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Inverse Problems of Recovering Lower-Order Coefficients from Boundary Integral Data

Sergey Pyatkov [1,2,*] and Oleg Soldatov [1]

1 Engineering School of Digital Technologies, Yugra State University, Chekhov St. 16, 628011 Khanty-Mansiysk, Russia; oleg.soldatov.97@bk.ru
2 Academy of Sciences of the Republic of Sakha (Yakutia), 33 Lenin Ave., 677007 Yakutsk, Russia
* Correspondence: s_pyatkov@ugrasu.ru

Abstract: We study inverse problems of identification of lower-order coefficients in a second-order parabolic equation. The coefficients are sought in the form of a finite series segment with unknown coefficients, depending on time. The linear case is also considered. Overdetermination conditions are the integrals over the boundary of a solution's domain with weights. We focus on existence and uniqueness theorems and stability estimates for solutions to these inverse problems. An operator equation to which the problem is reduced is studied with the use of the contraction mapping principle. A solution belongs to some Sobolev space and has all generalized derivatives occurring into the equation summable to some power. The method of the proof is constructive, and it can be used for developing new numerical algorithms for solving the problem.

Keywords: inverse problem; parabolic equation; convection–diffusion; heat and mass transfer; integral measurements

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1. Introduction

We study the question on the identification of lower-order coefficients and the right-hand side in a parabolic equation. The equation is of the form

$$Mu = u_t + Au = f(t,x), \ \ (t,x) \in Q = (0,T) \times G, \tag{1}$$

where $G \in \mathbb{R}^n$ is a bounded domain with boundary Γ. Let $S = (0,T) \times \Gamma$. The function f and the elliptic operator A are representable as follows:

$$A(t,x,D) = A_0(t,x,D_x) + \textstyle\sum_{i=1}^{r} q_i(t) A_i(t,x,D_x), \ f = f_0(t,x) + \sum_{i=r+1}^{s} f_i(t,x) q_i(t),$$
$$A_0 = -\textstyle\sum_{k,l=1}^{n} a_{kl}(t,x)\partial_{x_k x_l} + \sum_{k=1}^{n} a_k(t,x)\partial_{x_k} + a_0(t,x),$$
$$A_i = \textstyle\sum_{k=1}^{n} a_k^i(t,x)\partial_{x_k} + a_0^i(t,x), \ i = 1,2,\ldots,r.$$

Equation (1) is furnished with the initial and boundary conditions

$$u|_{t=0} = u_0, \ \ Bu|_S = \frac{\partial u}{\partial \gamma} + \sigma(t,x)u = g(t,x), \ \frac{\partial u}{\partial \gamma} = \sum_{i=1}^{n} \gamma_i(t,x) u_{x_i}. \tag{2}$$

The additional conditions for recovering the coefficients are as follows:

$$\int_\Gamma u(t,x)\varphi_j(x) = \psi_j(t), \ \ j = 1,2,\ldots,s. \tag{3}$$

The unknowns in the problem, (1)–(3), are a solution u and the functions $q_i(t)$ ($i = 1, 2, \ldots, s$).

Problems (1)–(3) arise when describing heat and mass transfer processes, diffusion and filtration processes, in ecology and many other fields [1,2]. The overdetermination conditions (3) are not standard. We mention the monograph [3] dealing with inverse parabolic problems and the monographs [4–6] containing basic statements of inverse problems. We can also refer to the monograph [4] (see also [7]), where inverse coefficient parabolic problems are studied, and the coefficients in question are independent of some part of spatial variables. Here, a large number of existence and uniqueness theorems is obtained. Due to the method, the coefficients in the equations are independent of some spatial variables. Overdetermination conditions are values of a solution on some hyperplanes. More detailed results are presented in [8–10], where well-posedness of inverse problems of the identification of coefficients from the values of a solutions on some manifolds are investigated. Pointwise or integral data of the form (3) (see the article [11], where the problem of identification of the heat transfer coefficient is considered) were studied in the articles authored by Prilepko A.I., and a series of interesting problems was described in [3]. Analogous results were obtained in [12–15], where other conditions on the data and other spaces were employed. Additional integral conditions with integrals over Domain G are considered in [16–18]. They are used to determine either the heat transfer coefficient or the flux on the boundary. These conditions are also involved in the articles [11,19,20] devoted to numerical methods for solving the problem. The problem of simultaneous identification of the heat transfer coefficient and a lower-order coefficient depending on an integral of a solution with weight is studied in [21]. The integral conditions (3) are often met as some approximations of pointwise conditions. This is noted, in particular, in [19,20]. The integral condition (3) is often used in some articles devoted to model problems and the numerical solving of these particular problems (see [22–24]). The main approaches to constructing numerical algorithms are discussed in [25,26]). The problem of recovering the heat transfer coefficient with the overdetermination condition (3) is considered in [27]. Note that we do not know articles devoted to the theoretical results for Problems (1)–(3), except for our article [28]. Here, attention is paid to the case of recovering coefficients of higher-order derivatives, which are sought in the class of continuous functions. Existence and uniqueness theorems are obtained. However, if we deal with real problems of heat and mass transfer, then it is more interesting to consider the case of non-smooth coefficients. By the same approach, we obtain a similar result for lower-order coefficients, which belong to some Lebesgue space. Note that the operator equations to which these two problems are reduced do not coincide.

