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On the Axiomatic of GV-Fuzzy Metric Spaces and Its Completion

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Abstract: The concept of fuzzy metric space introduced by Kramosil and Michalek was later slightly modified by George and Veeramani who imposed three additional restrictions on it. A significant difference between these two concepts of fuzzy metrics is that fuzzy metric spaces in the sense of George and Veeramani do not admit completion, in general. This paper is devoted to go into detail on completable fuzzy metric spaces by means of the study of the impact on the completion of each one of the restrictions imposed by George and Veeramani in their definition of fuzzy metric. In this direction, we characterize those completable fuzzy metric spaces, in which just one of the three restrictions imposed by George and Veeramani is required. Various examples illustrate and justify the main results.

Keywords: GV-fuzzy metric space; KM-fuzzy metric space; Cauchy sequence; convergence; completeness; completion

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1. Introduction

The concept of the fuzzy set introduced by L.A. Zadeh in [1] constitutes a generalization of the classical notion of sets. Recall that, give a non-empty (crisp) set A, a fuzzy set F on A can be defined (simply) as a mapping $F : A \to [0,1]$. This concept has turned out essential for various branches of mathematics as topology, algebra, and analysis. Thus, many researchers have focused their interest in adapting classical theories to the fuzzy context. In this direction, we can find different notions of fuzzy metrics defined with the aim of giving a fuzzy version of the classical concept of metrics. Concretely, in the literature, we can find distinct approaches to define fuzzy metrics, such as the Kaleva and Seikkala's one in [2], or another one introduced by Kramosil and Michalek in [3], which was slightly modified later by George and Veeramani in [4]. Fuzzy metric spaces in the Kramosil and Michalek's sense are usually known currently following their reformulation provided by Grabiec in [5] which, on account of the expounded by Miñana and Valero in [6], can be defined as follows.

Definition 1. *A fuzzy metric space is an ordered triple $(\mathcal{X}, \mathcal{M}, *)$ such that $\mathcal{X}$ is a (non-empty) set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times \mathbb{R}^+$ satisfying the following conditions, for all $x, y, z \in \mathcal{X}$ and $t, s > 0$*

(KM1) $\mathcal{M}(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$;

(KM2) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;

(KM3) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \le \mathcal{M}(x, z, t + s)$;

(KM4) *The function* $\mathcal{M}_{xy} : \mathbb{R}^+ \to [0,1]$ *is left-continuous, where* $\mathcal{M}_{xy}(t) = \mathcal{M}(x,y,t)$ *for each* $t > 0$.

Subsequently, George and Veeramani slightly modified the preceding notion in [4] with the aim of defining a concept of fuzzy metric which induces a Hausdorff topology. Concretely, George and Veeramani replaced axioms **(KM1)** and **(KM4)** in the preceding notion by two more restrictive ones. In addition, they imposed another requirement. So, fuzzy metrics in the George and Veeramani's sense can be defined as follows.

Definition 2. *A GV-fuzzy metric space is a fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *such that* $\mathcal{M}$ *satisfies (in addition) the following conditions:*

(GV1) $\mathcal{M}(x,y,t) > 0$, *for all* $x, y \in \mathcal{X}$ *and* $t > 0$;

(GV2) $\mathcal{M}(x,y,t) < 1$, *for all* $x, y \in \mathcal{X}$, *with* $x \neq y$, *and* $t > 0$;

(GV3) *The function* $\mathcal{M}_{xy} : \mathbb{R}^+ \to [0,1]$ *is right-continuous (and so it is continuous), where* $\mathcal{M}_{xy}(t) = \mathcal{M}(x,y,t)$ *for each* $t > 0$.

Both notions of fuzzy metrics defined above have been studied deeply in the literature for different authors since they were introduced. Indeed, we can currently find works that approach different subjects in this kind of fuzzy metrics such as convergence and Cauchyness, completion, the asymptotic dimension, or the Wijsman topology (see, for instance, [7–13]).

George and Veeramani showed in [4] that given a GV-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, then $\mathcal{M}$ induces a topology $\mathcal{T}_{\mathcal{M}}$ on $\mathcal{X}$, which has the family of open balls $\{B_{\mathcal{M}}(x,r,t) : x \in \mathcal{X}, r \in]0,1[, t > 0\}$ as a base, where $B_{\mathcal{M}}(x,r,t) = \{y \in \mathcal{X} : \mathcal{M}(x,y,t) > 1 - r\}$ for each $x \in \mathcal{X}$. Then, Gregori and Romaguera proved in [14] that the aforementioned topology $\mathcal{T}_{\mathcal{M}}$ is metrizable (see also [15]). Conversely, for each metrizable topology, there exists a fuzzy metric that induces such a topology (see [4,14]). Moreover, it is well-known that these conclusions are retrieved in fuzzy metrics in the sense of Kramosil and Michalek. So, from the topological point of view, fuzzy metrics (in both senses) and classical metrics are the same. Nonetheless, fuzzy metrics show some differences to their classical counterparts in "purely metrics" issues as the fixed point theory, which is currently an active topic of research in fuzzy metric spaces (see, for instance, [16–22]). Even more, the restrictions imposed in their definition of fuzzy metric by George and Veeramani provide a significant difference on completion between GV-fuzzy metrics and fuzzy metrics introduced by Kramosil and Michalek. Indeed, each fuzzy metric space (in the sense of Kramosil and Michalek) admits completion (see Remark 3 in [23]) whereas there exists GV-fuzzy metric spaces which are not completable (see [24]). Moreover, a characterization of those GV-fuzzy metric spaces which are completable was provided in [25], which was slightly modified later in [26]. On account of this last one characterization (see Theorem 1), a GV-fuzzy metric space is completable if and only if three properties are satisfied by each pair of Cauchy sequences (see Theorem 1).

The aim of this paper is to go deep in the study on completion of fuzzy metrics in both sense above detailed. Concretely, we are focused on looking into the impact on the completion of each one of the restrictions imposed by George and Veeramani in their definition of fuzzy metric space. In this direction, we study what are the conclusions on completion of a fuzzy metric space when just one of the restrictions **(GV1)–(GV3)** is required. The study carried out throughout the paper concludes that the aforesaid restrictions are directly related with the conditions demanded in a GV-fuzzy metric space to admit completion attending the characterization of completable GV-fuzzy metric spaces provided in [26].

