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Article

Some Aspects of Extended Kinetic Equation

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Abstract: Motivated by the pathway model of Mathai introduced in 2005 [*Linear Algebra and Its Applications*, *396*, 317–328] we extend the standard kinetic equations. Connection of the extended kinetic equation with fractional calculus operator is established. The solution of the general form of the fractional kinetic equation is obtained through Laplace transform. The results for the standard kinetic equation are obtained as the limiting case.

Keywords: kinetic equation; fractional derivatives and integrals; Mellin convolution; G-function; Mittag-Lefler function

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1. Introduction

The chemical evolution of a star like sun could be effectively explained by kinetic equations. The kinetic equations explain the rate of change of chemical composition of a star in terms of the thermonuclear reaction rates for destruction and production of the species involved. An arbitrary reaction is characterized by the rate of change $\frac{\mathrm{d}N}{\mathrm{d}t}$ of a time dependent quantity $N(t)$ between the destruction rate d and production rate p. Here the destruction or production at time t depends not only on $N(t)$ but also on the past history $N(\tau), \tau < t$ of the variable N. This may be formally represented by following [1,2]

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -d(N_t) + p(N_t) \tag{1}$$

where N_t denotes the function defined by $N_t(t^*) = N(t - t^*), t^* > 0$. It should be noted that d and p are functionals and Equation (1) represents a functional-differential equation. If we consider a simplified form of Equation (1) we could consider the decay rate of a radio-active substance which is given by a homogeneous differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N \tag{2}$$

where N is the number density of the radio-active substance and λ is the decay constant. The solution of this differential equation with initial condition $N = N_0$ at $t = 0$ is

$$N(t) = N_0 \mathrm{e}^{-\lambda t} \tag{3}$$

If we consider a more general form of the differential Equation (2) for the decay rate of a radio-active substance as

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N^{\alpha} \tag{4}$$

we have the solution of the form

$$N_{\alpha}(t) = N_0[1 + a(\alpha - 1)t]^{-\frac{1}{\alpha-1}} \tag{5}$$

where a is a constant. One may get the solution in Equation (3) from Equation (5) as $\alpha \to 1$. These types of problems arise in many experimental situations where one needs to switch from one family of functions to another family. In 2005, Mathai [3,4] introduced the pathway model by which one can switch among three different families of functions, say, type-1 beta families, type-2 beta families and gamma families. We get three different functional forms by varying the pathway parameter α. The pathway model in the real scalar case is defined as

$$f(x) = \begin{cases} c_1|x|^{\gamma}[1 - a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}},\ 1 - a(1-\alpha)|x|^{\delta} > 0, \alpha < 1 \\ c_2|x|^{\gamma}[1 + a(\alpha-1)|x|^{\delta}]^{-\frac{\eta}{\alpha-1}}, -\infty < x < \infty, \alpha > 1 \\ c_3|x|^{\gamma}\mathrm{e}^{-a\eta|x|^{\delta}}, -\infty < x < \infty, \alpha \to 1 \end{cases} \tag{6}$$

where $a > 0, \delta > 0, \gamma > 0, \eta > 0$. c_1, c_2 and c_3 are the normalizing constants when we consider the functions as statistical densities. The three different functional forms are respectively generalized type-1 beta, generalized type-2 beta and generalized gamma forms. By writing $1 - \alpha = -(\alpha - 1)$, the generalized type-2 beta form can be obtained from generalized type-1 beta form. Both generalized type-1 beta form and generalized type-2 beta form reduce to generalized gamma form as $\alpha \to 1$.

Due to this switching property, the pathway model has been widely used in many areas. In this paper, we use the pathway model to extend kinetic equations. The present paper is organized as follows: In the next section we discuss the extended kinetic equation and its solution with a brief description of the extended reaction rate probability integral. Connection of the extended kinetic equation with fractional calculus is examined in Section 3. In Section 4 we try to solve fractional kinetic equations and their various generalizations. Concluding remarks are given in Section 5.

2. Extended Kinetic Equations

The following discussion is based on [1,2]. If we consider a production and destruction of nuclei in the proton-proton chain reaction, we can describe it by the equation

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -\sum_{j} N_i N_j \langle\sigma v\rangle_{ij} + \sum_{k,l\neq i} N_k N_l \langle\sigma v\rangle_{kl} \tag{7}$$

where N_i is the number density of the species i over time. Here the summation is taken over all reactions, productions or destructions of the species i. The number density N_i of the species i can be expressed by the relation $N_i = \rho N_A \frac{X_i}{A_i}$ where ρ is the mass density, X_i is the mass abundance, N_A is the Avogadro number and A_i is the mass of species i in mass units. The mean life time $\tau_j(i)$ of species i for destruction by species j is given by the relation [2]

$$\lambda_j(i) = \frac{1}{\tau_j(i)} = N_j \langle \sigma v \rangle_{ij} = \rho N_A \frac{X_j}{A_j} \langle \sigma v \rangle_{ij} \tag{8}$$

where $\lambda_j(i)$ is the decay rate of i for interaction with j. $\langle \sigma v \rangle_{ij}$ denotes the reaction probability for an interaction involving species i and j defined as