We expose existence and uniqueness theorems for Problems (1)–(3) and some remarks on the stability of solutions. The method of the proof relies on the fixed point theorems and a priori estimates. The method is constructive and allows to develop quickly converging numerical methods based on an iteration procedure, finite element method, and the method of finite differences, an approach that allows to diminish the number of calculations in contrast to gradient methods. In the proof, we employ the known results on solvability of boundary value problems for parabolic equations, embedding theorems, and interpolation inequalities.

The rest of this paper is organized as follows. In Section 2, we expose some definitions and notations, and we briefly recall some results from the parabolic theory, which are the starting point of our study. Section 3 consists of four subsections. Section 3.1 provides additional conditions on the data and reducing the problem to an equivalent one. Section 3.2 deals with the main result of this study, which is the existence and uniqueness of solutions in Sobolev spaces; Section 3.3 provides the stability estimate and the statement of the results in the linear case. In Section 3.4, an example is displayed, where some applied problems are described. Section 4 comprises a discussion, and concluding remarks are provided in Section 5.

2. Preliminaries

Let E be a Banach space. By $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$), we mean the space of measurable functions defined on G with values in E and the finite norm $\|\|u(x)\|_E\|_{L_p(G)}$ [29]. We use Sobolev spaces $W_p^s(G;E)$, $W_p^s(Q;E)$ (see [30,31]) and Hölder spaces $C^{\alpha,\beta}(\overline{Q})$, $C^{\alpha,\beta}(\overline{S})$ (see [32]). By a norm of a vector, we mean the sum of the norms of coordinates. Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^s(J;L_p(G)) \cap L_p(J;W_p^r(G))$. Respectively, $W_p^{s,r}(S) = W_p^s(J;L_p(\Gamma)) \cap L_p(J;W_p^r(\Gamma))$. All function spaces and coefficients of Equation (1) are assumed to be real. Next, we suppose that $p > n+2$ and $\Gamma \in C^3$ (see the definition in [32]). Given sets X, Y, the symbol $\rho(X,Y)$ stands for the distance between them. Introduce the notations $Q^\tau = (0,\tau) \times G$, $S^\tau = (0,\tau) \times \Gamma$. $G_\delta = \{x \in G : \rho(x,\Gamma) < \delta\}$, $Q_\delta = (0,T) \times G_\delta$, $Q_\delta^\tau = (0,\tau) \times G_\delta$. Construct a function $\varphi(x) \in C^\infty(\overline{G})$ such that $\varphi(x) = 1$ in $G_{\delta/2}$ and $\varphi(x) = 0$ in $G \setminus G_{3\delta/4}$. In what follows, we fix a parameter $\delta > 0$ (it can be arbitrary small). Consider the auxiliary problem

$$Mu = u_t + A_0 u = f_0(t,x), \;\; A_0 u = -\sum_{i,j=1}^{n} a_{ij} u_{x_i,x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u, \tag{4}$$

$$\frac{\partial u}{\partial \gamma} + \sigma u\Big|_S = g, \; u|_{t=0} = u_0, \tag{5}$$

We assume that

$$u_0 \in W_p^{2-2/p}(G), \; B(0,x,D)u_0|_\Gamma = g(0,x), g \in W_p^{s_0,2s_0}(S), \; f_0 \in L_p(Q), \tag{6}$$

where $s_0 = 1/2 - 1/2p$;

$$\varphi u_0(x) \in W_p^{3-2/p}(G), \; \varphi f_0 \in L_p(0,T;W_p^1(G)), \; g \in W_p^{s_1,2s_1}(S) \; (s_1 = 1 - 1/2p); \tag{7}$$

$$a_{ij} \in C(\overline{Q}), \; \gamma_i, \sigma \in W_p^{s_0,2s_0}(S), a_k \in L_p(Q), i,j = 1,2,\ldots,n, \; k \le n; \tag{8}$$

$$a_{ij} \in L_\infty(0,T;W_\infty^1(G_\delta)), \; a_k \in L_\infty(0,T;W_p^1(G_\delta)), \; \gamma_i, \sigma \in W_p^{s_1,2s_1}(S); \tag{9}$$

where $i,j = 1,\ldots,n,$ k = 0, 1, . . . , n, l = 1, 2, . . . , r. Next, we suppose that

$$|\sum_{i=1}^{n} \gamma_i \nu_i| \ge \varepsilon_0 > 0, \; \forall (t,x) \in S, \tag{10}$$

where ν is the exterior unit normal to Γ, and ε_0 is a positive constant. The operator A_0 is elliptic, i. e., there exists a constant $\delta_0 > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \ge \delta_0 |\xi|^2 \; \forall (t,x) \in Q, \; \forall \xi \in \mathbb{R}^n. \tag{11}$$

The next theorem follows on from Theorem 1 in [33].