The remainder of this paper is organized as follows. In Section 2, we recall the main results on completion of a GV-fuzzy metric space that we need. Then, in Section 3 the main results of the paper are expounded. Throughout the paper, as usual, $\mathbb{N}$ and $\mathbb{R}$ denote the set of the positive integers and the real numbers, respectively.

2. Preliminaries

In this section, we compile the necessary definitions and results for completion of fuzzy metrics that will be essential to the remainder of the paper. With this aim, we begin recalling two well-known results.

Lemma 1 (Grabiec [5])**.** *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. Then, the function $\mathcal{M}_{xy} : \mathbb{R}^+ \to [0,1]$ is non-decreasing for each $x, y \in \mathcal{M}$.*

Proposition 1 (George and Veeramani [4])**.** *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space, and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then, $\{x_n\}$ converges to $x \in \mathcal{X}$ (in $\mathcal{T}_{\mathcal{M}}$) if and only if $\lim_n \mathcal{M}(x_n, x, t) = 1$ for all $t > 0$.*

Now, we continue compiling the notion of Cauchy sequence, complete fuzzy metric space, and completion of a fuzzy metric space.

Definition 3 (George and Veeramani [4])**.** *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. We will say that a sequence $\{x_n\}$ in $\mathcal{X}$ is Cauchy if, for each $\varepsilon \in]0,1[$ and $t > 0$, there exists $n_{\varepsilon,t} \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_{\varepsilon,t}$. A fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence converges (in the topology $\mathcal{T}_{\mathcal{M}}$).*

Definition 4 (Gregori and Romaguera [24])**.** *Given two fuzzy metric spaces $(\mathcal{X}, \mathcal{M}, *)$ and $(\mathcal{Y}, \mathcal{N}, \diamond)$, we will say that a mapping f from $\mathcal{X}$ to $\mathcal{Y}$ is an isometry if $\mathcal{M}(x, y, t) = \mathcal{N}(f(x), f(y), t)$, for all $x, y \in \mathcal{X}$ and $t > 0$. $(\mathcal{X}, \mathcal{M}, *)$ and $(\mathcal{Y}, \mathcal{N}, \diamond)$ will be called isometric whenever there exists an isometry from $\mathcal{X}$ to $\mathcal{Y}$.*

Definition 5 (Gregori and Romaguera [24])**.** *Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space. We will say that a complete fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a fuzzy metric completion of $(\mathcal{X}, \mathcal{M}, *)$ if $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$.*

The following is an interesting result concerning the uniqueness of the fuzzy metric completion of a fuzzy metric space.

Proposition 2 (Gregori and Romaguera [24])**.** *If a fuzzy metric space has a fuzzy metric completion, then it is unique up to isometry.*

In [23], was pointed out in Remark 3 that each fuzzy metric admits completion. Indeed, in [23] was provided the construction of a fuzzy metric completion for an arbitrary fuzzy quasi-metric space, a generalization of fuzzy metric in which symmetry (axiom **(KM2)**) is not required. For the sake of completeness, we recall the aforementioned construction in the context of fuzzy metric spaces below.

Let $(\mathcal{X}, \mathcal{M}, *)$ be a fuzzy metric space, and denote by $\mathcal{S}$ the set made up of all Cauchy sequences in $\mathcal{X}$. Define a relation $\sim$ on $\mathcal{S}$ as follows:

$$\{a_n\} \sim \{b_n\} \text{ if and only if } \sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s) = 1, \text{ for all } t > 0,$$

where $\liminf_n \mathcal{M}(a_n, b_n, s)$ denotes the lower limit of the sequence $\{\mathcal{M}(a_n, b_n, s)\}$ of real numbers.

In [23], it was proved that $\sim$ is an equivalence relation and that, if we denote by $\tilde{\mathcal{X}}$ the quotient set $\mathcal{S}/\sim$, then $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ is a fuzzy metric completion of $(\mathcal{X}, \mathcal{M}, *)$, where

$$\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t) = \sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s), \quad (1)$$

for each $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$ and $t > 0$. As usual, in the preceding formula $\{\tilde{a_n}\}$ and $\{\tilde{b_n}\}$ denote the class under the equivalence relation $\sim$ to which the Cauchy sequence $\{a_n\}$ and $\{b_n\}$ in $\mathcal{X}$ belong, respectively.

However, if we adapt the preceding study to the context of fuzzy metrics in the sense of George and Veeramani, the same conclusion is not retrieved. For such an adaptation, we are referring to obtain a GV-fuzzy metric completion of an arbitrary GV-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, i.e., a complete GV-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$ satisfying that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$. In the literature, we can find GV-fuzzy metrics which do not admit a GV-fuzzy metric completion (see [24–26]). So, the next definition makes sense.

Definition 6 (Gregori and Romaguera [25])**.** *We will say that a GV-fuzzy metric space is completable if it admits a GV-fuzzy metric completion.*

Completable GV-fuzzy metric spaces were characterized in [25]. Later, Gregori et al. in [26] slightly modified the statement of such a characterization as follows.

Theorem 1 (Gregori et al. [26])**.** *A GV-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is completable if and only if, for each pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X, the following three conditions are fulfilled:*

- **(c1)** $\lim_n \mathcal{M}(x_n, y_n, t_0) = 1$ *for some* $t_0 > 0$ *implies* $\lim_n \mathcal{M}(x_n, y_n, t) = 1$ *for all* $t > 0$.
- **(c2)** $\lim_n \mathcal{M}(x_n, y_n, t) > 0$ *for all* $t > 0$.
- **(c3)** *The assignment* $t \to \lim_n \mathcal{M}(x_n, y_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$, *provided with the usual topology of* $\mathbb{R}$.

It should be noted that, in [27], it was proved that none of the conditions of the preceding theorem can be obtained from the remaining two. So, these three conditions constitute an independent axiomatic system.

Finally, for the sake of completeness, and for better understanding, we recall the following well-known definition.

