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Article

Fixed Points of Local Actions of Lie Groups on Real and Complex 2-Manifolds

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Abstract: I discuss old and new results on fixed points of local actions by Lie groups G on real and complex 2-manifolds, and zero sets of Lie algebras of vector fields. Results of E. Lima, J. Plante and C. Bonatti are reviewed.

Keywords: Lie group; Lie algebra; vector fields; fixed point index; dynamical systems

1. Introduction

Classical results of Poincaré [1] (1885), Hopf [2] (1925) and Lefschetz [3] (1937) yield the archetypal fixed point theorem for Lie group actions:

Theorem 1. *Every flow on a compact manifold of non-zero Euler characteristic has a fixed point.*

Here the Lie group is the group $\mathbb{R}$ of real numbers.

The earliest papers I have found on fixed points for actions of other non-discrete Lie group are those of P. A. Smith [4] (1942) and H. Wang [5] (1952). Then came Armand Borel's landmark paper of 1956:

Theorem 2 (Borel [6])**.** *If H is a solvable, irreducible affine algebraic group over an algebraically closed field $\mathbb{K}$, every algebraic action of H on a complete algebraic variety over $\mathbb{K}$ has a fixed point.*

Over the field of complex numbers, completeness is equivalent to compactness in the classical topology, and complete nonsingular varieties are compact Kähler manifolds.

In 1973, A. Sommese [7] extended Borel's theorem to solvable holomorphic actions on compact Kähler manifolds with first Betti number 0. In contrast to the results below, these have no explicit restrictions on dimensions or Euler characteristics.

2. Actions and Local Actions

If $f\colon\; A \to B$ denotes a map, its domain is $\mathcal{D}f := A$ and its range is $\mathcal{R}f := f(A)$.

Let g, f denote maps. Regardless of their domains and ranges, the composition $g \circ f$ is defined as the map $x \mapsto g(f(x))$ whose domain, perhaps empty, is $f^{-1}(\mathcal{D}g)$. The associative law holds for these compositions: The maps $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have the same domain

$$D := \{x \in \mathcal{D}f\colon\; f(x) \in \mathcal{D}g, \quad g(f(x)) \in \mathcal{D}f\},$$

and

$$x \in D \implies (h \circ g)(f(x)) = h((g \circ f)(x)).$$

Henceforth M denotes a manifold with boundary ∂M, and G denotes a connected Lie group with Lie algebra $\mathfrak{g}$.

A *local homeomorphism* f on M is a homeomorphism between open subsets of M. The set of these homeomorphisms is denoted by $\mathsf{LH}(M)$.

A *local action* of G on M is a triple (α, G, M), where $\alpha\colon\; G \to \mathsf{LH}(M)$ is a function having the following properties:

- The set $\Omega(\alpha) := \big\{(g, p) \in G \times M\colon\; p \in \mathcal{D}\alpha(g)\big\}$ is an open neighborhood of $\{e_G\} \times M$.
- The *evaluation map*

  $$\mathsf{ev}_\alpha\colon\; \Omega(\alpha) \to M, \qquad (g, p) \mapsto \alpha(g) \cdot p$$

  is continuous.
- $\alpha(e_G)$ is the identity map of M.
- The maps $\alpha(fg) \circ \alpha(h)$ and $\alpha(f) \circ \alpha(gh)$ agree on the intersection of their domains.
- $\alpha(g^{-1}) = \alpha(g)^{-1}$.

Notation of α may be omitted.

When $\Omega(\alpha) = G \times M$ the local action is a *global action*. If G is simply connected and M is compact, every local action extends to a unique global action.

When α has been specified, we define the *fixed-point sets*

$$\mathsf{Fix}(g) := \{x \in \mathcal{D}g\colon\; g(x) = x\},$$
$$\mathsf{Fix}(G) := \bigcap_{g \in G} \mathsf{Fix}(g)$$

The local action is *effective* if $\mathsf{Fix}(g) \neq M$ for all $g \neq e_G$.

