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New Fixed-Point Results in Controlled Metric Type Spaces with Applications

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Abstract: In this manuscript, we present several novel results in fixed-point theory for a complete controlled metric space. The first presented result is inspired from the Caristi contraction where we explore the existence and uniqueness of fixed points under specific conditions. Furthermore, we propose a graphical form of it by endowing the considered space with a graph and develop a new fixed-point theorem, which is illustrated by two examples. Also, we establish a theorem for the α-admissible mapping. To demonstrate its effectiveness, the last theorem proposes an approach to solve a second-order differential equation.

Keywords: fixed point; controlled metric space; graph theory; differential equation

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1. Introduction

The Banach contraction principle is regarded as one of the most renowned results in fixed-point (FP) theory. It focused on the uniqueness of the FP of a self-mapping φ on a complete metric space (Ω, ϱ) [1]. This basic result motivated researchers to develop several FP theorems in different metric spaces [2–7].

FP theory, with its broad range of applications, has driven researchers to develop innovative results and explore its use across diverse mathematical disciplines. A prominent application lies in the solving of differential equations, where fixed-point theorems play a crucial role in establishing the existence and uniqueness of solutions, particularly for non-linear problems [8,9]. These results are instrumental in transforming complex differential equations into solvable integral equations, providing a framework for both analytical and numerical approaches.

On the other hand, the exploration of the interplay between fixed-point theory and graph theory has garnered significant attention, leading to the development of numerous research contributions. Many researchers have investigated the integration of graph-based constraints into metric spaces, yielding novel fixed-point results. This emerging field has not only enriched the theoretical framework of fixed-point theory but also expanded its applicability to a wide range of disciplines, including network theory, optimization, and dynamical systems. We will provide a brief historical overview of the development of the concept of metric spaces endowed with a graph structure.

It is essential to note that early research was focused on equipping metric spaces with a partial ordering. The first significant result in this area was presented by Ran and

Reurings [10]. They presented an analogue of Banach's FP theorem in partially ordered sets and several applications. Their results were extended by Petrusel and Rus in [11] by introducing FP results in ordered L-spaces. Next, in [12], Jachymski established the foundational framework for metric spaces endowed with graph structures. This pioneering approach created new possibilities for investigating the interactions between distance functions and relational structures, leading to significant advancements in FP theory within these spaces [13–18].

One of the most important FP results was introduced by Caristi in [19]. He established an FP theorem for a mapping φ satisfying the following condition

$$\varrho(\tau, \varphi\tau) \leq \phi(\tau) - \phi(\varphi\tau),$$

where ϕ is a nonnegative real function which is lower semi-continuous. The mapping φ is called a Caristi map on (Ω, ϱ). Next, Caristi's result was considered and developed to obtain more general results. Recently, Karapinar et al. [20] proposed a new FP theorem that combined both Banach and Caristi type theorems in a b-metric space. The concept of the b-metric is based on the generalization of the triangle inequality of the standard metric by inserting a constant coefficient $b \geq 1$ to the right-hand side of the triangle inequality. More studies of the b-metric can be found in [21–25]. One of the recent extensions of the b-metric was introduced by Mlaiki et al. [26] and is called controlled metric type space (CMS for short). They inserted a controlled function of the right-hand side of the b-triangle inequality. Subsequently, numerous studies have investigated this new space, demonstrating fixed-point results for various contraction mappings under a range of conditions [27,28].

In this paper, we introduce a new FP theorem inspired by the Caristi contraction in the CMS. Inspired by the exploration of the interplay between FP theory and graph theory, we endow the considered metric space with a graph and we propose a graphical form of the first theorem established. We illustrate the obtained result by two examples. Additionally, we present a theorem for α-admissible mappings. Since FP theory provides a robust mathematical framework for tackling a wide range of problems in differential equations and integral equations, our last result offers an approach to solve a second-order differential equation using FP theory to highlight its practical utility.

Let us revisit the definition and essential topological properties of the CMS.

Definition 1 ([26])**.** *Let $\Omega \neq \emptyset$, $\varpi : \Omega \times \Omega \to [1, \infty)$, and a function $\varrho : \Omega \times \Omega \to [0, \infty)$ satisfying the following hypothesis for all $s_1, s_2, s_3 \in \Omega$:*

(d1) $\varrho(s_1, s_2) = 0$ *if and only if* $s_1 = s_2$;

(d2) $\varrho(s_1, s_2) = \varrho(s_2, s_1)$;

(d3) $\varrho(s_1, s_2) \leq \varpi(s_1, s_3)\varrho(s_1, s_3) + \varpi(s_3, s_2)\varrho(s_3, s_2)$.

The triplet $(\Omega, \varrho, \varpi)$ is called a controlled metric type space, and the function ϱ is called a controlled metric.

It is clear that the CMS is an extension for b-metric spaces. Every b-metric space is a CMS, though the reverse may not hold.

Notation 1. *In the rest of this paper, we adopt the following notations:*
$\mathbb{R}$ *represents the set of real numbers.*
$\mathbb{N}$ *represents the set of natural numbers.*

Definition 2 ([26])**.** *Let $(\Omega, \varrho, \varpi)$ be a CMS and $\{\tau_n\}_{n\geq 0}$ be a sequence in Ω.*

1. *The sequence $\{\tau_n\}$ converges to some τ in Ω, if for every $\delta > 0$, there exists $n_0 = n_0(\delta)$ such that $\varrho(\tau_n, \tau) < \delta$ for all $n \geq n_0$.*

2. $\{\tau_n\}$ *is a Cauchy sequence, if* $\lim_{m,n\to\infty} \varrho(\tau_n, \tau_m) = 0$.
3. *The space* $(\Omega, \varrho, \varpi)$ *is said to be complete if every Cauchy sequence* $\{\tau_n\}$ *in* Ω *is convergent.*

Definition 3. *Let* $(\Omega, \varrho, \varpi)$ *be a CMS. Let* $\tau \in \Omega$ *and* $\delta > 0$.
(i) *The open ball* $B(\tau, \delta)$ *is*

$$B(\tau, \delta) = \{s \in \Omega, \varrho(\tau, s) < \delta\}.$$

(ii) *The mapping* $\varphi : \Omega \to \Omega$ *is said to be continuous at* $\tau \in \Omega$ *if* $\forall\, r > 0$, *there exists* $\gamma > 0$ *such that* $\varphi(B_p(\tau, \gamma)) \subseteq B_p(\varphi\tau, r)$.

Evidently, if a mapping g is continuous at v in the space $(\Omega, \varrho, \varpi)$, then $v_n \to v$ implies that $gv_n \to gv$ as $n \to \infty$.