Definition 7. *Let A and B subsets of $\mathbb{R}$. A function $f : A \to B$ is said to be left-continuous (right-continuous) at $x_0 \in A$ if, for each $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x_0) - f(x)| < \varepsilon$ whenever $x \in]x_0 - \delta, x_0] \cap A$ ($x \in [x_0, x_0 + \delta[\cap A$).*

3. The Results

This section is devoted to delving into the study on the completion of GV-fuzzy metric spaces. Concretely, we analyze in more detail the impact of each one of the three additional axioms imposed in their definition of fuzzy metric space by George and Veeramani. Concretely, we are focused in studying the consequences on the completion when we impose only one of the aforementioned three axioms to the notion of fuzzy metric space due to Kramosil and Michalek. Such an study is carried out considering one by one of them. First of all, we make some observations on the construction of the completion of a fuzzy metric space detailed in the preceding section.

Attending to Formula (1), to construct the fuzzy metric completion of a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, we use the lower limit of the sequence $\{\mathcal{M}(a_n, b_n, s)\}$, and then the supremum of all $s \in]0, t[$ of $\liminf_n \mathcal{M}(a_n, b_n, s)$ for each $t > 0$. These two facts could avoid, on the one hand, the possibility that would not exist the limit of the sequence $\{\mathcal{M}(a_n, b_n, s)\}$ and that, on the other hand, the fuzzy set constructed being left-continuous on the parameter. So, we wonder if these two considerations are actually necessary. That is, can we find a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ in which the limit of the sequence $\{\mathcal{M}(a_n, b_n, s)\}$ does not exist for two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in $\mathcal{X}$ and some $s > 0$? Even more so, in case that such a limit exists for all $s > 0$, could the assignment $t \to \lim_n \mathcal{M}(a_n, b_n, t)$ not be left-continuous?

In the next example, we tackle the first question proposed above by showing a fuzzy metric space in which there exist two Cauchy sequences such that the aforesaid limit does not exist. Before that, we recall a celebrated example of fuzzy metric defined from a classical one that was introduced in [4].

Consider a metric space $(\mathcal{X}, d)$, and define the fuzzy set $\mathcal{M}_d$ on $\mathcal{X} \times \mathcal{X} \times]0, \infty[$ by

$$\mathcal{M}_d(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for all } x, y \in \mathcal{X} \text{ and } t \in]0, \infty[. \tag{2}$$

Then, $(\mathcal{X}, \mathcal{M}_d, \cdot)$ is a fuzzy metric space, where $\cdot$ denotes the product t-norm. In fact, it is a GV-fuzzy metric space.

Example 1. *Let $\mathcal{X} = \mathbb{R}$ and denote by d_u the usual metric of $\mathbb{R}$, i.e., $d_u(x, y) = |x - y|$, where $|\cdot|$ denotes the absolute value. Define the fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X} \times]0, \infty[$ as follows:*

$$\mathcal{M}(x, y, t) = \begin{cases} \mathcal{M}_{d_u}(x, y, t), & \text{if } 0 < t \leq d_u(x, y); \\ \mathcal{M}_{d_u}(x, y, 2t), & \text{if } t > d_u(x, y). \end{cases} \tag{3}$$

We claim that $(\mathcal{X}, \mathcal{M}, \cdot)$ is a fuzzy metric space. Indeed, it is not hard to check that, for all $x, y \in \mathcal{X}$ and $t > 0$, **(KM1)**, **(KM2)** *and* **(KM4)** *are satisfied. Below, we show that, for all $x, y, z \in \mathcal{X}$ and $t, s > 0$, axiom* **(KM3)** *also holds.*

With this aim, let $x, y, z \in \mathcal{X}$ and $t, s > 0$. We distinguish two possibilities:

1. *Assume $0 < t + s \leq d_u(x, z)$. Then, $\mathcal{M}(x, z, t + s) = \mathcal{M}_{d_u}(x, z, t + s) = \frac{t+s}{t+s+d_u(x,z)}$. Obviously, if $0 < t \leq d_u(x, y)$ and $0 < s \leq d_u(y, z)$* **(KM3)** *is fulfilled due to the fact that, in such a case, we have that*

$$\mathcal{M}(x, z, t + s) = \mathcal{M}_{d_u}(x, z, t + s) \geq$$

$$\geq \mathcal{M}_{d_u}(x, y, t) \cdot \mathcal{M}_{d_u}(y, z, s) = \mathcal{M}(x, y, t) \cdot \mathcal{M}(y, z, s).$$

Now, suppose $d_u(x, y) < t$. Then, $0 < s \leq d_u(y, z)$, and so $\mathcal{M}(x, y, t) = \mathcal{M}_{d_u}(x, y, 2t)$ and $\mathcal{M}(y, z, s) = \mathcal{M}_{d_u}(y, z, s)$. Therefore, we must show that the next inequality holds.

$$\mathcal{M}(x, z, t + s) = \frac{t + s}{t + s + d_u(x, z)} \geq \frac{2ts}{2ts + sd_u(x, y) + 2td_u(y, z) + d_u(x, y)d_u(y, z)} =$$

$$= \frac{2t}{2t + d_u(x, y)} \cdot \frac{s}{s + d_u(y, z)} = \mathcal{M}(x, y, t) \cdot \mathcal{M}(y, z, s).$$

A simple computation brings us to the preceding inequality is fulfilled if and only if the next one is

$$s(t + s)d_u(x, y) + 2t(t + s)d_u(y, z) + (t + s)d_u(x, y)d_u(y, z) \geq 2tsd_u(x, z).$$

Now, by assumption, $0 < s \leq d_u(y,z)$*; then,*

$$s(t+s)d_u(x,y) + 2t(t+s)d_u(y,z) + (t+s)d_u(x,y)d_u(y,z) \geq$$

$$\geq s(t+s)d_u(x,y) + 2t(t+s)d_u(y,z) + (t+s)d_u(x,y)s =$$

$$= 2s(t+s)d_u(x,y) + 2t(t+s)d_u(y,z) > 2tsd_u(x,y) + 2tsd_u(y,z) \geq 2tsd_u(x,z).$$

Then, **(KM3)** *is also satisfied.*

Even more, in an analogous way, **(KM3)** *is proved when* $d_u(y,z) < s$*. Hence,* **(KM3)** *holds for all cases when* $0 < t+s \leq d_u(x,z)$*.*

