when $\lambda = 1/c$ (see (6)), we see that the Althammer coefficients $\{a'_n\}_{n=2}^{\infty}$ satisfy the *exact* same recurrence relation as the Krein–Sobolev coefficients $\{a_n\}_{n=2}^{\infty}$ in (19). Moreover, since

$$a'_0 = a'_2 = a_0 = a_2,$$

it follows that

$$a'_{2n} = a_{2n} \quad (n \in \mathbb{N}_0),$$

and, consequently, when $\lambda = 1/c$, we see that

$$K_{2n}(x) = A_{2n}(x).$$

However, since $a'_3 = 1 + 1/c \neq a_3 = 1$, the odd connection coefficients $\{a_{2n-1}\}_{n=1}^{\infty}$ for the Krein–Sobolev polynomials are different from the Althammer coefficients $\{a'_{2n-1}\}_{n=1}^{\infty}$ given explicitly in (21). In fact, the unique solution to (19) when n is odd, that is to say,

$$a_{2n+1} = \left(\frac{8n-6}{4n-5} + \frac{16n^2-16n+3}{c}\right) a_{2n-1} - \left(\frac{4n-1}{4n-5}\right) a_{2n-3}.$$

with the initial conditions $a_1 = a_3 = 1$, is given explicitly by

$$a_{2n+1} = \sum_{k=0}^{n-1} \left(\frac{\lambda}{4}\right)^k \frac{(n-k)}{(n+k)} \frac{(2n+2k+1)!}{(2n-2k+1)!(2k)!} \quad (n \in \mathbb{N}_0). \tag{22}$$

An algebraic proof (22) that satisfies this initial value difference problem is tedious to calculate. Remarkably, we note that Wolfram's Mathematica *instantly* verifies the solution (22) with the command FullSimplify.

We summarize our main result in the following theorem.

Theorem 5. *With the polynomials $\{S_n\}_{n=0}^{\infty}$ defined in (15) we have the following formula for each Krein-Sobolev polynomial of degree $n \in \mathbb{N}$:*

$$K_n(x) = \sum_{k=0}^{n} a_{2n-2k} S_{2n-2k}(x), \tag{23}$$

where

$$a_0 = a_2 = 1 \text{ and } a_{2m} = \sum_{k=0}^{m-1} \left(\frac{\lambda}{4}\right)^k \frac{(2m+2k-1)!}{(2m-2k-1)!(2k)!}$$

and

$$K_{2n+1}(x) = \sum_{k=0}^{n} a_{2n-2k+1} S_{2n-2k+1}(x), \tag{24}$$

where

$$a_1 = a_3 = 1 \text{ and } a_{2m+1} = \sum_{k=0}^{m-1} \left(\frac{\lambda}{4}\right)^k \binom{m-k}{m+k} \frac{(2m+2k+1)!}{(2m-2k+1)!(2k)!}.$$

As discussed in Theorem 3 and Theorem 4, these polynomials $\{K_n\}_{n=0}^{\infty}$ form a complete orthogonal set in the Sobolev space $(H^1[-1,1], (\cdot,\cdot)_{1,c})$ and, for $n \in \mathbb{N}$, $K_n(x)$ has exactly n distinct real roots in $(-1,1)$.

We list the first few Krein–Sobolev polynomials $K_n(x)$:

$$\begin{aligned}
K_0(x) &= 1 \\
K_1(x) &= x \\
K_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\
K_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \\
K_4(x) &= \frac{(35c+525)}{8c}x^4 - \frac{(30c+630)}{8c}x^2 + \frac{(3c+105)}{8c} \\
K_5(x) &= \frac{(126c+3969)}{16c}x^5 - \frac{(140c+5670)}{16c}x^3 + \frac{(30c+1701)}{16c}x \\
K_6(x) &= \frac{(231c^2+24255c+218295)}{16c^2}x^6 - \frac{(315c^2+39375c+363825)}{16c^2}x^4 \\
&\quad + \frac{(105c^2+16065c+155925)}{16c^2}x^2 - \frac{(-5c^2+945c+10395)}{16c^2}.
\end{aligned}$$

6. Conclusions

The first left-definite space $(H^1[-1,1], (\cdot,\cdot)_{1,c})$, defined in (1) and (2), for the self-adjoint shifted Krein operator $\mathcal{K}_c$ contains polynomials as a dense subspace. Furthermore, the Krein–Sobolev polynomials $\{K_n\}_{n=0}^{\infty}$ form a (complete) orthogonal set in $(H_1, (\cdot,\cdot)_{1,c})$ and, for $n \geq 1$, $K_n(x)$ has n simple roots in $(-1,1)$. These results were established by Littlejohn and Quintero in the earlier contribution [1]. In this present paper, the main objective was to present an explicit form of these Krein–Sobolev polynomials; this explicit form is given in (23) and (24).

A natural question to ask regards the set of polynomials that are dense in the *second* left-definite space $(H_2[-1,1], (\cdot,\cdot)_{2,c})$ given explicitly by

$$\begin{aligned}
H_2[-1,1] = \{f : [-1,1] \to \mathbb{C} \mid f, f' \in AC[-1,1]; f'' \in L^2(-1,1); \\
f'(-1) = f'(1) = \tfrac{f(1)-f(-1)}{2}\}
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
(f,g)_{2,c} = &-c(f(1)-f(-1))(\overline{g}(1)-\overline{g}(-1)) \\
&+ \int_{-1}^{1} \Big(f''(x)\overline{g}''(x) + 2cf'(x)\overline{g}'(x) + c^2 f(x)\overline{g}(x)\Big)dx?
\end{aligned}$$

We do not know the answer to this question. A straightforward calculation shows that a basis for polynomials in $H_2[-1,1]$ is given by

$$\begin{aligned}
p_{2n}(x) &= x^{2n} - n/(n-1)x^{2n-2} \\
p_{2n+1} &= x^{2n+1} - n/(n-1)x^{2n-1}
\end{aligned}$$

for $n \in \mathbb{N}_0, n \neq 1$. Consequently, there is no polynomial of degree 2 or 3 belonging to $H_2[-1,1]$. Even though this basis is not algebraically complete, it is not clear if they are analytically complete in $H_2[-1,1]$. Note that the theory of exceptional orthogonal polynomials provides several examples of algebraically incomplete sequences of polynomials that are analytically complete. This problem suggests a larger open problem with regard to finding conditions for the density of polynomials in Hilbert spaces constrained by boundary conditions (as is the case in (25)).

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