2. Main Results

In this section, we present the main results concerning fixed-point theorems for Caristi contractions and α-admissible mappings. Some examples and an applications are presented.

2.1. Fixed-Point Theorems for Caristi Contractions

Our first main result is a fixed-point theorem for Caristi contraction mappings. We demonstrate the existence and uniqueness of fixed points under certain conditions. Then, in Theorem 2, we extend this framework to providing a new fixed-point theorem that incorporates the structure of a graph on the underlying space. We propose two examples to illustrate the practical application of the FP results in metric spaces with a graph.

Theorem 1. *Let* $(\Omega, \varrho, \varpi)$ *be a complete CMS. Consider the mapping* $\varphi\colon \Omega \to \Omega$ *such that*

$$\varrho(\varphi d, \varphi t) \leq (\hbar(d) - \hbar(\varphi d))\varrho(d, t) \quad \text{for all } t, d \in \Omega \tag{1}$$

where $\hbar : \Omega \to \mathbb{R}$ *is a bounded function from below.*

For $d_0 \in \Omega$, *take* $d_n = \varphi^n d_0$. *Moreover, assume that, for every* $d \in \Omega$, *we have*

$$\lim_{n\to\infty} \varpi(d_n, d) \;\text{ and }\; \lim_{n\to\infty} \varpi(d, d_n) \;\text{ exist and are finite} \tag{2}$$

and ϖ *satisfies the following condition*

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\varpi(d_{i+1}, d_{i+2})}{\varpi(d_i, d_{i+1})} \varpi(d_{i+1}, d_m) < \frac{1}{k} \text{ where } k \in (0,1). \tag{3}$$

Then, φ *has a unique FP.*

Proof. Case 1: Assume that there exists $n \geq 0$ such that $\varrho(d_n, \varphi d_n) = 0$, which implies that $\varphi d_n = d_n$. Then, d_n is a FP of φ.

Case 2: Assume that $\varrho(d_n, \varphi d_n) > 0$ for all $n \in \mathbb{N}$. Let us denote $b_n = \varrho(d_{n-1}, d_n)$. From (1), we obtain

$$\begin{aligned} b_{n+1} = \varrho(d_n, d_{n+1}) &= \varrho(\varphi d_{n-1}, \varphi d_n) \\ &\leq (\hbar(d_{n-1}) - \hbar(\varphi d_{n-1}))\varrho(d_{n-1}, d_n) \\ &= (\hbar(d_{n-1}) - \hbar(d_n))b_n. \end{aligned}$$

Hence,

$$0 < \frac{b_{n+1}}{b_n} \leq \hbar(d_{n-1}) - \hbar(d_n), \;\; \forall n \in \mathbb{N}. \tag{4}$$

Therefore, the sequence $(\hbar(d_n))$ is required to be positive and non-increasing. Thus, $\lim_{n\to\infty} \hbar(d_n) = r > 0$. Now, using (4), we obtain

$$\begin{aligned}\sum_{i=1}^{n} \frac{b_{i+1}}{b_i} &\leq \sum_{i=1}^{n}(\hbar(d_{i-1}) - \hbar(d_i)) \\ &= \hbar(d_0) - \hbar(d_1) + \hbar(d_1) - \ldots + \hbar(d_{n-1}) - \hbar(d_n) \\ &= \hbar(d_0) - \hbar(d_n)\end{aligned}$$

which means that $\sum_{i=1}^{\infty} \frac{b_{i+1}}{b_i} < \infty$. Consequently, we have

$$\lim_{i\to\infty} \frac{b_{i+1}}{b_i} = 0. \tag{5}$$

Taking into account (5), there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$\frac{b_{i+1}}{b_i} \leq k \quad \text{for } k \in (0,1). \tag{6}$$

This yields that

$$\varrho(d_{i+1}, d_i) \leq k\varrho(d_i, d_{i-1}) \ \ \forall i \geq i_0. \tag{7}$$

Now, we show that $\{d_i\}$ is a Cauchy sequence. From (7), we obtain

$$\varrho(d_{i+1}, d_i) \leq k^i \varrho(d_0, d_1) \ \ \forall i \geq i_0. \tag{8}$$

Let $m, n \in \mathbb{N}$ with $n < m$. By using the condition $(d3)$ in Definition 1, we have

$$\begin{aligned}\varrho(d_n, d_m) &\leq \varpi(d_n, d_{n+1})\varrho(d_n, d_{n+1}) + \varpi(d_{n+1}, d_m)\varrho(d_{n+1}, d_m) \\ &\leq \varpi(d_n, d_{n+1})\varrho(d_n, d_{n+1}) + \varpi(d_{n+1}, d_m)\varpi(d_{n+1}, d_{n+2})\varrho(d_{n+1}, d_{n+2}) \\ &+ \varpi(d_{n+1}, d_m)\varpi(d_{n+2}, d_m)\varrho(d_{n+2}, d_m) \\ &\leq \varpi(d_n, d_{n+1})\varrho(d_n, d_{n+1}) + \varpi(d_{n+1}, d_m)\varpi(d_{n+1}, d_{n+2})\varrho(d_{n+1}, d_{n+2}) \\ &+ \varpi(d_{n+1}, d_m)\varpi(d_{n+2}, d_m)\varpi(d_{n+2}, d_{n+3})\varrho(d_{n+2}, d_{n+3}) \\ &+ \varpi(d_{n+1}, d_m)\varpi(d_{n+2}, d_m)\varpi(d_{n+3}, d_m)\varrho(d_{n+3}, d_m) \\ &\leq \cdots \\ &\leq \varpi(d_n, d_{n+1})\varrho(d_n, d_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} \varpi(d_j, d_m)\right)\varpi(d_i, d_{i+1})\varrho(d_i, d_{i+1}) \\ &+ \prod_{k=n+1}^{m-1} \varpi(d_k, d_m)\varrho(d_{m-1}, d_m).\end{aligned}$$

Using (8), we obtain

$$
\begin{aligned}
\varrho(d_n, d_m) &\leq \varpi(d_n, d_{n+1})k^n \varrho(d_0, d_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i \varrho(d_0, d_1) \\
&+ \prod_{i=n+1}^{m-1} \varpi(d_i, d_m) k^{m-1} \varrho(d_0, d_1) \\
&\leq \varpi(d_n, d_{n+1})k^n \varrho(d_0, d_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i \varrho(d_0, d_1) \\
&+ \left(\prod_{i=n+1}^{m-1} \varpi(d_i, d_m) \right) k^{m-1} \varpi(d_{m-1}, d_m) \varrho(d_0, d_1) \\
&= \varpi(d_n, d_{n+1})k^n \varrho(d_0, d_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i \varrho(d_0, d_1) \\
&\leq \varpi(d_n, d_{n+1})k^n \varrho(d_0, d_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i \varrho(d_0, d_1).
\end{aligned}
$$