2. *Assume* $d_u(x,z) \leq t+s$*. Then,* $\mathcal{M}(x,z,t+s) = \mathcal{M}_{d_u}(x,z,2(t+s))$*. Now, taking into account that, for all* $t,s > 0$ *we have, on the one hand,* $\mathcal{M}_{d_u}(x,z,2(t+s)) \geq \mathcal{M}_{d_u}(x,y,2t) * \mathcal{M}_{d_u}(y,z,2s)$*, since* $\mathcal{M}_{d_u}$ *is a fuzzy metric on* $\mathcal{X}$*, and, on the other hand, by definition of* $\mathcal{M}$ *and Lemma 1 successively* $\mathcal{M}(x,y,t) \leq \mathcal{M}_{d_u}(x,y,2t)$ *and* $\mathcal{M}(y,z,s) \leq \mathcal{M}_{d_u}(y,z,2s)$*, we conclude that* **(KM3)** *is also satisfied when* $d_u(x,z) \leq t+s$*.*

Hence, $(\mathcal{X},\mathcal{M},\cdot)$ *is a fuzzy metric space.*

Consider the sequences $\{x_n\}$ *and* $\{y_n\}$*, where* $x_n = \frac{1}{n}$ *and* $y_n = 1 + \frac{(-1)^n}{n}$*, respectively, for all* $n \in \mathbb{N}$*. Then,*

$$d_u(x_n,y_n) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 1 - \frac{2}{n}, & \text{if } n \text{ is odd,} \end{cases} \tag{4}$$

and so we have

$$\mathcal{M}(x_n,y_n,1) = \begin{cases} \frac{1}{1+1}, & \text{if } n \text{ is even;} \\ \frac{2}{2+1-\frac{2}{n}}, & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

Obviously, the limit of the sequence $\{\mathcal{M}(x_n,y_n,1)\}$ *does not exist. Indeed,*

$$\liminf_n \mathcal{M}(x_n,y_n,1) = \frac{1}{2} \neq \frac{2}{3} = \limsup_n \mathcal{M}(x_n,y_n,1).$$

The second question proposed above was actually answered in [26]. Indeed, Example 12 in [26] shows a fuzzy metric space in which we can find two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ for which the limit of the sequence $\{\mathcal{M}(a_n,b_n,t)\}$ exists, for all $t > 0$, but the assignment $t \to \lim_n \mathcal{M}(a_n,b_n,t)$ is not a left-continuous function. For the sake of completeness, we include it below.

Example 2. *Let* $\mathcal{X} =]0,1]$*, and denote again by* d_u *the usual metric on* $\mathbb{R}$ *restricted to* $\mathcal{X}$*. We define on* $\mathcal{X} \times \mathcal{X} \times]0,\infty[$ *the fuzzy set* $\mathcal{M}$ *as follows:*

$$\mathcal{M}(x,y,t) = \begin{cases} \mathcal{M}_{d_u}(x,y,t), & 0 < t \leq d_u(x,y) \\ \mathcal{M}_{d_u}(x,y,2t) \cdot \frac{t-d_u(x,y)}{1-d_u(x,y)} + \mathcal{M}_{d_u}(x,y,t) \cdot \frac{1-t}{1-d_u(x,y)}, & d_u(x,y) < t \leq 1 \\ \mathcal{M}_{d_u}(x,y,2t), & t > 1 \end{cases}$$

Then, $(\mathcal{X},\mathcal{M},\cdot)$ *is a fuzzy metric space. Concretely, it is a GV-fuzzy metric space (see [26]). Additionally, the sequences* $\{a_n\}$ *and* $\{b_n\}$*, where* $a_n = \frac{1}{n}$ *and* $b_n = 1$*, for* $n \in \mathbb{N}$*, are Cauchy sequences in* $(\mathcal{X},\mathcal{M},\cdot)$*, such that the limit of the real sequence* $\{\mathcal{M}(a_n,b_n,t)\}$ *exists for all* $t > 0$*. Nonetheless, the assignment* $t \to \lim_n \mathcal{M}(a_n,b_n,t)$ *is not left-continuous. Indeed,*

$$\lim_n M(a_n,b_n,t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases},$$

which is not a left-continuous function at $t = 1$*.*

Now, we are able to approach the study of the significance of the completion of each one of the restrictions required in the definition of a GV-fuzzy metric space. With this aim, we define three new concepts of fuzzy metric spaces in which just one of the aforesaid conditions is imposed. We have summarized them in the next definition.

Definition 8. *For $i = 1, 2, 3$, we will say that a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is a GVi-fuzzy metric space if, for all $x, y \in \mathcal{X}$ and $t > 0$, $\mathcal{M}$ satisfies condition* **(GVi)**.

As usual, a GVi-fuzzy metric completion of a GVi-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is a complete GVi-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$ such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$, for $i = 1, 2, 3$. If confusion does not arise, we will just say that, for $i = 1, 2, 3$, a GVi-fuzzy metric space is completable if it admits a GVi-fuzzy metric completion.

3.1. Completion of GV1-Fuzzy Metric Spaces

This subsection is devoted to detailing the completion of fuzzy metric spaces that satisfy condition **GV1** in the definition of GV-fuzzy metric space. So, we focus on the $GV1$-completion of a $GV1$-fuzzy metric space, i.e., a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$, such that $\mathcal{M}(x, y, t) > 0$ for each $x, y \in \mathcal{X}$ and $t > 0$. The next example shows that, in general, $GV1$-fuzzy metric spaces do not admit $GV1$-fuzzy metric completion.

Example 3. *Let $\mathcal{X} =]0, 1]$, and define the fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X} \times]0, \infty[$ by $\mathcal{M}(x, y, t) = 1 - d_u(x, y)$ (again, d_u denotes the usual metric of $\mathbb{R}$ restricted to $\mathcal{X}$). An easy computation shows that $(\mathcal{X}, \mathcal{M}, \mathfrak{L})$ is a $GV1$-fuzzy metric space, where $\mathfrak{L}$ denotes the Lukasievicz t-norm, i.e., $a \mathfrak{L} b = \max\{a + b - 1, 0\}$ for each $a, b \in [0, 1]$. Below, we prove by contradiction that $(\mathcal{X}, \mathcal{M}, \mathfrak{L})$ does not admit a $GV1$-fuzzy metric completion.*