Theorem 1. *Let Conditions* (6), (8), (10) *and* (11) *hold. Then, there exists a unique solution* $u \in W_p^{1,2}(Q)$ *to Problems* (4) *and* (5). *The following estimates is valid:*

$$\|u\|_{W_p^{1,2}(Q)} \le c\big[\|u_0\|_{W_p^{2-2/p}(G)} + \|f_0\|_{L_p(Q)} + \|g\|_{W_p^{s_0,2s_0}(S)}\big]. \tag{12}$$

If Conditions (7) and (9) also hold, then a solution is such that $\varphi u_t \in L_p(0,T;W_p^1(G))$, $\varphi u \in L_p(0,T;W_p^3(G))$. *In the case of* $g=0$ *and* $u_0=0$, *we have the estimates*

$$\|u\|_{W_p^{1,2}(Q^\tau)} \le c\|f\|_{L_p(Q^\tau)}, \tag{13}$$

$$\|u\|_{W_p^{1,2}(Q^\tau)} + \|\varphi u_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi u\|_{L_p(0,\tau;W_p^3(G))} \le c(\|f_0\|_{L_p(Q^\tau)} + \|\varphi f_0\|_{L_p(0,\tau;W_p^1(G))}), \tag{14}$$

where the constant c *is independent of* $f, \tau \in (0,T]$.

3. Existence and Uniqueness Theorems

3.1. Additional Conditions on the Data

The problem is to find a solution u to Equation (1) and the functions $(q_1, q_2, \dots, q_s)$ satisfying (1)–(3), and such that $u \in W_p^{1,2}(Q)$ and $q_j \in L_p(0,\tau_0)$, $j=1,2,\dots,s$. In this case, Equation (1) is satisfied almost everywhere in Q. We reduce the problem to an operator equation whose solvability is proven with the help of the fixed point theorem and a priori estimates. First, we describe additional conditions on the data. We assume that

$$\psi_i \in W_p^1(0,T),\ \psi_i(0) = \int_\Gamma u_0\varphi_i\,d\Gamma,\ \varphi_i \in L_q(\Gamma),\ \frac{1}{q}+\frac{1}{p}=1,\ i=1,\dots,s, \tag{15}$$

$$f_m \in L_\infty(0,T;L_p(G)) \cap L_\infty(0,T;W_p^1(G_\delta)) \cap C([0,T];L_p(\Gamma)),\ m=r+1,\dots,s, \tag{16}$$

$$a_k^l \in L_\infty(0,T;W_p^1(G_\delta)) \cap L_\infty(0,T;L_p(G)) \cap C([0,T];L_p(\Gamma)), \tag{17}$$

for all $l=1,\dots,r$, $k=0,1,\dots,n$. Assuming that the conditions of Theorem 1 and the above conditions are fulfilled, we can construct a solution to Problems (4) and (5). It possesses the properties $\Phi \in W_p^{1,2}(Q)$, $\varphi\Phi_t \in L_p(0,T;W_p^1(G))$, $\varphi\Phi \in L_p(0,T;W_p^3(G))$. Consider the matrix B_0 of dimension $s \times s$ with the rows

$$-<A_1(t,x,D)\Phi,\varphi_j>,\dots,-<A_r(t,x,D)\Phi,\varphi_j>,$$
$$<f_{r+1}(t,x),\varphi_j>,\dots,<f_s(t,x),\varphi_j>,\ j\le s,$$

where $<u,v>=\int_\Gamma u(x)v(x)\,d\Gamma$. The embedding theorems imply that $\Phi \in C^{1-(n+2)/2p,2-(n+2)/p}(\overline{Q})$ (see Lemma 3.3, Ch.2 [32]). In this case, the entries of the matrix B_0 are continuous functions. We require that

$$det\,B_0 \neq 0\ \forall t \in [0,T]. \tag{18}$$

Since, in Theorem 2 below, we prove existence of solutions locally in time, Condition (18) can be replaced with the following condition:

$$det\,B \neq 0,\ \forall t \in [0,T]. \tag{19}$$

where Matrix B has the rows

$$-<A_1(t,x,D)u_0,\varphi_j>,\dots,-<A_r(t,x,D)u_0,\varphi_j>,$$
$$<f_{r+1}(t,x),\varphi_j>,\dots,<f_s(t,x),\varphi_j>,\ j\le s.$$

This condition is easier to check in contrast to that in (18). If we replace Condition (18) with Condition (19) in Theorem 2, then the claim of Theorem 2 remains valid.

Make the change of variables $u = v + \Phi$. Problems (1)–(3) are reduced to an equivalent problem as follows:

$$v_t + A(\vec{q})v = \sum_{i=r+1}^{s} q_i f_i - \sum_{i=1}^{r} q_i A_i \Phi = f^1, \tag{20}$$

where $A(\vec{q})v = A_0 v + A^1(\vec{q})v$, $A^1(\vec{q}) = \sum_{i=1}^{r} q_i A_i$;

$$v|_{t=0} = 0,\ Bv|_S = 0; \tag{21}$$

$$< v, \varphi_j >= \psi_j(t) - < \Phi, \varphi_j >= \tilde{\psi}_j,\ \ i = 1, \dots, s. \tag{22}$$

We use the smoothness conditions on the data, (6)–(9) and (15)–(17), and the well-posedness conditions, (10), (11) and (18).