Let

$$R_q = \sum_{i=0}^{q} \left(\prod_{j=0}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i.$$

Then, we obtain

$$\varrho(d_n, d_m) \leq \varrho(d_0, d_1)[k^n \varpi(d_n, d_{n+1}) + (R_{m-1} - R_n)]. \tag{9}$$

Now, in order to obtain the limit of R_q, we will study the convergence of the ratio $\dfrac{R_{q+1}}{R_q}$. We have

$$
\begin{aligned}
\lim_{q\to\infty} \frac{R_{q+1}}{R_q} &= \lim_{q\to\infty} \frac{\sum_{i=0}^{q+1} \left(\prod_{j=0}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i}{\sum_{i=0}^{q} \left(\prod_{j=0}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i} \\
&= \lim_{q\to\infty} \frac{\prod_{j=0}^{q+1} \varpi(d_j, d_m) \varpi(d_i, d_{i+1}) k^{q+1}}{\sum_{i=0}^{q} \left(\prod_{j=0}^{i} \varpi(d_j, d_m) \right) \varpi(d_i, d_{i+1}) k^i} \\
&\leq \lim_{q\to\infty} \frac{\prod_{j=0}^{q+1} \varpi(d_j, d_m) \varpi(d_i, d_{i+1}) k^{q+1}}{\prod_{j=0}^{q} \varpi(d_j, d_m) \varpi(d_i, d_{i+1}) k^{q}} \\
&= \lim_{q\to\infty} \frac{\varpi(d_{q+1}, d_m) \varpi(d_{q+1}, d_{q+2})}{\varpi(d_q, d_{q+1})} .k \big(\text{ by condition (3)}\big) \\
&< \frac{1}{k}.k = 1.
\end{aligned}
$$

Therefore, by the ratio test, we deduce that $\lim_{n\to\infty} R_n$ exists and is finite. Hence, $\{R_n\}$ is a Cauchy sequence. Subsequently, by applying the limit to the inequality (9), we obtain

$$\lim_{n,m\to\infty} \varrho(d_n, d_m) = 0. \tag{10}$$

Therefore, $\{d_n\}$ is a Cauchy sequence, and by the completeness of the space Ω, we can affirm that $d_n \longrightarrow d^*$ as $n \longrightarrow \infty$.

Now, we claim that d^* is an FP of φ. From (1), we have

$$\varrho(d^*, d_{n+1}) \leq \varpi(d^*, d_n)\varrho(d^*, d_n) + \varpi(d_n, d_{n+1})\varrho(d_n, d_{n+1}). \tag{11}$$

Knowing that the limits of $\varpi(d^*, d_n)$ and $\varpi(d_n, d_{n+1})$ exist and are finite from (2) and using (10), we can affirm that

$$\lim_{n\to\infty} \varrho(d^*, d_n) = 0. \tag{12}$$

On the other hand, we have

$$\begin{aligned}\varrho(d^*, \varphi d^*) &\leq \varpi(d^*, d_{n+1})\varrho(d^*, d_{n+1}) + \varpi(d_{n+1}, \varphi d^*)\varrho(d_{n+1}, \varphi d^*) \\ &= \varpi(d^*, d_{n+1})\varrho(d^*, d_{n+1}) + \varpi(d_{n+1}, \varphi d^*)\varrho(\varphi d_n, \varphi d^*) \\ &\leq \varpi(d^*, d_{n+1})\varrho(d^*, d_{n+1}) + \varpi(d_{n+1}, \varphi d^*)(\hbar(d_n) - \hbar(d_{n+1}))\varrho(d_n, d^*). \end{aligned} \tag{13}$$

If we take the limit in (13) as n goes toward ∞ and from (2) and (12), we obtain $\varrho(d^*, \varphi d^*) = 0$, that is, d^* is an FP of φ.

Assume that φ has two FPs, e.g., d_1 and d_2 (that is, $\varphi d_1 = d_1$ and $\varphi d_2 = d_2$). Then,

$$\begin{aligned}\varrho(d_1, d_2) &= \varrho(\varphi d_1, \varphi d_2) \\ &\leq (\hbar(d_1) - \hbar(\varphi d_1))\varrho(d_1, d_2) \\ &= (\hbar(d_1) - \hbar(d_1))\varrho(d_1, d_2) = 0.\end{aligned}$$

Therefore $\varrho(d_1, d_2) = 0$ and $d_1 = d_2$. □

Now, we present the graphical version of Theorem 1 by endowing the CMS with a graph. We propose a corollary by relaxing the condition of the continuity. Also, two examples are introduced. In order to get into the topic, let us begin by recalling some concepts and definitions from graph theory which will be necessary later.

In accordance with the work of Jachymski in [12], we endow the CMS $(\Omega, \varrho, \varpi)$ with a graph G, where G is characterized by its set of vertices $U = U(G)$, which is identical to Ω, and its set of edges $E = E(G)$. Assuming that G contains no parallel edges, it can be identified as the pair (U, E).

Additionally, the graph G can be interpreted as a weighted graph by assigning a weight to each edge based on the distance between its vertices.

Definition 4. *Let v_1 and v_2 be two vertices in a graph G. A path from v_1 to v_2 in G of length j (where $j \in \mathbb{N} \cup \{0\}$) is a sequence $(w_i)_{i=0}^{j}$ of $j+1$ distinct vertices where $w_0 = v_1$ and $w_j = v_2$, and each pair of consecutive vertices $(w_i, w_{i+1}) \in E(G)$, for all $i = 1, 2, \ldots, j$.*

Definition 5 ([12])**.** *Consider a vertex u in a graph G. The subgraph G_u, which consists of all the vertices and edges that are part of some path in G starting at u, is referred to as the component of G that contains u. The equivalence class $[u]_G$ on the vertex set $V(G)$, defined by the relation R (where uRv if there is a path from u to v), satisfies the property that the set of vertices in G_u, denoted by $V(G_u)$, is equal to $[u]_G$.*

Definition 6. *Let $\varphi : \Omega \longrightarrow \Omega$ be a mapping. We denote $\Omega^f = \{\tau \in \Omega / (\tau, \varphi\tau) \in E(G)$ or $(\varphi\tau, \tau) \in E(G)\}$.*