Assume that $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a $GV1$-fuzzy metric completion of $(\mathcal{X}, \mathcal{M}, \mathfrak{L})$. Then, there exists an isometry $f : \mathcal{X} \to \mathcal{Y}$ such that $f(\mathcal{X})$ is a dense subset of $\mathcal{Y}$. Taking into account that the sequence $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$, where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, we conclude that $\{f(x_n)\}$ is a Cauchy sequence in $\mathcal{Y}$. So, $\{f(x_n)\}$ converges to some $y \in \mathcal{Y}$. Therefore, for each $t > 0$ we obtain

$$\mathcal{M}(x_n, 1, 2t) = \mathcal{N}(f(x_n), f(1), 2t) \geq \mathcal{N}(f(x_n), y, t) \diamond \mathcal{N}(y, f(1), t),$$

for all $n \in \mathbb{N}$. Now, by definition, $\mathcal{M}(x_n, 1, 2t) = 1 - d_u(x_n, 1) = \frac{1}{n}$, and then $\frac{1}{n} \geq \mathcal{N}(f(x_n), y, t) \diamond \mathcal{N}(y, f(1), t)$ for all $n \in \mathbb{N}$. Since $\lim_n \mathcal{N}(f(x_n), y, t) = 1$, taking limits as n tends to ∞ on both sides of the preceding inequality, we conclude that $\mathcal{N}(y, f(1), t) = 0$ for all $t > 0$, which contradicts the fact that $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a $GV1$-fuzzy metric space.

On account of the preceding example, we will say that a $GV1$-fuzzy metric space is completable if it admits a $GV1$-fuzzy metric completion. So, the following theorem characterizes those $GV1$-fuzzy metric spaces which are completable.

Theorem 2. *A $GV1$-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is completable if and only if, for each pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$, we have that $\liminf_n \mathcal{M}(x_n, y_n, t) > 0$ for all $t > 0$.*

Proof. For the direct implication, let $(\mathcal{X}, \mathcal{M}, *)$ be a completable $GV1$-fuzzy metric space. Then, there exist a complete $GV1$-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$, such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$ for the isometry f. Now, consider a pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$. Then, $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy sequences in $\mathcal{Y}$, and so, since $(\mathcal{Y}, \mathcal{N}, \diamond)$ is complete, there exist $x, y \in \mathcal{Y}$ such that $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to x and y, respectively. Moreover, for each $t > 0$, we have

$$\mathcal{M}(x_n, y_n, t) = \mathcal{N}(f(x_n), f(y_n), t) \geq \mathcal{N}\left(f(x_n), x, \frac{t}{3}\right) \diamond \mathcal{N}\left(x, y, \frac{t}{3}\right) \diamond \mathcal{N}\left(y, f(y_n), \frac{t}{3}\right),$$

for all $n \in \mathbb{N}$. Taking the lower limit as n tends to ∞ on the both sides of the previous inequality we have, due to $\liminf_n \mathcal{N}(f(x_n), x, \frac{t}{3}) = \liminf_n \mathcal{N}(y, f(y_n), \frac{t}{3}) = 1$, that $\liminf_n \mathcal{M}(x_n, y_n, t) \geq \mathcal{N}(x, y, \frac{t}{3}) > 0$, since $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a $GV1$-fuzzy metric space. So, the direct implication has been showed.

Conversely, let $(\mathcal{X}, \mathcal{M}, *)$ be a $GV1$-fuzzy metric space such that for each pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$, we have that $\liminf_n \mathcal{M}(x_n, y_n, t) > 0$ for all $t > 0$. Consider the fuzzy metric completion $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ provided in Section 2. Obviously, if we show that $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ is a $GV1$-fuzzy metric space, the proof is over. Then, we need to show that, for each $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$, we have that $\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t) > 0$ for all $t > 0$. So, let $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$, and fix an arbitrary $t > 0$. Therefore, by Formula (1), we have

$$\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t) = \sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s),$$

and taking into account that $\{a_n\}$ and $\{b_n\}$ is a pair of Cauchy sequences, by our assumption, we obtain $\liminf_n \mathcal{M}(a_n, b_n, s) > 0$ for all $s \in]0, t[$. Hence, $\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t) > 0$, and we conclude that $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ is a $GV1$-fuzzy metric space. □

3.2. Completion of GV2-Fuzzy Metric Spaces

In this subsection, we are focused in carrying out the study provided in the previous one for fuzzy metrics that satisfy now condition **(GV2)** of the notion of GV-fuzzy metric space. Concretely, for $GV2$-fuzzy metric spaces, we are referring to fuzzy metric spaces $(\mathcal{X}, \mathcal{M}, *)$ satisfying $\mathcal{M}(x, y, t) < 1$ for each $x, y \in \mathcal{X}$, with $x \neq y$, and $t > 0$. Again, in general, $GV2$-fuzzy metric spaces do not admit $GV2$-fuzzy metric completion, as the next example, which was given in [25], shows.

Example 4. *Consider two strictly increasing sequences $\{a_n\}$ and $\{b_n\}$ of positive real numbers converging to 1 with respect to the usual topology of $\mathbb{R}$, such that $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\} = \emptyset$. Define the fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X} \times]0, \infty[$, where $\mathcal{X} = \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}$, as follows:*

$$\mathcal{M}(x, y, t) = \begin{cases} 1, & \text{if } x = y; \\ \min\{x, y\}, & \text{if } x, y \in A \text{ or } x, y \in B; \\ \min\{x, y, t\}, & \text{otherwise.} \end{cases} \quad (6)$$

On account of Example 2 in [25] $(\mathcal{X}, \mathcal{M}, \wedge)$ is a GV-fuzzy metric space, where $\wedge$ denotes the minimum t-norm (i.e., $a \wedge b = \min\{a, b\}$ for each $a, b \in [0, 1]$). Therefore, $(\mathcal{X}, \mathcal{M}, \wedge)$ is a $GV2$-fuzzy metric space.

We will show by contradiction that $(\mathcal{X}, \mathcal{M}, \wedge)$ does not admit $GV2$-fuzzy metric completion. So, assume there exists a complete $GV2$-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$ such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of Y. So, there exists an isometry $f : \mathcal{X} \to \mathcal{Y}$ such that $f(\mathcal{X})$ is a dense subset of $\mathcal{Y}$. Taking into account that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in $(\mathcal{X}, \mathcal{M}, \wedge)$ (it is easy to verify), then $\{f(a_n)\}$ and $\{f(b_n)\}$ are so in $(\mathcal{Y}, \mathcal{N}, \diamond)$. Therefore, there exists $a, b \in \mathcal{Y}$ such that $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to a and b, respectively.