$$\langle \sigma v \rangle_{ij} = \sqrt{\frac{2}{\mu}} \int_0^\infty E^{\frac{1}{2}} f(E) \sigma(E) \mathrm{d}E \tag{9}$$

where μ is the reduced mass of the particles given by $\mu = \frac{m_1 m_2}{m_1 + m_2}$, $E = \frac{\mu v^2}{2}$ is the kinetic energy of the particles in the center of mass system. Consider the cross section $\sigma(E)$ for low-energy non-resonant reactions given by

$$\sigma(E) = \sum_{\nu=0}^{2} \frac{S^{(\nu)}(0)}{\nu!} E^{\nu-1} \mathrm{e}^{-2\pi \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{Z_i Z_j e^2}{\hbar E^{\frac{1}{2}}}} \quad \left(\frac{E}{B} \ll 1\right) \tag{10}$$

where Z_i and Z_j are the atomic numbers of the nuclei i and j, e is the quantum of electric charge, $\hbar$ is the Planck's quantum of action, B the nuclear barrier height, $S(E)$ is the cross-section factor which is a slowly varying function of energy over a limited energy range and which can be characterized depending on the nuclear reaction. The density function of the relative velocities of the nuclei for a non-degenerate and non-relativistic gas is assumed to be Maxwell-Boltzmann given as

$$f_{MBD}(E) \mathrm{d}E = 2\pi \left(\frac{1}{\pi k T}\right)^{\frac{3}{2}} e^{-\frac{E}{kT}} \sqrt{E} \mathrm{d}E \tag{11}$$

By substituting Equation (11) and Equation (10) in Equation (9) the reaction probability $\langle \sigma v \rangle_{ij}$ is obtained as

$$\langle \sigma v \rangle_{ij} = \left(\frac{8}{\pi \mu}\right)^{\frac{1}{2}} \left(\frac{1}{kT}\right)^{\frac{3}{2}} \sum_{\nu=0}^{2} \frac{S^{(\nu)}(0)}{\nu!} \int_0^\infty E^{\nu} \mathrm{e}^{-\frac{E}{kT} - 2\pi \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{Z_i Z_j e^2}{\hbar E^{\frac{1}{2}}}} \mathrm{d}E \tag{12}$$

If a deviation from the thermodynamic equilibrium with regard to their velocities is considered then it results in a deviation from the Maxwell-Boltzmann velocity. In this context, we consider a more general density function than Maxwell-Boltzmann density function by using the pathway model defined in Equation (6). The pathway energy density function has the form

$$f_{PD}(E) \mathrm{d}E = \frac{2\pi (\alpha - 1)^{\frac{3}{2}}}{(\pi k T)^{\frac{3}{2}}} \frac{\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{1}{\alpha-1} - \frac{3}{2}\right)} \sqrt{E} \left[1 + (\alpha - 1) \frac{E}{kT}\right]^{-\frac{1}{\alpha-1}} \mathrm{d}E \tag{13}$$

for $\alpha > 1$, $\frac{1}{\alpha-1} - \frac{3}{2} > 0$. Replacing the Maxwell-Boltzmann density Function (11) by the pathway energy density Equation (13), we get the extended thermonuclear reaction probability integral in the form

$$\begin{aligned} \langle \sigma v \rangle_{ij} = {} & \left(\frac{8}{\pi \mu}\right)^{\frac{1}{2}} \left(\frac{\alpha - 1}{kT}\right)^{\frac{3}{2}} \frac{\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{1}{\alpha-1} - \frac{3}{2}\right)} \sum_{\nu=0}^{2} \frac{S^{(\nu)}(0)}{\nu!} \\ & \times \int_0^\infty E^{\nu} \left[1 + (\alpha - 1) \frac{E}{kT}\right]^{-\frac{1}{\alpha-1}} \exp\left[-2\pi \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \frac{Z_i Z_j e^2}{\hbar E^{\frac{1}{2}}}\right] \mathrm{d}E \end{aligned} \tag{14}$$

Putting $y = \frac{E}{kT}$ and $x = 2\pi\left(\frac{\mu}{2kT}\right)^{\frac{1}{2}} \frac{Z_i Z_j e^2}{\hbar}$ we get

$$\langle \sigma v \rangle_{ij} = \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} (\alpha - 1)^{\frac{3}{2}} \frac{\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{1}{\alpha-1} - \frac{3}{2}\right)} \sum_{\nu=0}^{2} \left(\frac{1}{kT}\right)^{-\nu+\frac{1}{2}} \frac{S^{(\nu)}(0)}{\nu!} I_{1\alpha}(\nu, 1, x, \frac{1}{2}) \tag{15}$$

where

$$I_{1\alpha}(\nu, 1, x, \frac{1}{2}) = \int_0^\infty y^\nu [1 + (\alpha - 1)y]^{-\frac{1}{\alpha-1}} \mathrm{e}^{-xy^{-\frac{1}{2}}} \mathrm{d}y \tag{16}$$