Definition 7. *Let $(\Omega, \varrho, \varpi)$ be a complete CMS equipped with a graph G. We name the mapping $\varphi : \Omega \longrightarrow \Omega$ a G-Caristi mapping if it satisfies the following hypothesis:*

1. *(G-edge preserving)*

$$\forall s_1, s_2 \in E(G), (s_1, s_2) \in E(G) \Longrightarrow (\varphi s_1, \varphi s_2) \in E(G). \tag{14}$$

2. *There exists a function $\phi : \Omega \longrightarrow \mathbb{R}$ bounded from below satisfying*

$$\varrho(\varphi s_1, \varphi s_2) \leq (\phi(s_1) - \phi(\varphi s_1)) \times \varrho(s_1, s_2) \text{ for all } (\varphi s_1, \varphi s_2) \in E(G) \text{ or } (s_1, s_2) \in E(G). \tag{15}$$

Theorem 2. *Let $(\Omega, \varrho, \varpi)$ be a complete CMS equipped with a graph G. Let $\varphi : \Omega \longrightarrow \Omega$ be a continuous G-Caristi mapping. Assume that there exists $\tau_0 \in \Omega$ such that*

$$(\varphi\tau_0, \tau_0) \in E(G). \tag{16}$$

We take $\tau_n = \varphi^n \tau_0$ and we assume that for each $\tau \in \Omega$, we have

$$\lim_{i \to \infty} \varpi(\tau_i, \tau) \text{ and } \lim_{i \to \infty} \varpi(\tau_i, \tau_{i+1}) \text{ which exist and are finite,} \tag{17}$$

and ϖ satisfies the following condition

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\varpi(\tau_{i+1}, \tau_{i+2})}{\varpi(\tau_i, \tau_{i+1})} \varpi(\tau_{i+1}, \tau_m) < \frac{1}{k} \text{ where } k \in (0, 1). \tag{18}$$

Therefore, φ has a unique FP.

Proof. Equation (16) implies that there exists $\tau_0 \in \Omega$ such that $(\varphi\tau_0, \tau_0) \in E(G)$. Since φ is G-edge-preserving, we obtain $(\varphi^{n+1}\tau_0, \varphi^n\tau_0) \in E(G)$ for all $n \geq 1$. Then, from (15), we have

$$\varrho(\varphi^n\tau_0, \varphi^{n+1}\tau_0) \leq (\phi(\varphi^{n-1}\tau_0) - \phi(\varphi^n\tau_0))\varrho(\varphi^{n-1}\tau_0, \varphi^n\tau_0),$$

which gives $\phi(\varphi^{n-1}\tau_0) - \phi(\varphi^n\tau_0) \geq 0$. Hence, $\{\phi(\varphi^n\tau_0)\}$ is a sequence of positive numbers that is decreasing. Let $\phi_0 = \lim_{n \to \infty} \phi(\varphi^n\tau_0)$. For any $m, n \geq 1$, we obtain

$$\begin{aligned}
\varrho(\varphi^n\tau_0, \varphi^{n+m}\tau_0) &\leq (\phi(\varphi^{n-1}\tau_0) - \phi(\varphi^n\tau_0))\varrho(\varphi^{n-1}\tau_0, \varphi^{n+m-1}\tau_0) \\
&\leq (\phi(\varphi^{n-1}\tau_0) - \phi(\varphi^n\tau_0))(\phi(\varphi^{n-2}\tau_0) - \phi(\varphi^{n-1}\tau_0))\varrho(\varphi^{n-2}\tau_0, \varphi^{n+m-2}\tau_0) \\
&\vdots \\
&\leq \prod_{k=1}^{n} \Big(\phi(\varphi^{n-k}\tau_0) - \phi(\varphi^{n-k+1}\tau_0)\Big)\varrho(\varphi^0\tau_0, \varphi^m\tau_0).
\end{aligned}$$

From the properties of ϕ, we have that $\prod_{k=1}^{n}(\phi(\varphi^{n-k}\tau_0) - \phi(\varphi^{n-k+1}\tau_0))$ converges to zero when $k \longrightarrow \infty$. We claim that $\lim_{m \to \infty} \varrho(\varphi^0\tau_0, \varphi^m\tau_0) = 0$. Using the triangle inequality of the controlled metric, we obtain

$$\begin{aligned}\varrho(\varphi^0\tau_0, \varphi^m\tau_0) &= \varrho(\tau_0, \tau_m)\\ &\leq \varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \varpi(\tau_1, \tau_m)\varrho(\tau_1, \tau_m)\\ &\leq \varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \varpi(\tau_1, \tau_m)\varpi(\tau_1, \tau_2)\varrho(\tau_1, \tau_2) + \varpi(\tau_1, \tau_m)\varpi(\tau_2, \tau_m)\varrho(\tau_2, \tau_m)\\ &\vdots\\ &\leq \varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \sum_{i=1}^{m-2}\left(\prod_{j=1}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})\varrho(\tau_i, \tau_{i+1})\\ &+ \prod_{k=1}^{m-1}\varpi(\tau_k, \tau_m)\varrho(\tau_{m-1}, \tau_m).\end{aligned}$$

Similarly to Theorem 1, using (15), we have

$$\varrho(\tau_i, \tau_{i+1}) \leq k^i\varrho(\tau_0, \tau_1). \tag{19}$$

Then,

$$\begin{aligned}\varrho(\varphi^0\tau_0, \varphi^m\tau_0) &\leq \varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \sum_{i=1}^{m-2}\left(\prod_{j=1}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})k^i\varrho(\tau_0, \tau_1)\\ &+ \prod_{k=1}^{m-1}\varpi(\tau_k, \tau_m)k^{m-1}\varrho(\tau_0, \tau_m)\\ &\leq \varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})k^i\varrho(\tau_0, \tau_1)\\ &\leq \varpi(\tau_0, \tau_m)\varpi(\tau_0, \tau_1)\varrho(\tau_0, \tau_1) + \sum_{i=1}^{m-1}\left(\prod_{j=1}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})k^i\varrho(\tau_0, \tau_1)\\ &\leq \sum_{i=0}^{m-1}\left(\prod_{j=0}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})k^i\varrho(\tau_0, \tau_1)\\ &= \sum_{i=0}^{m-1}\gamma_i\varrho(\tau_0, \tau_1),\end{aligned}$$

where $\gamma_i = \left(\prod_{j=0}^{i}\varpi(\tau_j, \tau_m)\right)\varpi(\tau_i, \tau_{i+1})k^i$. Using condition (18) similarly to Theorem 1, we can deduce that $\lim_{m\to\infty}\varrho(\varphi^0\tau_0, \varphi^m\tau_0) = 0$. Thereafter, $\lim_{m\to\infty}\varrho(\varphi^n\tau_0, \varphi^{n+m}\tau_0) = 0$. Consequently, $\{\varphi^n\tau_0\}$ is a Cauchy sequence within the space Ω. Then, there exists $u_* \in \Omega$ such that $lim_{n\longrightarrow\infty}\varphi^n\tau_0 = u_*$. The continuity of φ implies that $\varphi u_* = u_*$ and hence u_* is an FP of φ.