Observe that, by definition of $\mathcal{M}$, for each $\varepsilon \in]0, 1[$ we can find $n_0 \in \mathbb{N}$ such that $\mathcal{M}(a_n, b_n, 1) > 1 - \varepsilon$ for all $n \geq n_0$. Moreover, given $0 < t < 1$, there exists $n_1 \in \mathbb{N}$ such that $\mathcal{M}(a_n, b_n, t) = t$ for all $n \geq n_1$. Therefore, we conclude that

$$\lim_n \mathcal{M}(a_n, b_n, t) = \begin{cases} t, & \text{if } 0 < t < 1; \\ 1, & \text{otherwise.} \end{cases}$$

On the one hand, fix $0 < t < 1$ *and let* $t < s < 1$. *Then,*

$$\mathcal{M}(a_n, b_n, s) = \mathcal{N}(f(a_n), f(b_n), s) \geq \mathcal{N}\left(a, f(a_n), \frac{s-t}{2}\right) \diamond \mathcal{N}(a, b, t) \diamond \mathcal{N}\left(f(b_n), b, \frac{s-t}{2}\right)$$

and, taking limits as n tends to ∞ *on both sides of the preceding inequality, we obtain* $\mathcal{N}(a, b, t) \leq s < 1$.

On the other hand, we fix $t > 1$ *and have*

$$\mathcal{N}(a, b, t) \geq \mathcal{N}\left(a, f(a_n), \frac{t-1}{2}\right) \diamond \mathcal{N}(f(a_n), f(b_n), 1) \diamond \mathcal{N}\left(f(b_n), b, \frac{t-1}{2}\right) =$$

$$= \mathcal{N}\left(a, f(a_n), \frac{t-1}{2}\right) \diamond \mathcal{M}(a_n, b_n, 1) \diamond \mathcal{N}\left(f(b_n), b, \frac{t-1}{2}\right).$$

Again, taking limits as n tends to ∞ *on the above inequality, we obtain* $\mathcal{N}(a, b, t) = 1$. *It leads us to a contradiction since, in such a case,* $\mathcal{N}$ *does not satisfy axiom* **(GV2)**. *Observe that axiom* **(GV2)** *implies that if* $\mathcal{N}(a, b, t_0) = 1$ *for some* $t_0 > 0$, *then* $\mathcal{N}(a, b, t) = 1$ *for all* $t > 0$.

The next theorem characterizes those $GV2$-fuzzy metric spaces that admit a $GV2$-fuzzy metric completion.

Theorem 3. *A* $GV2$*-fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *is completable if and only if, for each pair of Cauchy sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* $\mathcal{X}$, *the condition* **(c1)** *in Theorem 1 is satisfied, i.e.,* $\lim_n \mathcal{M}(x_n, y_n, t_0) = 1$ *for some* $t_0 > 0$ *implies* $\lim_n \mathcal{M}(x_n, y_n, t) = 1$ *for all* $t > 0$.

Proof. For the direct implication, suppose that $(\mathcal{X}, \mathcal{M}, *)$ is a $GV2$-fuzzy metric space that admits completion, and consider an arbitrary pair of Cauchy sequences in $\mathcal{X}$, $\{x_n\}$ and $\{y_n\}$, such that $\lim_n \mathcal{M}(x_n, y_n, t_0) = 1$ for some $t_0 > 0$. Taking into account that, for each $n \in \mathbb{N}$, it is satisfied $\mathcal{M}(x_n, y_n, t) \geq \mathcal{M}(x_n, y_x, t_0)$ for all $t > t_0$, we just have to see that $\lim_n \mathcal{M}(x_n, y_n, t) = 1$ for all $t \in]0, t_0[$.

First, of all, by our assumption, $(\mathcal{X}, \mathcal{M}, *)$ admits $GV2$-fuzzy metric completion, so there exists a complete $GV2$-fuzzy metric $(\mathcal{Y}, \mathcal{N}, \diamond)$ such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$. Assume that $f : \mathcal{X} \to \mathcal{Y}$ is an isometry satisfying $f(\mathcal{X})$ is a dense subset of $\mathcal{Y}$. Then, $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy sequences in $\mathcal{Y}$, so they are convergent in $\mathcal{Y}$ to some x and y, respectively. Then, we fixe $\delta > 0$ and obtain

$$\mathcal{N}(x, y, t_0 + \delta) \geq \mathcal{N}\left(x, f(x_n), \frac{\delta}{2}\right) \diamond \mathcal{N}(f(x_n), f(y_n), t_0) \diamond \mathcal{N}\left(f(y_n), y, \frac{\delta}{2}\right) =$$

$$= \mathcal{N}\left(x, f(x_n), \frac{\delta}{2}\right) \diamond \mathcal{M}(x_n, y_n, t_0) \diamond \mathcal{N}\left(f(y_n), y, \frac{\delta}{2}\right),$$

and taking limits as n tends to ∞ on the both sides of the inequality we obtain $\mathcal{N}(x, y, t_0 + \delta) = 1$. Therefore, since $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a $GV2$-fuzzy metric space, by axiom **(GV2)**, we conclude that $\mathcal{N}(x, y, s) = 1$ for all $s > 0$ and, consequently, $x = y$. So, we can write

$$\mathcal{M}(x_n, y_n, t) = \mathcal{N}(f(x_n), f(y_n), t) \geq \mathcal{N}\left(f(x_n), x, \frac{t}{2}\right) \diamond \mathcal{N}\left(y, f(y_n), \frac{t}{2}\right).$$

By taking limits as n tends to ∞ on the preceding inequality, we obtain $\lim_n \mathcal{M}(x_n, y_n, t) = 1$ and, so, the direct implication is proved.

Conversely, suppose that for each pair of Cauchy sequences in $(\mathcal{X}, \mathcal{M}, *)$ satisfies Condition **(c1)** in Theorem 1. Consider the fuzzy metric completion $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ provided in

Section 2. We must show that, for each $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$, with $\{\tilde{a_n}\} \neq \{\tilde{b_n}\}$, we have that $\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t) < 1$ for all $t > 0$. We will make this demonstration by contradiction.