3.2. The Main Result

Theorem 2. *Let Conditions (6)–(11) and (15)–(18) hold. Then, there exists a number $\tau_0 \in (0, T]$ such that on segment $[0, \tau_0]$, there exists a unique solution $(u, q_1, q_2, \dots, q_s)$ to Problem (1)–(3), such that $u \in W_p^{1,2}(Q^{\tau_0})$, $\varphi u \in L_p(0, \tau_0; W_p^3(G))$, $\varphi u_t \in L_p(0, \tau_0; W_p^1(G))$, $q_j \in L_p(0, \tau_0)$, $j = 1, 2, \dots, s$.*

Proof. Reduction of the problem to the operator equation. We have reduced our problem, (1)–(3), to an equivalent simpler problem, (20)–(22). Denote by H_τ the space of functions v satisfying the Condition (21), such that $v \in W_p^{1,2}(Q^\tau)$, $\varphi v_t \in L_p(0, \tau; W_p^1(G))$, $\varphi v \in L_p(0, \tau; W_p^3(G))$. Let $\|v\|_{H_\tau} = \|v\|_{W_p^{1,2}(Q^\tau)} + \|\varphi v_t\|_{L_p(0,T;W_p^1(G))} + \|\varphi v\|_{L_p(0,T;W_p^3(G))}$. Moreover, define the space W_τ of functions $f \in L_p(Q^\tau)$ such that $\varphi f \in L_p(0, \tau; W_p^1(G))$. This space is endowed with the norm $\|f\|_{W_\tau} = \|f\|_{L_p(Q^\tau)} + \|\varphi f\|_{L_p(0,\tau;W_p^1(G))}$. By Theorem 1, given a function $f^1 \in W_\tau$, there exists a unique solution $v = (\partial_t - A_0)^{-1} f^1$ to the equation $v_t - A_0 v = f^1$, satisfying Condition (21) and the estimate

$$\|v\|_{H_\tau} \le c_1 \|f^1\|_{W_\tau}, \tag{23}$$

where the constant c_1 is independent of τ. Reduce our problem to an operator equation. Multiply Equation (20) by φ_j and integrate over Γ. Note that in the class of solutions u described in Theorem 2, traces of summands occurring in (20) on Γ exist. We obtain the equality

$$\tilde{\psi}_j' + < A(\vec{q})v, \varphi_j >= -\sum_{i=1}^{r} q_i < A_i \Phi, \varphi_j > + \sum_{i=r+1}^{s} q_i < f_i, \varphi_j > . \tag{24}$$

The right-hand side of this equality coincides with the j-th coordinate of the vector $B_0(t)\vec{q}$. In this case, System (24) can be written in the form

$$\vec{q}(t) = B_0^{-1} H(\vec{q})(t) = R(\vec{q}),\ H(\vec{q}) = (\tilde{\psi}_1' + < A(\vec{q})v, \varphi_1 >,$$
$$\tilde{\psi}_2' + < A(\vec{q})v, \varphi_2 >, \dots, \tilde{\psi}_s' + < A(\vec{q})v, \varphi_s >)^T, \tag{25}$$

where v is a solution to Problem (20), (21).

A priori estimates. Let $\vec{q} = 0$. In this case, the right-hand side in (25) is written as follows:

$$R(0) = B_0^{-1} \vec{\Psi},\ \vec{\Psi} = (\tilde{\psi}_1', \tilde{\psi}_2', \dots, \tilde{\psi}_s')^T.$$

Assign $R_0 = 2\|R(0)\|_{L_p(0,T)}$. Introduce that ball $B_\tau = \{\vec{q} \in L_p(0, \tau) : \ \|\vec{q}\|_{L_p(0,\tau)} \le R_0\}$. Next, our aim is to prove that there exists a solution to Equation (25). We employ the fixed point theorem. To this end, we first obtain estimates for solutions to Problems (20) and (21), assuming that $\vec{q} \in B_\tau$. We have that

$$v = -(\partial_t + A_0)^{-1} A^1(\vec{q})v + (\partial_t + A_0)^{-1} f^1. \tag{26}$$

From (23), it follows that

$$\|(\partial_t + A_0)^{-1} A^1(\vec{q})v\|_{H_\tau} \le c_1 \|A^1(\vec{q})v\|_{W_\tau}. \tag{27}$$

Our conditions on the coefficients ensure the estimate

$$\|A^1(\vec{q})v\|_{W_\tau} \le \|\sum_{j=1}^{r} q_j A_j(t,x,D)v\|_{W_\tau} \le \|\vec{q}\|_{L_p(0,\tau)} \sum_{j=1}^{r} (\|A_j(t,x,D)v\|_{L_\infty(0,\tau;L_p(G))} +$$
$$\|\varphi A_j(t,x,D)v\|_{L_\infty(0,\tau;W_p^1(G))}) \le c_3 \|\vec{q}\|_{L_p(0,\tau)} (\|v\|_{L_\infty(0,\tau;W_\infty^1(G))} + \|\varphi v\|_{L_\infty(0,\tau;W_\infty^2(G))}). \tag{28}$$