Suppose that there exist two FPs, u_*^1 and v_*^1, in Ω such that $\varphi u_*^1 = u_*^1$ and $\varphi v_*^1 = v_*^1$. We have

$$\begin{aligned}\varrho(u_*^1, v_*^1) &= \varrho(\varphi u_*^1, \varphi v_*^1)\\ &\leq (\phi(u_*^1) - \phi(\varphi u_*^1))\varrho(u_*^1, v_*^1)\\ &= (\phi(u_*^1) - \phi(u_*^1))\varrho(u_*^1, v_*^1) = 0,\end{aligned}$$

and therefore $\varrho(u_*^1, v_*^1) = 0 \Longrightarrow u_*^1 = v_*^1$. $\square$

Example 1. *Let $\Omega = [0,1]$, $\varrho(u,v) = |u-v|^2$, and $\varpi(u,v) = u+v+2$. It is easy to prove that $(\Omega, \varrho, \varpi)$ is a complete CMS. Note that this space is neither a metric space in the usual sense nor a b-metric space.*

Consider the mappings $\varphi : \Omega \Longrightarrow \Omega$ and $\phi : \Omega \longrightarrow \mathbb{R}$ defined by $\varphi(u) = \frac{u}{5} + \frac{1}{2}$ and $\phi(u) = 2u^2 + 3u$. We claim that φ is G-Caristi mapping in $(\Omega, \varrho, \varpi)$ endowed with a graph G. Indeed, $V(G) = [0,1]$ and $E(G) = [0,1]^2$; then, φ satisfies the G-edge preserving condition.

On the other hand, for all $u, v \in \Omega$, we have

$$\begin{aligned}(\phi(u) - \phi(\varphi(u))\varrho(u,v) &= (2u^2 + 3u + 2(\frac{u}{5} + \frac{1}{2})^2 + 3(\frac{u}{5} + \frac{1}{2}))|u-v|^2 \\ &= (2u^2 + 3u + 2(\frac{u^2}{25} + \frac{u}{5} + \frac{1}{4}) + \frac{3u}{5} + \frac{3}{2}))|u-v|^2 \\ &= (\frac{52}{25}u^2 + 4u + 2)|u-v|^2.\end{aligned}$$

By a simple calculus, we can verify that $\frac{52}{25}u^2 + 4u + 2 \geq \frac{1}{4}$. Consequently, we obtain

$$\begin{aligned}(\phi(u) - \phi(\varphi(u))\varrho(u,v) &\geq \frac{1}{4}|u-v|^2 \\ &= \varrho(\varphi(u), \varphi(v)).\end{aligned}$$

Therefore, assumption (15) is satisfied and φ is G-Caristi mapping. Additionally, conditions (17) and (23) are met. Finally, all the conditions of Theorem 2 are satisfied, so the mapping φ has a unique fixed point that is $\varphi(\frac{5}{8}) = \frac{5}{8} \in [0,1]$.

Example 2. *Let $\Omega = \{0,1,2,3\}$, $\varrho(u,v) = |u-v|^2$, and $\varpi(u,v) = u+v+2$. Then, $(\Omega, \varrho, \varpi)$ is a complete CMS. Consider the mappings $\varphi : \Omega \Longrightarrow \Omega$ and $\phi : \Omega \longrightarrow \mathbb{R}$ defined, respectively, by $\varphi(0) = 2$, $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 0$, and*

$$\phi(u) = \begin{cases} 6 - u^2 & \text{if } u \leq 1 \\ 5 & \text{if } u = 2 \\ 9 & \text{if } u = 3. \end{cases}$$

Consider the following set of edges $E(G) = \{(0,0); (1,1); (2,2); (1,2); (0,2); (2,1); (3,0)\}$. We represent the graph G composed by the vertices $\Omega = \{0,1,2,3\}$ and the edges $E(G)$ in Figure 1.

Let us begin by verifying the condition for G-edge preserving. We have

$(0,0) \in E(G)$, $(\varphi(0), \varphi(0)) = (2,2) \in E(G)$,
$(2,2) \in E(G)$, $(\varphi(2), \varphi(2)) = (1,1) \in E(G)$,
$(1,1) \in E(G)$, $(\varphi(1), \varphi(1)) = (1,1) \in E(G)$,
$(1,2) \in E(G)$, $(\varphi(1), \varphi(2)) = (1,1) \in E(G)$,
$(0,2) \in E(G)$, $(\varphi(0), \varphi(2)) = (2,1) \in E(G)$,
$(2,1) \in E(G)$, $(\varphi(2), \varphi(1)) = (1,1) \in E(G)$,
$(3,0) \in E(G)$, $(\varphi(3), \varphi(0)) = (0,2) \in E(G)$,

Therefore, condition (14) holds. Now, we will prove that the mapping φ satisfies condition (15) for all the edges in $E(G)$.

- *For the edge $(0,0)$, we have $\varrho(\varphi(0), \varphi(0)) = |2-2|^2 = 0 = (\phi(0) - \phi(\varphi 0))|0-0|^2$. Also, we obtain the same result in a similar manner for the edges $(1,1)$ and $(2,2)$.*
- *For the edge $(1,2)$, we have $\varrho(\varphi(1), \varphi(2)) = \varrho(1,1) = |1-1|^2 = 0 = (\phi(1) - \phi(\varphi 1))|1-2|^2 = (\phi(1) - \phi(1))|1-2|^2$.*

- *For the edge* $(0,2)$, *we have* $\varrho(\varphi(0),\varphi(2)) = \varrho(2,1) = |2-1|^2 = 1 \leq (\phi(0)-\phi(\varphi 0))|0-2|^2 = 4$.
- *For the edge* $(2,1)$, *we have* $\varrho(\varphi(2),\varphi(1)) = \varrho(1,1) = |1-1|^2 = 0 = (\phi(2)-\phi(\varphi 2))|2-1|^2 = (\phi(2)-\phi(1))|2-1|^2 = (5-5).1^2 = 0$.
- *For the edge* $(3,0)$, *we have* $\varrho(\varphi(3),\varphi(0)) = \varrho(0,2) = |0-2|^2 = 4 = (\phi(3)-\phi(\varphi 3))|3-0|^2 = (\phi(3)-\phi(0)).3^2 = (9-6)3^2 = 27$.