A *local flow* is a local action $(\Psi, \mathbb{R}, M)$. In this case we set $\Psi_t := \Psi(t)$ and identify Ψ with the indexed family of $\{\Psi_t\}_{t \in \mathbb{R}}$ of local maps in M. If (α, G, M) is a local action, to every $X \in \mathfrak{g}$ there corresponds a local flow $(\alpha^*\Psi, \mathbb{R}, M)$ defined in the following. Consider X as a 1-parameter subgroup of G, i.e., a homomorphism $X\colon \mathbb{R} \to G$, and set $\alpha^*\Psi = \{\alpha(X(t))\}_{t\in\mathbb{R}}$. The local flow induced by a C^1 vector field X on M tangent to ∂M is denoted by $\Phi^X := \{\Phi_t^X\}_{t\in\mathbb{R}}$.

A *block* for a local flow Ψ (a Ψ-block) is a compact $K \subset \mathsf{Fix}(\Psi)$ having a precompact open neighborhood $U \subset M$, termed *isolating*, such that $\mathsf{Fix}(\Psi) \cap \overline{U} = K$. When this holds, the *index* $\mathsf{i}(\Psi, U)$ of Ψ in U is defined as the fixed point index of $\Psi_t|U\colon U \to M$ for sufficiently small $t > 0$, as defined by Dold [8] (see also Brown [9] and Granas and Dugundji [10]). This integer depends only on K, and we set $\mathsf{i}_K(\Psi) := \mathsf{i}(\Psi, U)$. When $\mathsf{i}_K(\Psi) \neq 0$ then K is *essential*. If K is a block for the local flow Φ^X of a vector field X, an equivalent definition of $\mathsf{i}_K(\Phi^X)$ as the Poincaré–Hopf index of X at K is given in Section 4.

3. Fixed Points of Local Actions on Surfaces

In the rest of this section M denotes a real closed surface (compact with empty boundary) and G is a connected Lie group acting continuously on M.

An important role is played by the group $ST_\circ(n, \mathbb{R})$, the solvable group of real, upper triangulable $n \times n$ matrices with positive diagonal entries. In his pioneering 1964 paper, E. Lima [11] constructed fixed-point free actions of $ST_\circ(2, \mathbb{R})$ on the compact 2-cell and the 2-sphere, but he also showed that every abelian Lie group action on a compact surface M of nonzero Euler characteristic $\chi(M)$ has a fixed point. These results were extended in 1986 by Plante:

Theorem 3 (Plante [12])**.** *Let M be a compact surface whose boundary may be nonempty.*

(i) *$ST_\circ(2, \mathbb{R})$ has a fixed-point free action on M.*

(ii) *If $\chi(M) \neq 0$, every action on M by a connected nilpotent Lie group has a fixed point.*

Many facts about existence of fixed points for continuous actions on closed surfaces can be derived from the results of M. Belliart summarized in the following theorem. If $H \subset GL(n, \mathbb{F})$ denotes a group of matrices, PH denotes the quotient of H by its center.

Theorem 4 (Belliart [13])**.** *There is a fixed-point free action of G on M iff one of the following conditions* (a), (b), (c) *holds:*

(a) *$\chi(M) > 0$ and G is solvable but not nilpotent.*

(b) *$\chi(M) < 0$ and G has $ST_\circ(2, \mathbb{R})$ as a quotient.*

(c) *$\chi(M) \geq 0$, G is semisimple, and either:*

- **(i)** *G has $PSL(2, \mathbb{R})$ as a quotient, or*
- **(ii)** *$\chi(M) > 0$, $\partial M = \varnothing$, and G has as a quotient one of the groups $PSL(3, \mathbb{R})$, $PSL(2, \mathbb{C})$ or $PSO(3)$.*

A Lie algebra is *supersolvable* if it is faithfully represented as upper triangular real matrices. A Lie group is supersolvable if its Lie algebra is.

Theorem 5.

(i) *$ST_\circ(3,\mathbb{R})$ has an effective analytic action on M.*

(ii) *If G has an effective, fixed-point free analytic action on M, then $\chi(M) \geq 0$, with equality when G is a supersolvable and $\partial M = \varnothing$.*

Part (i) and the first conclusion in (ii) are due to Turiel [14]. The second conclusion in (ii) is due to Hirsch and Weinstein [15].