The constant c_3 depends on the norm of coefficients in $L_\infty(0,\tau;L_p(G)) \cap L_\infty(0,\tau;W_p^1(G_\delta))$. Note that $W_p^{1,2}(Q^\tau) \subset C([0,\tau];W_p^{2-2/p}(G))$. Moreover, if $\varphi v_t \in L_p(0,\tau;W_p^1(G))$, $\varphi v \in L_p(0,\tau;W_p^3(G))$, then $\varphi u \in C([0,\tau];W_p^{3-2/p}(G))$. These embeddings result from [31] [Theorem III 4.10.2]. Additionally, in both cases, the embedding constant is independent of τ. Next, we have the estimate (see Theorem 4.6.1 and 4.6.2 [29])

$$\|v\|_{L_\infty(0,\tau;W_\infty^1(G))} + \|\varphi v\|_{L_\infty(0,\tau;W_\infty^2(G))} \le c_4(\|v\|_{L_\infty(0,\tau;W_p^{1+s}(G))} + \|\varphi v\|_{L_\infty(0,\tau;W_\infty^{2+s}(G))}) \le$$
$$c_5(\|v\|^{\theta}_{L_\infty(0,\tau;W_p^{2-2/p}(G))} \|v\|^{1-\theta}_{L_\infty(0,\tau;L_p(G))} + \|\varphi v\|^{\theta_1}_{L_\infty(0,\tau;W_p^{3-2/p}(G))} \|v\|^{1-\theta_1}_{L_\infty(0,\tau;L_p(G))}), \tag{29}$$

where $n/p < s < 1-2/p$, $1+s = \theta(2-2/p)$, $2+s = \theta_1(3-2/p)$, and we have used the interpolation inequalities [29]. The Newton–Leibnitz formula yields

$$\|v\|_{L_\infty(0,\tau;L_p(G))} \le \tau^{(p-1)/p} \|v_t\|_{L_p(Q^\tau)}. \tag{30}$$

Estimates (28)–(30) ensure the inequality

$$\|A^1(\vec{q})v\|_{W_\tau} \le c_6 \|\vec{q}\|_{L_p(0,\tau)} \|v\|_{H_\tau} \tau^{\beta},\ \beta = \min(\beta_0, \beta_1), \tag{31}$$

where $\beta_0 = (1-\theta)(p-1)/p$, $\beta_1 = (1-\theta_1)(p-1)/p$. Choose τ_0 from the condition $c_1 c_6 R_0 \tau^{\beta} = 1/2$. In this case, Relation (26) yields

$$\|v\|_{H_\tau} \le 2\|(\partial_t + A_0)^{-1} f^1\|_{H_\tau}. \tag{32}$$

In view of (23), we can write out the estimate

$$\|v\|_{H_\tau} \le c_7 \|f^1\|_{W_\tau} \le c_8 \|\vec{q}\|_{L_p(0,\tau)},\ \tau \le \tau_0, \tag{33}$$

where we can assume that the constant c_8 is independent of $\tau \le \tau_0$. It depends on the norms of the data. Next, we consider two vectors $\vec{q}_i = (q_1^i, q_2^i, \ldots, q_s^i)$, $i = 1,2$, and we construct the corresponding solutions v_i to Problems (20) and (21). By subtracting two Equations (20), we conclude that the difference $w = v_1 - v_2$ is a solution to the problem

$$w_t + A((\vec{q}_1 + \vec{q}_2)/2)w + A^1(\vec{q}_1 - \vec{q}_2)(v_1 + v_2)/2 = \sum_{i=r+1}^{s} (q_1^i - q_2^i) f_i - \sum_{i=1}^{r} (q_1^i - q_2^i) A_i \Phi, \tag{34}$$

$$w|_{t=0} = 0,\ Bw|_S = 0. \tag{35}$$

As before, in view of Inequality (23), we have the estimate

$$\|w\|_{H_\tau} \le c_1\|A^1((\vec{q}_1+\vec{q}_2)/2)w\|_{W_\tau} + c_1\|A^1(\vec{q}_1-\vec{q}_2)\frac{(v_1+v_2)}{2}\|_{W_\tau} + c_{10}\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}. \tag{36}$$

Applying Estimate (31), we infer

$$\|w\|_{H_\tau} \le c_1c_6R_0\tau^\beta\|w\|_{H_\tau} + c_1c_6\tau^\beta\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}\|\frac{(v_1+v_2)}{2}\|_{H_\tau} + c_{10}\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}. \tag{37}$$

Since $\tau \le \tau_0$, the last estimate implies that

$$\|w\|_{H_\tau} \le 2c_1c_6\tau^\beta\|\vec{q}^1-\vec{q}_2\|_{L_p(0,\tau)}\|\frac{(v_1+v_2)}{2}\|_{H_\tau} + 2c_{10}\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}.$$

Next, involving Estimate (33), written for the functions v_i, we conclude that

$$\|v_1-v_2\|_{H_\tau} \le c_{11}\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}, \tag{38}$$

with a constant c_{11} independent of $\tau \le \tau_0$. Estimate the norm $\|R(\vec{q}_1)-R(\vec{q}_2)\|_{L_p(0,\tau)}$ with $\tau \le \tau_0$. Actually, the required estimate was already established. Consider the expressions $I_j =< (A(\vec{q}_1)v_1 - A(\vec{q}_2))v_2, \varphi_j >$ occurring in the difference $R(\vec{q}_1)-R(\vec{q}_2)$. The Hölder inequality yields

$$\begin{aligned}|I_j| = |< A_0w + A^1((\vec{q}_1+\vec{q}_2)/2)w + A^1(\vec{q}_1-\vec{q}_2)(v_1+v_2)/2, \varphi_j >| \le \\ c_{12}(\|A_0w\|_{L_p(\Gamma)} + \|A^1((\vec{q}_1+\vec{q}_2)/2)w\|_{L_p(\Gamma)} + \|A^1(\vec{q}_1-\vec{q}_2)(v_1+v_2)/2\|_{L_p(\Gamma)}.\end{aligned}$$