Then, Equation (15) is fulfilled. Additionally, it is easy to see that the function $\varpi(u,v) = u+v+2$ *obeys conditions (17) and (23). Finally, all the conditions of Theorem 2 are met. Then,* φ *has a unique FP that is* $\varphi(1) = 1$.

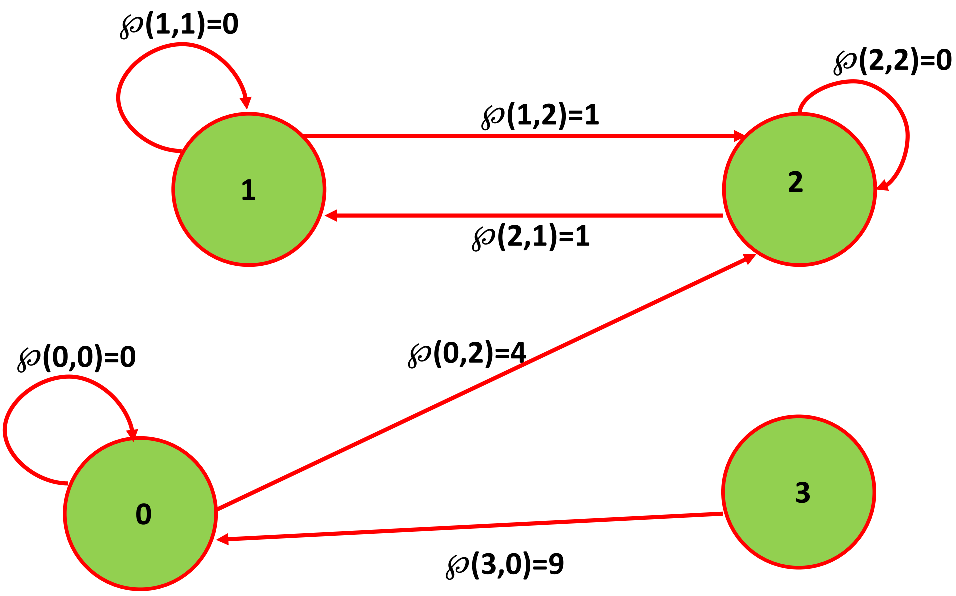

Figure 1. The graph of Example 2.

The following result is obtained by relaxing the condition of the continuity of the contraction. We use orbital continuity which is weaker than continuity.

Definition 8. *A mapping* $\varphi : \Omega \longrightarrow \Omega$ *is called orbitally G-continuous if for all* $s_1, s_2 \in \Omega$ *and any positive sequence* $\{s_n\}_{n\in\mathbb{N}}$,

$$f^{s_n}s_1 \longrightarrow s_2,\ (f^{s_n}s_1, f^{s_{n+1}}s_2) \in E(G) \Longrightarrow \lim_{n\to\infty} f(f^{s_n}s_1) = fs_2. \tag{20}$$

Corollary 1. *Let* (Ω,ϱ,G) *be a complete CMS equipped with a graph G. Let* $\varphi : \Omega \longrightarrow \Omega$ *be a G-Caristi mapping orbitally G-continuous. We assume the following property* $(\mathbb{P})$*: for any* $\{\tau_n\}_{n\in\mathbb{N}}$ *in* $\mathcal{S}$*, if* $\tau_n \longrightarrow s$ *and* $(\tau_n,\tau_{n+1}) \in E(G)$*, then there is a subsequence* $\{\tau_{k_n}\}_{n\in\mathbb{N}}$ *with* $(\tau_{k_n}, s) \in E(G)$.

Moreover, suppose that there exists $\tau_0 \in \Omega$ *such that*

$$(\varphi\tau_0, \tau_0) \in E(G). \tag{21}$$

We take $\tau_n = \varphi^n\tau_0$ *and we assume that for each* $\tau \in \Omega$*, we have*

$$\lim_{i\to\infty} \varpi(\tau_i,\tau) \text{ and } \lim_{i\to\infty} \varpi(\tau_i,\tau_{i+1}) \text{ which exist and are finite,} \tag{22}$$

and ϖ satisfies the following condition

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\varpi(\tau_{i+1},\tau_{i+2})}{\varpi(\tau_i,\tau_{i+1})}\varpi(\tau_{i+1},\tau_m)<\frac{1}{k} \text{ where } k\in(0,1). \tag{23}$$

Therefore, the restriction $\varphi_{|[\tau]_G}$ has a unique FP.

Proof. Similarly to the proof of Theorem 2, we demonstrate that $\{\varphi^n\tau_0\}$ is a Cauchy sequence. Then, there exists $u_*\in\Omega$ such that

$$\lim_{n\longrightarrow\infty}\varphi^n\tau_0=u_*. \tag{24}$$

Since $\tau_0\in\Omega^f$, $\varphi^n\tau_0\in\Omega^f$ for all $n\in\mathbb{N}$, then $(\tau_0,\varphi\tau_0)\in E(G)$. From property $(\mathbb{P})$, there exists a subsequence $\{\varphi^{k_n}\tau_0\}_n$ of $\{\varphi^n\tau_0\}_n$ such that $(\varphi^{k_n}\tau_0,u_*)\in E(G)$ for all $n\in\mathbb{N}$.
On the other hand, a path P_G can be constructed by using the points $\tau_0,\varphi\tau_0,\dots,\varphi^{k_1}\tau_0,u_*$, which allows us to affirm that $u_*\in[\tau_0]_G$. From the orbitally G-continuous φ, we obtain

$$\lim_{n\longrightarrow\infty}\varphi(\varphi^{k_n})=\varphi u_*. \tag{25}$$

Therefore, from (24) and (25), we conclude that u_* is an FP of $\varphi_{|[\tau]_G}$. The uniqueness of the FP is similar to Theorem 2. □

2.2. Fixed-Point Results for α-Admissible Mappings

In this subsection, we introduce a new FP theorem concerning the α-admissible mappings under suitable hypotheses. Moreover, to highlight the potential application in various mathematical contexts, we propose Theorem 4 to solve second-order differential equations. Let us start by defining the α-admissible mappings that will be involved in the next theorem.