The following result gives upper and lower bounds on the number of fixed points of analytic actions of $ST_\circ(3,\mathbb{R})$:

Proposition 1 (Hirsch [16], Cor. 17, Thm 22)**.**

(i) *Let M have genus g. For every $k \in \mathbb{N}$ there is an effective analytic action β of $ST_\circ(3,\mathbb{R})$ on M such that:*

$$\#\,\mathsf{Fix}(\beta) = \begin{cases} 2(g+k+1) & \text{if } M \text{ is orientable,} \\ g+k & \text{if } M \text{ is nonorientable and } g \geq 1. \end{cases} \tag{1}$$

(ii) *If G is not supersolvable and has an effective analytic action on M,*

$$0 \leq \#\,\mathsf{Fix}(G) \leq \chi(M) \leq 2.$$

Question. Can the right hand side of Equation (1) can be lowered?

4. Indices of Vector Fields

Let $\mathcal{V}(M)$ denote the vector space of vector fields (continuous sections of the tangent bundle) on a smooth manifold M, endowed with the compact open topology.

The *zero set* of $X \in \mathcal{V}(M)$ is

$$\mathsf{Z}(X) := \{p \in M\colon\ X_p = 0\}.$$

A *block* for X (an X-*block*) is a compact, relatively open set $K \subset \mathsf{Z}(X)$. Every sufficiently small open neighborhood $U \subset M$ of K is *isolating* for X, meaning its closure $\overline{U}$ is compact and $\mathsf{Z}(X) \cap \overline{U} = K$. This implies that U is isolating for every vector field Y sufficiently close to X.

Let K be an X-block. When K is finite, the *Poincaré–Hopf index* of X at K, and in U, is the integer $i_K^{PH}(X) = \mathsf{i}^{PH}(X,U)$ defined as follows. For each $p \in K$ choose an open set $W \subset U$ meeting K only at p, such that W is the domain of a C^1 chart

$$\phi\colon\ W \approx W' \subset \mathbb{R}^n, \quad \phi(p) = p'.$$

The transform of X by ϕ is

$$X' := T\phi \circ X \circ \phi^{-1} \in \mathcal{V}(W').$$

There is a unique map of pairs

$$F_p\colon\ (W', 0) \to \mathbb{R}^n, 0)$$

that expresses X' by the formula

$$X'_x = \big(x, F_p(x)\big) \in \{x\} \times \mathbb{R}^n, \qquad (x \in W').$$

Noting that $F^{-1}(0) = p$, we define $\mathsf{i}_p^{PH}(X) \in \mathbb{Z}$ as the degree of the map defined for any sufficiently small $\epsilon > 0$ as

$$\mathbf{S}^{n-1} \to \mathbf{S}^{n-1}, \quad u \mapsto \frac{F_p(\epsilon u)}{\|F_p(\epsilon u)\|}$$

where $\|\cdot\|$ is the norm defined by any Riemannian metric on M. This degree is independent of ϵ and the chart ϕ, by standard properties of the degree function. Therefore the integer

$$\mathsf{i}_K^{PH}(X) = i^{PH}(X, U) := \begin{cases} \sum_{p \in K} i_p^{PH}(X) & \text{if } K \neq \varnothing, \\ 0 & \text{if } K = \varnothing. \end{cases}$$

is well defined and depends only on X and K.

The *index* of an arbitrary X-block K is the integer $\mathsf{i}_K(X) := \mathsf{i}(X, U)$ defined as the Poincaré–Hopf index of any sufficiently close approximation to X having only finitely many zeros in U [17].

This number is independent of U and is stable under perturbations of X. The X-block K is *essential* when $\mathsf{i}_K(X) \neq 0$. This implies $\mathsf{Z}(X) \cap K \neq \varnothing$ because every isolating neighborhood of K meets $\mathsf{Z}(X)$.

Theorem 6 (Poincaré–Hopf)**.** *If M is compact, $\mathsf{i}(X, M) = \chi(M)$ for all continuous vector fields X on M.*

For calculations of the index in more general settings see Morse [18], Pugh [19], Gottlieb [20], Jubin [21].

Theorem 7 (Bonatti [22])**.** *Assume M is a real manifold of dimension ≤ 4 with empty boundary, and X, Y are analytic vector fields on M such that $[X, Y] = 0$. Then $\mathsf{Z}(Y)$ meets every essential X-block [23].*

This implies certain local actions of 2-dimensional abelian Lie groups have fixed points. The results below are analogs for local actions of nonabelian Lie groups.