The second summand on the right-hand side of I_j is estimated as follows (see (31)):

$$\|A^1((\vec{q}_1+\vec{q}_2)/2)w\|_{L_p(0,\tau;L_p(\Gamma))} \le c_{12}\|A^1((\vec{q}_1+\vec{q}_2)/2)w\|_{W_\tau} \le c_{13}\tau^\beta R_0\|w\|_{H_\tau}. \tag{39}$$

In view of Inequality (33), written for the functions v_i, the third summand admits the estimate

$$\begin{aligned}\|A^1(\vec{q}_1-\vec{q}_2)(v_1+v_2)/2\|_{L_p(0,\tau;L_p(\Gamma))} \le c_{14}\|A^1(\vec{q}_1-\vec{q}_2)(v_1+v_2)/2\|_{W_\tau} \le \\ c_{13}\tau^\beta C_1(R_0)\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}.\end{aligned} \tag{40}$$

At last, the first summand is estimated as follows:

$$\|A_0w\|_{L_p(0,\tau;L_p(\Gamma))} \le c_{15}\|\varphi A_0w\|_{L_p(0,\tau;W_p^s(G))} \le c_{16}\|\varphi A_0w\|^s_{L_p(0,\tau;W_p^1(G))}\|\varphi A_0w\|^{1-s}_{L_p(Q^\tau)}, \tag{41}$$

where $n/p < s < 1$. Here, we have used the interpolation inequality [29]. Using the conditions on the coefficients, we obtain the estimate

$$\|\varphi A_0w\|_{L_p(G)} \le c_{16}(\|\varphi w\|_{W_p^2(G)} + \|w\|_{L_\infty(0,\tau;W_\infty^1(G)}). \tag{42}$$

Next, we repeat the arguments used in the proof of Estimate (31) (see Inequality (29)). Finally, we arrive at the estimate

$$\|\varphi A_0w\|_{L_p(G)} \le c_{17}\tau^{\beta_2}\|\varphi w\|_{H_\tau},\ \beta_2 > 0. \tag{43}$$

Estimates (39), (40), (43) and (38) ensure the following estimate:

$$\|R(\vec{q}_1)-R(\vec{q}_2)\|_{L_p(0,\tau)} \le c_{18}\tau^{\beta_3}\|\vec{q}_1-\vec{q}_2\|_{L_p(0,\tau)}, \tag{44}$$

where β_3 is a positive constant and c_{18} is a constant depending on R_0, but it is independent of τ.

Solvability of the problem. Choose a quantity $\tau_1 \leq \tau_0$ such that $c_{18}\tau^{\beta_3} \leq 1/2$ for $\tau \leq \tau_1$. In this case, the operator $R(\vec{q})$ takes the ball B_{τ_1} into itself and is a contraction. The fixed point theorem implies that Equation (25) has a unique solution in the ball B_{τ_1}. Let $v = v(\vec{q})$. Demonstrate that this functions satisfies (22). Integrating (20) with the weight φ_j over Γ, we conclude that

$$< v_t, \varphi_j > + < A(\vec{q})v, \varphi_j >= -\sum_{j=1}^{r} q_j < A_j\Phi, \varphi_j > + \sum_{j=r+1}^{s} < f_j, \varphi_j > q_j(t). \tag{45}$$

Subtracting them from (24), we conclude that $< v_t, \varphi_j > -\tilde{\psi}_j' = 0$ for all j; thereby, Condition (22) holds. The uniqueness of solutions follows from the estimates obtained in the above proof. □

3.3. Some Applications of the Results

Stability estimates for solutions also hold. Actually, by repeating the proof, we can validate the following statement:

Lemma 1. *Assume that Conditions (8)–(11), (16) and (17) hold and the data $(\tilde{u}_0, \tilde{f}_0, g, \tilde{\psi}_1, \ldots, \tilde{\psi}_s)$ satisfy the conditions (6), (7), (15) and (19). Given $\varepsilon > 0$, there exists a small parameter $\delta > 0$ such that if the data $(\tilde{u}_0, \tilde{f}_0, \tilde{g}, \tilde{\phi}_1, \ldots, \tilde{\psi}_s)$ satisfy Conditions (8)–(11), (16) and (17) and*

$$\|\tilde{u}_0 - u_0\|_{W_p^{2-2/p}(G)} + \|\varphi(\tilde{u}_0 - u_0)\|_{W_p^{3-2/p}(G)} + \|\tilde{f}_0 - f_0\|_{L_p(G)} +$$
$$\|\varphi(\tilde{f}_0 - f_0)\|_{L_p(0,T;W_p^1(G))} + \|\tilde{g} - g\|_{W_p^{s_0,2s_0}(S)} + \sum_{i=1}^{s} \|\tilde{\psi}_i - \psi_i\|_{W_p^1(0,T)} < \delta,$$

then a solution $\tilde{u}$ to Problems (1)–(3) with the above new data exists on some segment $(0, \tau_1)$ (the quantity τ_1 depends only on δ and $\tau_1 \to \tau_0$ as $\delta \to 0$) and

$$\|\tilde{u} - u\|_{W_p^{1,2}(Q^{\tilde{\tau}})} + \|\varphi(\tilde{u} - u)_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi(\tilde{u} - u)\|_{L_p(0,\tau;W_p^3(G))} \leq \varepsilon.$$