Definition 9. *Let $\varphi:\Omega\longrightarrow\Omega$ and let $\alpha:\Omega\times\Omega\to[0,+\infty)$. The mapping φ is called α-admissible if $\forall\,\tau,s\in\Omega$, $\alpha(\tau,s)\geq 1$ implies that $\alpha(\varphi\tau,\varphi s)\geq 1$.*

Definition 10 ([4]). *Let $\varphi:\Omega\longrightarrow\Omega$ and $\alpha,\beta:\Omega\times\Omega\to[0,+\infty)$. The mapping φ is called α-admissible with respect to β if $\forall\,\tau,s\in\Omega$, $\alpha(s,\tau)\geq\beta(s,\tau)$, we have $\alpha(\varphi s,\varphi\tau)\geq\beta(\varphi s,\varphi\tau)$.*

Now, we consider a new class of families Ψ of mappings $g:[0,+\infty)\to[0,+\infty)$ satisfying the following assumptions:

(i) g is an upper semi-continuous mapping from the right;
(ii) $g(s)<s\ \forall s>0$;
(iii) $g(0)=0$.

Theorem 3. *Let (Ω,ϱ,ϖ) be a complete CMS and $\psi\in\Psi$. Supppose that $\varphi:\Omega\to\Omega$ is a continuous mapping that meets the following hypotheses:*

(c_1) *φ is α-admissible with respect to β;*
(c_2) *if $\tau,s\in\Omega$ and $\alpha(\tau,s)\geq\beta(\tau,s)$, then $\varrho(\varphi\tau,\varphi s)\leq\psi(\varrho(\tau,s))$;*
(c_3) *there exists $\tau_0\in\Omega$ such that $\alpha(\tau_0,\varphi\tau_0)\geq\beta(\tau_0,\varphi\tau_0)$.*

Therefore, φ has an FP.

Proof. Consider τ_0 as an element in Ω. We denote $\tau_1 = \varphi\tau_0$. Therefore, we construct the sequence $\{\tau_n\} \in \Omega$ defined as follows

$$\tau_{n+1} = \varphi\tau_n, \forall n \in \mathbb{N}. \tag{26}$$

Assume that $\tau_n \neq \tau_{n+1}$ $\forall n \in \mathbb{N}$; otherwise, φ has an FP.

From condition (c_2), we have $\alpha(\tau_0, \tau_1) = \alpha(\tau_0, \varphi\tau_0) \geq \beta(\tau_0, \varphi\tau_0)$ and taking into account that φ is α-admissible with respect to β, we obtain that

$$\alpha(\tau_1, \tau_2) = \alpha(\varphi\tau_0, \varphi\tau_1) \geq \beta(\varphi\tau_0, \varphi\tau_1) = \beta(\tau_1, \tau_2).$$

By extending this process, we obtain

$$\alpha(\tau_n, \tau_{n+1}) \geq \beta(\tau_n, \tau_{n+1}) \ \ \forall n \in \mathbb{N}. \tag{27}$$

Applying (c_2) and the property of ψ, we obtain that

$$\varrho(\tau_n, \tau_{n+1}) = \varrho(\varphi\tau_{n-1}, \varphi\tau_n) \leq \psi(\varrho(\tau_{n-1}, \tau_n)) < \varrho(\tau_{n-1}, \tau_n) \ \ \forall n \in \mathbb{N}. \tag{28}$$

Therefore, $\{\varrho(\tau_n, \tau_{n+1})\}$ is a nonincreasing sequence. As a result, there exists $r \geq 0$ fulfilling

$$\lim_{n\to\infty} \varrho(\tau_n, \tau_{n+1}) = r.$$

We claim that $r = 0$. Suppose that $r > 0$. Since ψ is upper semi-continuous from the right, using (28), we obtain

$$\begin{aligned}&\lim_{n\to\infty} \varrho(\tau_n, \tau_{n+1}) = r \\ &\leq \lim\sup_{n\to\infty} \psi(\varrho(\tau_{n-1}, \tau_n)) \leq \psi(r) < r,\end{aligned}$$

This leads to a contradiction. Hence,

$$\lim_{n\to\infty} \varrho(\tau_n, \tau_{n+1}) = 0. \tag{29}$$

From (29), we can affirm the existence of some $n_l \in \mathbb{N}$ for every $l \in \mathbb{N}$, such that

$$\varrho(\tau_{n_l}, \tau_{n_l+1}) \leq \frac{1}{2^l}.$$

Then, we obtain

$$\sum_{l=1}^{\infty} \varrho(\tau_{n_l}, \tau_{n_l+1}) \leq \infty.$$

Consequently, $\{\tau_n\}$ forms a Cauchy sequence and thus converges to some $\tau \in \Omega$. Owing to the continuity of φ, we have

$$\tau = \lim_{l\to\infty} \tau_{n_l+1} = \lim_{l\to\infty} \varphi\tau_{n_l} = \varphi\tau. \tag{30}$$

Thus, τ is an FP of φ. □

To illustrate that the FP result is a powerful tool in various mathematical fields, we apply it to solve a second-order differential equation using Theorem 3. Indeed, by carefully verifying the assumptions outlined in Theorem 3, we ensure that the FP serves as a valid and effective tool for deriving solutions to second-order differential equations for the given problem, as demonstrated in the following theorem (Theorem 4).

Consider the following problem $(\mathbb{P})$:

$$-\frac{d^2z}{d\mu^2} = h(\mu, z(\mu)), \ \ \mu \in [0,1] \tag{31}$$
$$z(0) = z(1) = 0,$$

where $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The Green function associated to (31) is defined by

$$\mathcal{H}(\mu,\eta) = \begin{cases} \mu(1-\eta), & 0 \le \mu \le \eta \le 1 \\ \eta(1-\mu), & 0 \le \eta \le \mu \le 1. \end{cases}$$

Denote $\mathcal{C}_o([0,1]) := \{h : [0,1] \to [0,1]/h \text{ continuous}\}$. Let $\varrho : \mathcal{C}_o([0,1] \times \mathcal{C}_o([0,1] \to \mathbb{R}$ be defined by

$$\varrho(\mu,\eta) = ||\mu - \eta||_\infty = \sup_{k\in[0,1]} |\mu(k) - \eta(k).| \tag{32}$$

It is easy to see that $(\mathcal{C}_o([0,1], \varrho)$ is a complete CMS.