Assume that there exist $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$, with $\{\tilde{a_n}\} \neq \{\tilde{b_n}\}$, such that $\tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t_0) = 1$ for some $t_0 > 0$. Then,

$$1 = \tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t_0) = \sup_{0<s<t_0} \liminf_n \mathcal{M}(a_n, b_n, s).$$

Taking into account that, for each $x, y \in \mathcal{X}$, we have $\mathcal{M}(x, y, s) \leq \mathcal{M}(x, y, t)$, whenever $0 < s < t$, we obtain $\liminf_n \mathcal{M}(a_n, b_n, t_0) = 1$. Therefore, by definition of lower limit, we obtain $\lim_n \mathcal{M}(a_n, b_n, t_0) = 1$. Now, our assumption ensures that $\lim_n \mathcal{M}(a_n, b_n, t) = 1$ for all $t > 0$. Thus, $\sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s) = 1$, for all $t > 0$, and, by definition of $\sim$, we conclude that $\{a_n\} \sim \{b_n\}$, a contradiction with our assumption on $\{\tilde{a_n}\} \neq \{\tilde{b_n}\}$. Hence, $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ is a $GV2$-fuzzy metric space. □

3.3. Completion of GV3-Fuzzy Metric Spaces

Finally, in this subsection we study the completion of $GV3$-fuzzy metric spaces, i.e., those fuzzy metrics where the function $\mathcal{M}_{xy} : \mathbb{R}^+ \to]0, 1]$ defined by $\mathcal{M}_{xy}(t) = \mathcal{M}(x, y, t)$ for each $t > 0$, is (right-)continuous. First, we will show, below, that the $GV3$-fuzzy metric space of Example 2 does not admit $GV3$-completion.

Example 5. *Let $(\mathcal{X}, \mathcal{M}, *)$ be the GV-fuzzy metric space of Example 2. Then, it is a $GV3$-fuzzy metric space. Suppose that $(\mathcal{X}, \mathcal{M}, *)$ admits a $GV3$-fuzzy metric completion, then there exists a complete $GV3$-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$, such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of Y. Let an isometry $f : \mathcal{X} \to \mathcal{Y}$, such that $f(\mathcal{X})$ is a dense subset of $\mathcal{Y}$.*

Taking into account that $\{a_n\}$ and $\{b_n\}$, where $a_n = \frac{1}{n}$ and $b_n = 1$, for $n \in \mathbb{N}$, are Cauchy sequences in $(\mathcal{X}, \mathcal{M}, \cdot)$, we conclude that $\{f(a_n)\}$ and $\{f(b_n)\}$ are Cauchy sequences in $(\mathcal{Y}, \mathcal{N}, \diamond)$. Therefore, there exist $a, b \in \mathcal{Y}$ such that $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to a and b, respectively. So, fix $t > 0$ and consider an arbitrary $\delta \in]0, t[$. Then, on the one hand,

$$\mathcal{N}(a, b, t + \delta) \geq \mathcal{N}\left(a, f(a_n), \frac{\delta}{2}\right) \diamond \mathcal{N}(f(a_n), f(b_n), t) \diamond \mathcal{N}\left(f(b_n), b, \frac{\delta}{2}\right) =$$

$$= \mathcal{N}\left(a, f(a_n), \frac{\delta}{2}\right) \diamond \mathcal{M}(a_n, b_n, t) \diamond \mathcal{N}\left(f(b_n), b, \frac{\delta}{2}\right).$$

Taking limits on the previous inequality we obtain that $\mathcal{N}(a, b, t + \delta) \geq \lim_n \mathcal{M}(a_n, b_n, t)$, for each $\delta \in]0, t[$.

On the other hand,

$$\mathcal{M}(a_n, b_n, t) = \mathcal{N}(f(a_n), f(b_n), t) \geq$$

$$\geq \mathcal{N}\left(f(a_n), a, \frac{\delta}{2}\right) \diamond \mathcal{N}(a, b, t - \delta) \diamond \mathcal{N}\left(b, f(b_n), \frac{\delta}{2}\right).$$

Taking limits now in both sides of the above inequality, we obtain $\lim_n \mathcal{M}(a_n, b_n, t) \geq \mathcal{N}(a, b, t - \delta)$ for each $\delta \in]0, t[$. So,

$$\mathcal{N}(a, b, t + \delta) \geq \lim_n \mathcal{M}(a_n, b_n, t) \geq \mathcal{N}(a, b, t - \delta), \textit{ for all } \delta \in]0, t[.$$

On account that $(\mathcal{Y}, \mathcal{N}, \diamond)$ is a $GV3$-fuzzy metric space, we have that the function $\mathcal{N}_{ab} : \mathbb{R}^+ \to [0, 1]$ is both left-continuous and right-continuous, where $\mathcal{N}_{ab}(t) = \mathcal{N}(a, b, t)$ for all

$t > 0$. *Thus, we conclude that* $\mathcal{N}(a,b,t) = \lim_n \mathcal{M}(a_n, b_n, t)$ *for each* $t > 0$. *This fact becomes a contradiction, since*

$$\mathcal{N}(a,b,t) = \lim_n \mathcal{M}(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases},$$

which is not a left-continuous function.

After showing that $GV3$-fuzzy metrics do not admit, in general, $GV3$-fuzzy metric completion, we provide a characterization of completable $GV3$-fuzzy metric spaces in the next theorem.

Theorem 4. *A* $GV3$*-fuzzy metric space* $(\mathcal{X}, \mathcal{M}, *)$ *is completable if and only if, for each pair of Cauchy sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* $\mathcal{X}$*, the assignment* $t \to \liminf_n \mathcal{M}(x_n, y_n, t)$ *for each* $t > 0$ *is a continuous function.*

Proof. Suppose that $(\mathcal{X}, \mathcal{M}, *)$ is a completable $GV3$-fuzzy metric space. Then, there exists a complete $GV3$-fuzzy metric space $(\mathcal{Y}, \mathcal{N}, \diamond)$, such that $(\mathcal{X}, \mathcal{M}, *)$ is isometric to a dense subset of $\mathcal{Y}$. Denote by f such an isometry, and consider a pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$. Then, $\{f(x_n)\}$ and $\{f(y_n)\}$ are also Cauchy sequences in $\mathcal{Y}$ and, since $(\mathcal{Y}, \mathcal{N}, \diamond)$ is complete, we can find $x, y \in \mathcal{Y}$ such that $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to x and y, respectively. We will see that $\liminf_n \mathcal{M}(x_n, y_n, t) = \mathcal{N}(x, y, t)$ for all $t > 0$.