Remark 1. *A numerical solution to the problem can be constructed by method of successive approximations with the use of Equation (25), and it will be stable under random perturbations of the data.*

In the linear case, i.e., the unknown functions occur only into the right-hand side; thus, the claim becomes global in time. In this case, $A = A_0$ and the matrix B_0 has the rows

$$< f_1(t,x), \varphi_j >, \ldots, < f_s(t,x), \varphi_j >, \; j \leq s, \tag{46}$$

i.e., $r = 0$.

More exactly, the following theorem is valid.

Theorem 3. *Let Conditions (6)–(11), (15), (16) and (18) hold. Then, there exists a unique solution $(u, q_1, q_2, \ldots, q_s)$ to Problems (1)–(3), such that $u \in W_p^{1,2}(Q)$, $\varphi u \in L_p(0,T;W_p^3(G))$, $\varphi u_t \in L_p(0,T;W_p^1(G))$, $q_j \in L_p(0,T)$, $j = 1, 2, \ldots, s$.*

The proof is in line with that of the main result in [10]. The idea of the proof is as follows. First, we prove solvability of the problem on some small segment of time $[0, \tau_0]$. Next, by repeating the arguments, we establish solvability on $[\tau_0, \tau_1]$, and so on. Since the problem is linear, it is possible to prove that the length $\tau_i - \tau_{i-1}$ does not tend to zero, which implies solvability on the whole segment $[0, T]$.

3.4. Example

As an example, we write out a one-dimensional heat equation of the form

$$c(t,x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a(t,x)u_x) + b(t,x)u_x + k(t,x)u = f.$$

Here, Coefficient b is an advection/convection coefficient and k is a reaction or perfusion coefficient in bio-heat conduction (see [22]). In different models, Coefficient k also serves as the control function of the heat process [23] or the first-order rate constant for microbial CH_4 oxidation [24].

4. Discussion

Mathematical models of heat and mass transfer we considered. We studied inverse problems of recovering lower-order coefficients and the right-hand side in a second-order parabolic equation. The coefficients were representable in the form of finite series segments with unknown coefficients depending on time. This approach is new. The linear case in which only the right-hand side is recovered was also considered. The overdetermination conditions are the integrals over the boundary of a domain of a solution with weights. Attention was paid to existence, uniqueness, and stability estimates for solutions to inverse problems of this type. The problems were reduced to an operator equation, which was studied with the use of the fixed point theorem and a priori estimates. A solution had all generalized derivatives occurring into the equation summable to some power. The method of the proof is constructive and can be used for developing new numerical algorithms for solving the problem. Moreover, the approach is applicable to the study of inverse problems for higher-order parabolic equations, and the results can be transferred to this case without any modifications. The smoothness conditions on the data are sharp and cannot be weakened.

5. Conclusions

The existence and uniqueness theorems in inverse problems of recovering lower-order coefficients in a parabolic equation from boundary integral measurements were proven locally in time. They were sought in the form of a finite segment of the Fourier series with coefficients depending on time. The proof relies on a priori bounds and the fixed point theorem. The conditions on the data ensuring existence and uniqueness of solutions in Sobolev classes are sharp. They are smoothness and consistency conditions on the data and additional conditions on the kernels of the integral operators used in additional measurements.