Theorem 4. *Let us examine the two-point boundary value problem* $(\mathbb{P})$*. Suppose that the following assumptions hold:*

1. *there exists a function* $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ *such that,* $\forall\ \mu \in [0,1]$*, and* $a_1, a_2 \in \mathbb{R}$ *with* $\Lambda(a_1,a_2) \ge 0$*, we have*
$$|h(\mu,a_1) - h(\mu,a_2)| \le 8\psi(\max_{a_1,a_2\in\mathbb{R},\Lambda(a_1,a_2)\ge 0} |a_1 - a_2|); \tag{33}$$
2. *There exists* $z_0 \in \mathcal{C}_o([0,1])$ *such that,* $\forall\ \mu \in [0,1]$*, we have*
$$\Lambda\Big(z_0(\mu), \mathcal{H}(\mu,\eta)h(\eta,z_0(\eta))\Big) \ge 0; \tag{34}$$
3. *If* $\{z_n\}$ *is a sequence in* $\mathcal{C}_o([0,1])$ *such that* $z_n \to z \in \mathcal{C}_o([0,1])$ *and* $\Lambda(z_n, z_{n+1}) \ge 0$*,* $\forall\ n \in \mathbb{N}$*, then* $\Lambda(z_n, z) \ge 0$ *for all* $n \in \mathbb{N}$*;*
4. *For all* $\mu \in [0,1]$*, for all* $z, y \in \mathcal{C}_o([0,1])$*,* $\Lambda(z(\mu), y(\mu)) \ge 0$ *implies that*
$$\Lambda\Big(\int_0^1 \mathcal{H}(\mu,\eta)h(\eta,z(\eta))d\eta, \int_0^1 \mathcal{H}(\mu,\eta)h(\eta,y(\eta))d\eta\Big) \ge 0. \tag{35}$$

Then, $(\mathbb{P})$ *possesses a solution in* $\mathcal{C}_o([0,1])$*.*

Proof. Solving the problem $(\mathbb{P})$ is tantamount to solving the following integral equation:

$$z(\mu) = \int_0^\mu \mathcal{H}(\mu,\eta)h(\eta,z(\eta))d\eta \ \ \forall \mu \in [0,1]. \tag{36}$$

Let φ be a self-mapping on $\mathcal{C}_o([0,1])$ defined by

$$\varphi z(\mu) = \int_0^1 \mathcal{H}(\mu,\eta)h(\eta,z(\eta))d\eta \ \ \forall \mu \in [0,1]. \tag{37}$$

Suppose that $z, y \in \mathcal{C}_o([0,1])$ such that $\Lambda(z(\mu), y(\mu)) \ge 0\ \forall\ \mu \in [0,1]$. Using the first assumption of the theorem, we obtain that

$$
\begin{aligned}
|\varphi z(\mu) - \varphi y(\mu)| &= \Big| \int_0^1 \mathcal{H}(\mu,\eta)[h(\eta,z(\eta)) - h(\eta,y(\eta))]d\eta \Big| \\
&\leq \int_0^1 \mathcal{H}(\mu,\eta)\Big|h(\eta,z(\eta)) - h(\eta,y(\eta))\Big|d\eta\Big| \\
&\leq 8\Big(\int_0^1 \mathcal{H}(\mu,\eta)d\eta\Big)(\psi(||z-y||_\infty)) \\
&\leq 8\Big(\sup_{\mu\in[0,1]} \int_0^1 \mathcal{H}(\mu,\eta)d\eta\Big)(\psi(||z-y||_\infty)). \qquad (38)
\end{aligned}
$$

As $\int_0^1 \mathcal{H}(\mu,\eta)d\eta = -\dfrac{\mu^2}{2} + \dfrac{\mu}{2}$, for all $\mu \in [0,1]$, we obtain $\sup\limits_{\mu\in[0,1]} \displaystyle\int_0^1 \mathcal{H}(\mu,\eta)d\eta = \dfrac{1}{8}$. Consequently,

$$||\varphi z - \varphi y||_\infty \leq \psi(||z-y||_\infty) \qquad (39)$$

for each $z, y \in \mathcal{C}_o([0,1])$, such that $\Lambda(z(\mu), y(\mu)) \geq 0$ for all $\mu \in [0,1]$. Therefore, condition (c_2) of Theorem 3 holds.

Now, let us prove that φ is α-admissible concerning β. Let $\alpha, \beta : \mathcal{C}_o([0,1]) \times \mathcal{C}_o([0,1]) \to [0,\infty)$ be mappings defined by

$$\alpha(z,y) = \begin{cases} 1, & \Lambda(z(\mu), y(\mu)) \geq 0, \mu \in [0,1] \\ 0, & otherwise. \end{cases}$$

$$\beta(z,y) = \begin{cases} \frac{1}{2}, & \Lambda(z(\mu), y(\mu)) \geq 0, \mu \in [0,1] \\ 2, & otherwise. \end{cases}$$

Let $z, y \in \mathcal{C}_o([0,1])$ such that $\alpha(z,y) \geq \beta(z,y)$. Hence, $\Lambda(z(\mu), y(\mu)) \geq 0, \forall \mu \in [0,1]$. Hence,

$$||\varphi z - \varphi y||_\infty \leq \psi(||z-y||_\infty). \qquad (40)$$

Moreover, if $z, y \in \mathcal{C}_o([0,1])$ such that $\alpha(z,y) \geq \beta(z,y)$, by applying assumption 4 of Theorem 4, we obtain $\Lambda(\varphi z(\mu), \varphi y(\mu)) \geq 0$, and this yields $\alpha(\varphi z, \varphi y) \geq \beta(\varphi z, \varphi y)$. Therefore, φ is α-admissible with respect to β. Using condition 2, there exists $z_0 \in \mathcal{C}_o([0,1])$ such that $\alpha(z_0, \varphi z_0) \geq \beta(z_0, \varphi z_0)$.

Finally, given that all the conditions of Theorem 3 are satisfied, then φ has an FP in $\mathcal{C}_o([0,1])$, e.g., z_{sol}, which is a solution of the problem $(\mathbb{P})$. □

3. Conclusions and Perspectives

In conclusion, this paper introduces several significant contributions to FP theory within the context of CMS. We have extended the classical Caristi contraction by exploring the existence and uniqueness of fixed points under specific conditions, and further developed a graphical representation of this result. Also, by introducing the concept of α-admissible mappings, we provided a new FP theorem with practical applications, including the solution to a second-order differential equation. These contributions provide valuable tools for tackling problems across various domains of mathematical analysis.

Moving forward, further research will aim to expand on these results, exploring broader applications and potential generalizations of the established theorems in the double controlled metric space which is more general than the CMS. Also, one promising direction is the generalization of the α-admissible mappings to a broader class of operators, potentially incorporating non-contractive and nonlinear mappings. This could pave the way for FP theorems in settings where traditional contraction mappings are not applicable, thereby extending the scope of fixed-point theory.

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