Fix $t > 0$ and let an arbitrary $\delta \in]0, t[$. On the one hand,

$$\mathcal{N}(x, y, t+\delta) \geq \mathcal{N}\left(x, f(x_n), \frac{\delta}{2}\right) \diamond \mathcal{N}(f(x_n), f(y_n), t) \diamond \mathcal{N}\left(f(y_n), y, \frac{\delta}{2}\right) =$$

$$= \mathcal{N}\left(x, f(x_n), \frac{\delta}{2}\right) \diamond \mathcal{M}(x_n, y_n, t) \diamond \mathcal{N}\left(f(y_n), y, \frac{\delta}{2}\right),$$

and taking the lower limit in the preceding inequality, we obtain $\mathcal{N}(x, y, t+\delta) \geq \liminf_n \mathcal{M}(x_n, y_n, t)$. On the other hand,

$$\mathcal{M}(x_n, y_n, t) = \mathcal{N}(f(x_n, f(y_n), t) \geq$$

$$\geq \mathcal{N}\left(f(x_n), x, \frac{\delta}{2}\right) \diamond \mathcal{N}(x, y, t-\delta) \diamond \mathcal{N}\left(x, f(x_n), \frac{\delta}{2}\right).$$

Taking limits on both sides of the previous inequality, we obtain $\liminf_n \mathcal{M}(x_n, y_n, t) \geq \mathcal{N}(x, y, t-\delta)$. So, $\mathcal{N}(x, y, t+\delta) \geq \liminf_n \mathcal{M}(x_n, y_n, t) \geq \mathcal{N}(x, y, t-\delta)$ for all $\delta \in]0, t[$. Due to the function $\mathcal{N}_{xy} : \mathbb{R}^+ \to [0,1]$ being continuous, where $\mathcal{N}_{xy}(t) = \mathcal{N}(x, y, t)$ for all $t > 0$, we conclude that $\liminf_n \mathcal{M}(x_n, y_n, t) = \mathcal{N}(x, y, t)$. Additionally, taking into account that $t > 0$ was arbitrary, we obtain such an equality for all $t > 0$. Hence, the assignment $t \to \liminf_n \mathcal{M}(x_n, y_n, t)$ for each $t > 0$ is a continuous function due to $\mathcal{N}_{xy}$.

Conversely, let $(\mathcal{X}, \mathcal{M}, *)$ be a $GV3$-fuzzy metric space such that for each pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$, the assignment $t \to \liminf_n \mathcal{M}(x_n, y_n, s)$ for each $t > 0$ is a continuous function. We will see that the fuzzy metric completion $(\tilde{\mathcal{X}}, \tilde{\mathcal{M}}, *)$ of $(\mathcal{X}, \mathcal{M}, *)$ provided in Section 2 is a $GV3$-fuzzy metric space. With this aim, let $\{\tilde{a_n}\}, \{\tilde{b_n}\} \in \tilde{\mathcal{X}}$, and we will show that the function $\tilde{\mathcal{M}}_{\{\tilde{a_n}\},\{\tilde{b_n}\}} : \mathbb{R}^+ \to [0,1]$ is right-continuous, where $\tilde{\mathcal{M}}_{\{\tilde{a_n}\},\{\tilde{b_n}\}}(t) = \tilde{\mathcal{M}}(\{\tilde{a_n}\}, \{\tilde{b_n}\}, t)$ for all $t > 0$.

Recall that $\tilde{\mathcal{M}}(\{\tilde{a_n}\},\{\tilde{b_n}\},t) = \sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s)$ for all $t > 0$. Moreover, by our assumption, the assignment $t \to \liminf_n \mathcal{M}(a_n, b_n, t)$ for each $t > 0$ is a continuous function and, due to for each $x, y \in \mathcal{X}$, we have $\mathcal{M}(x, y, s) \leq \mathcal{M}(x, y, t)$ when $0 < s < t$, and we obtain $\sup_{0<s<t} \liminf_n \mathcal{M}(a_n, b_n, s) = \liminf_n \mathcal{M}(a_n, b_n, t)$ for all $t > 0$. Therefore, we conclude that $\tilde{\mathcal{M}}_{\{\tilde{a_n}\},\{\tilde{b_n}\}}(t) = \liminf_n \mathcal{M}(a_n, b_n, t)$ for all $t > 0$, and so the above function $\tilde{\mathcal{M}}_{\{\tilde{a_n}\},\{\tilde{b_n}\}} : \mathbb{R}^+ \to [0, 1]$ is (right-)continuous. □

4. Conclusions and Future Work

This paper analyses in more detail the completion of fuzzy metric spaces in the sense of George and Veeramani. Specifically, we study how each axiom imposed in the definition of George and Veeramani, separately, affects to the completion of a fuzzy metric space. The main results provided in the paper conclude that each one of the aforesaid axioms is straight related with one condition included in the characterization of completable fuzzy metrics provided in [26] (see Theorem 1). Moreover, different examples justify and illustrate the study performed. In addition, attending to Theorems 2–4 we obtain the following immediate corollary, which provides a slightly different characterization of completable GV-fuzzy metric spaces.

Corollary 1. *A GV-fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ is completable if and only if, for each pair of Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X ,the following three conditions are fulfilled:*

(c1) $\lim_n \mathcal{M}(x_n, y_n, t_0) = 1$ *for some* $t_0 > 0$ *implies* $\lim_n \mathcal{M}(x_n, y_n, t) = 1$ *for all* $t > 0$.

(c2′) $\liminf_n \mathcal{M}(x_n, y_n, t) > 0$ *for all* $t > 0$.

(c3′) *The assignment* $t \to \liminf_n \mathcal{M}(x_n, y_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$, *provided with the usual topology of* $\mathbb{R}$.

Concerning the future work to continue the research performed in this article, we propose two distinguished lines. On the one hand, the study of the completion of fuzzy metric spaces, both for the Kramosil and Michalek sense, as well as for the George and Veeramani's one, when different notions of convergence or Cauchyness are under consideration. On the other hand, it could be an interesting issue to study the impact of each one of the restrictions imposed by George and Veeramani in their definition of fuzzy metric space when considering another ones of the differences between fuzzy metrics GV-fuzzy metrics. For instance, in [4], it was proved that in a GV-fuzzy metric space each closed ball is a closed set, whereas Example 3.8 in [28] provided a fuzzy metric space for which such a property is not satisfied.

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