Theorem 8 (Hirsch [24])**.** *Let M be a real surface, perhaps non-compact or having non-empty boundary. Let G be a connected nilpotent Lie group and (α, G, M) an effective local action. Assume given a continuous local action of G on M, and let K be an essential block for the local flow induced by a 1-parameter subgroup. Then $\mathsf{Fix}(G) \cap K \neq \varnothing$.*

This implies Plante's result, Theorem 3(ii).

Corollary 1. *Let G, M and X be as in Theorem 8.*

(i) *If $\Gamma \subset M$ is a compact attractor for Φ^X and $\chi(\Gamma) \neq 0$, then $\mathsf{Fix}(G) \cap \Gamma \neq \varnothing$.*

(ii) *If Φ^X has n essential blocks, then $\mathsf{Fix}(G)$ has n components.*

The counter-example in Theorem 3(i) show that fixed point results for broader classes of Lie groups, including supersolvable groups, need stronger hypotheses.

Henceforth M denotes either a real or complex 2-manifold, the corresponding ground field being $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{V}^\omega(M)$ denote the Lie algebra of vector fields on M that are analytic over $\mathbb{F}$. If $Y \in \mathcal{V}^\omega(M)$, $T\Phi^Y$ denotes the induced local flow on the tangent vector bundle of M.

Assume $X, Y \in \mathcal{V}^\omega(M)$. We say that Y *tracks* X if there exists a continuous map

$$f\colon\ M \to \mathbb{F}, \quad f^{-1}(0) = \mathsf{Z}(X), \quad [Y, X] = fX.$$

Equivalently: if $p \in M$ and $t \in \mathbb{R}$ there exists $g(t, p) \in \mathbb{F}$ such that:

$$\Phi^Y_t(p) = q(t) \implies T\Phi^Y_t(X_p) = g(t, p)X_{q(t)}.$$

For real M this means Φ^Y_t sends orbits of $X|\mathcal{D}\Phi^Y_t$ to orbits of $X|\mathcal{R}\Phi^Y_t$.

Let $\mathcal{G} \subset \mathcal{V}^\omega(M)$ denote a Lie algebra of vector fields. We say that $\mathcal{G}$ *tracks* X provided each $Y \in \mathcal{G}$ tracks X.

Example 1. If X spans an ideal in $\mathcal{G}$ then $\mathcal{G}$ tracks X, and the converse holds if $\mathcal{G}$ is finite dimensional.

Example 2. The set $\{Y \in \mathcal{V}^\omega(M)\colon\ Y \text{ tracks } X\}$ is a Lie algebra that tracks X.

The following result will be proved in a forthcoming paper [25]; a preliminary version is in [26].

Theorem 9. *Assume $X \in \mathcal{V}^\omega(M)$, K is an essential X-bloc, and $\mathcal{G} \subset \mathcal{V}^\omega(M)$ tracks X. Let one of the following conditions hold:*

(a) *M is complex,*

(b) *M is real and $\mathcal{G}$ is supersolvable.*

Then $\mathsf{Z}(\mathcal{G}) \cap K \neq \varnothing$.

Example 3. Here is a simple example in which the hypotheses hold. For M take complex projective 3-space. Let G be the solvable complex Lie group of unimodular $4\times\ 4$ upper triangular complex matrices. The natural action of G on $\mathbb{C}^4$ induces an effective holomorphic action of G on M, mapping the Lie algebra of G isomorphically onto a Lie algebra $\mathcal{G} \subset \mathcal{V}^\omega(M)$. Let $X \in \mathcal{G}$ have the block $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ in its upper right hand corner and all other elements equal to zero. X spans an ideal, the triple commutator subalgebra $\mathcal{G}'''$. The X-block $K := \mathsf{Z}(X)$, a copy of $\mathbb{C}\mathbf{P}^1$, is essential because $\chi(M) = 3$; and $\mathsf{Z}(\mathcal{G})$ is a singleton in $\mathsf{Z}(X)$.

Conflicts of Interest

The author declares no conflict of interest.

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