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References

1. Alifanov, O.M.; Artyukhin, E.A.; Nenarokomov, A.V. *Inverse Problems in the Study of Complex Heat Transfer*; Janus-K: Moscow, Russia, 2009.
2. Ozisik, M.N.; Orlande, H.R.B. *Inverse Heat Transfer*; Taylor & Francis: New York, NY, USA, 2000.
3. Prilepko, A.I.; Orlovsky, D.G.; Vasin, I.A. *Methods for Solving Inverse Problems in Mathematical Physics*; Marcel Dekker: New York, NY, USA, 1999.
4. Belov, Y.Y. *Inverse Problems for Parabolic Equations*; VSP: Utrecht, The Netherlands, 2002.
5. Isakov, V. *Inverse Problems for Partial Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2006.
6. Kabanikhin, S.I. *Inverse and Ill-Posed Problems*; Walter de Gruyter: Berlin/Heidelberg, Germany; Boston, MA, USA, 2012.
7. Frolenkov, I.V.; Romanenko, G.V. On Solving an Inverse Problem for a Multidimensional Parabolic Equation. *Sib. J. Ind. Math.* **2012**, *15*, 139–146.
8. Pyatkov, S.G.; Samkov, M.L. On Some Classes of Coefficient Inverse Problems for Parabolic Systems of Equations. *Sib. Adv. Math.* **2012**, *22*, 287–302.
9. Pyatkov, S.G.; Tsybikov, B.N. On Some Classes of Inverse Problems for Parabolic and Elliptic Equations. *J. Evol. Eq.* **2011**, *11*, 155–186.
10. Pyatkov, S.G. On Some Classes of Inverse Problems for Parabolic Equations. *J. Inv. Ill-Posed Probl.* **2011**, *18*, 917–934.
11. Kostin, A.B.; Prilepko, A.I. On Some Problems of the Reconstruction of a Boundary Condition for a Parabolic Equation, II. *Differ. Eq.* **1996**, *32*, 1515–1525.
12. Pyatkov, S.G. On Some Classes of Inverse Problems with Overdetermination Data on Spatial Manifolds. *Sib. Math. J.* **2016**, *57*, 870–880.
13. Pyatkov, S.G.; Rotko, V.V. On some parabolic inverse problems with the pointwise overdetermination. *Siber. Adv. Math.* **2020**, *30*, 124–142.
14. Pyatkov, S.G. Identification of Thermophysical Parameters in Mathematical Models of Heat and Mass Transfer. *J. Comput. Eng. Math.* **2022**, *9*, 52–66.
15. Pyatkov, S.G.; Rotko, V.V. Inverse Problems with Pointwise Overdetermination for some Quasilinear Parabolic Systems. *AIP Conf. Proc.* **2017**, *1907*, 020008.
16. Pyatkov, S.; Soldatov, O.; Fayazov, K. Inverse problems of recovering the heat transfer coefficient with integral data, *J. Math. Sci.* **2023**, *274*, 255–268.
17. Kozhanov, A.I. Linear Inverse Problems for Some Classes of Nonlinear Nonstationary Equations. *Sib. Electron. Rep.* **2015**, *12*, 264–275.
18. Verzhbitskiy, M.A.; Pyatkov, S.G. On Some Inverse Problems of Recovering Boundary Regimes. *Math. Notes NEFU* **2016**, *23*, 3–18.
19. Dihn, N.; Hao, D.N.; Thanh, P.X.; Lesnik, D. Determination of the Heat Transfer Coefficients in Transient Heat Conduction. *Inverse Probl.* **2013**, *29*, 095020.
20. Hao, D.N.; Huong, B.V.; Thanh, P.X.; Lesnik, D. Identification of Nonlinear Heat Transfer Laws from Boundary Observations. *Appl. Anal.* **2014**, *94*, 1784–1799.
21. Lorenzi, A.; Messina, F.; Identifying a Spherically Symmetric Conductivity in a Nonlinear Parabolic Equation. *Appl. Anal.* **2006**, **85**, 867–889.
22. Hussein, M.S.; Lesnik, D. Simultaneous Determination of Time-Dependent Coefficients and Heat Source. *Intern. J. Comput. Methods Eng. Sci. Mech.* **2016**, *17*, 401–411.
23. Huang, D.; Li, Y.; Pei, D. Identification of a Time-Dependent Coefficient in Heat Conduction Problem by New Iteration Method. *Adv. Math. Phys.* **2018**, *2018*, 4918256.
24. Murguia-Flores, F.; Arndt, S.; Ganesan, A.L.; Murray-Tortarolo, G.; Hornibrook, E.R.C. Soil Methanotrophy Model (MeMo v1.0): A Process-Based Model to Quantify Global Uptake of Atmospheric Methane by Soil. *Geosci. Model Dev.* **2018**, *11*, 2009–2032.
25. Samarskii, A.A.; Vabishchevich P.N. *Numerical Methods for Solving Inverse Problems of Mathematical Physics*; Walter de Gruyter GmbH & Co. KG: Berlin/Heidelberg, Germany; Boston, MA, USA, 2007.
26. Huntul, M.J. Identification of the Timewise Thermal Conductivity in a 2D Heat Equation from Local Heat Flux Conditions. *Inverse Probl. Sci. Eng.* **2021**, it 29, 903–919.
27. Pyatkov, S.G.; Soldatov, O.A.; Identification of the Heat Transfer Coefficient from Boundary Integral Data. *Sib. Math. J.* **2024**, *65*, 824–839.
28. Pyatkov, S.; Pronkina, T. Coefficient Inverse Problems of Identification of Thermophysical Parameters from Boundary Integral Data. *J. Math. Sci.* **2024**, *282*, 241–252.
29. Triebel, H. *Interpolation Theory. Function Spaces. Differential Operators*; VEB Deutscher Verlag der Wissenschaften: Berlin/Heidelberg, Germany, 1978.
30. Amann, H. Compact embeddings of vector-valued sobolev and besov spaces. *Glas. Mat.* **2000**, *35*, 161–177.
31. Amann, H. *Linear and Quasilinear Parabolic Problems*; Birkhauser Verlag: Basel, Switzerland, 1995.

32. Ladyzhenskaya, O.A.; Solonnikov, V.A.; Uraltseva, N.N. *Linear and Quasilinear Equations of Parabolic Type*; American Mathematical Society: Providence, RI, USA, 1968.
33. Baranchuk, V.A.; Pyatkov, S.G. On some classes of inverse problems with pointwise ovedetermination for mathematical models of heat and mass transfer. *Bull. Yugra State Univ.* **2020**, *3*, 38–48.

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