Following [5], by taking the Mellin transform of Equation (16) and simplifying, we get

$$M_{I_{1\alpha}}(s) = \frac{\Gamma(s)\Gamma(\nu + 1 + \frac{s}{2})\Gamma\left(\frac{1}{\alpha-1} - \nu - 1 - \frac{s}{2}\right)}{(\alpha - 1)^{\nu+1+\frac{s}{2}}\Gamma\left(\frac{1}{\alpha-1}\right)} \tag{17}$$

where $\Re(s) > 0, \Re(\nu + 1 + \frac{s}{2}) > 0, \Re\left(\frac{1}{\alpha-1} - \nu - 1 - \frac{s}{2}\right) > 0$. By taking the inverse Mellin transform we get,

$$\begin{aligned} I_{1\alpha}(\nu, 1, x, \frac{1}{2}) =& \frac{1}{(\alpha - 1)^{\nu+1}\Gamma\left(\frac{1}{\alpha-1}\right)} \frac{1}{2\pi i} \int_L \Gamma(s)\Gamma(\nu + 1 + \frac{s}{2}) \\ & \times \Gamma\left(\frac{1}{\alpha - 1} - \nu - 1 - \frac{s}{2}\right) [x(\alpha - 1)^{\frac{1}{2}}]^{-s} \mathrm{d}s \end{aligned} \tag{18}$$

where L is a suitable contour which separates the poles of $\Gamma(s)$ and $\Gamma(\nu + 1 + \frac{s}{2})$ from the poles of $\Gamma\left(\frac{1}{\alpha-1} - \nu - 1 - \frac{s}{2}\right)$. Putting $s = 2s'$ and using Legendre's duplication formula [6]

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), z \in \mathbf{C} \tag{19}$$

we get

$$\begin{aligned} I_{1\alpha}(\nu, 1, x, \frac{1}{2}) =& \frac{\pi^{-\frac{1}{2}}}{(\alpha - 1)^{\nu+1}\Gamma\left(\frac{1}{\alpha-1}\right)} \frac{1}{2\pi i} \int_{L'} \Gamma(s')\Gamma(\frac{1}{2} + s')\Gamma(\nu + 1 + s') \\ & \times \Gamma\left(\frac{1}{\alpha - 1} - \nu - 1 - s'\right) \left[\frac{(\alpha - 1)x^2}{4}\right]^{-s'} \mathrm{d}s' \end{aligned} \tag{20}$$

$$= \frac{\pi^{-\frac{1}{2}}}{(\alpha - 1)^{\nu+1}\Gamma\left(\frac{1}{\alpha-1}\right)} G^{3,1}_{1,3}\left(\frac{(\alpha - 1)x^2}{4}\Big|^{2-\frac{1}{\alpha-1}+\nu}_{0,\frac{1}{2},\nu+1}\right) \tag{21}$$

where $G^{3,1}_{1,3}(.)$ is the G-function originally introduced by C.S. Meijer in 1936, see [5,7,8]. The $G^{3,1}_{1,3}(.)$ used in Equation (21) converges for all $\frac{(\alpha-1)x^2}{4}, x \neq 0$. The contour line L' appearing in the integral in Equation (20) is $c - i\infty$ to $c + i\infty$ for $0 < c < \frac{1}{\alpha-1} - \nu - 1$ so that all the poles of $\Gamma(s'), \Gamma(\frac{1}{2} + s')$ and $\Gamma(\nu + 1 + s')$ lie to the left and all the poles of $\Gamma\left(\frac{1}{\alpha-1} - \nu - 1 - s'\right)$ lie to the right. $G^{3,1}_{1,3}(.)$ is evaluated as the sum of the residues at the poles of $\Gamma(s'), \Gamma(\frac{1}{2} + s')$ and $\Gamma(\nu + 1 + s')$.

In most of the cases the nuclear factor $S^{(\nu)}(0)$ used in Equation (15) is approximately constant across the fusion window. Hence taking $S^{(\nu)}(0) = 0$ for $\nu = 1$ and $\nu = 2$ and taking $S^0(0) = S(0)$ we get

$$\langle \sigma v \rangle_{ij} = \left(\frac{8(\alpha - 1)}{\mu kT}\right)^{\frac{1}{2}} \frac{1}{\pi\Gamma\left(\frac{1}{\alpha-1} - \frac{3}{2}\right)} S(0) G^{3,1}_{1,3}\left(\frac{(\alpha - 1)x^2}{4}\Big|^{2-\frac{1}{\alpha-1}}_{0,\frac{1}{2},1}\right) \tag{22}$$

The following derivations are adapted from [9]. From the Mellin-Barnes representation of the G-function, $G^{3,1}_{1,3}(\frac{(\alpha-1)x^2}{4})$ appearing in Equation (20) with $\nu = 0$, the poles of $\Gamma(s')$ are $s' = 0, -1, -2, \ldots$; the poles of $\Gamma\left(\frac{1}{2}+s'\right)$ are $s' = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$; and the poles of $\Gamma(1+s')$ are $s' = -1, -2, -3, \ldots$. Here the poles of $\Gamma(s')$ and $\Gamma(1+s')$ will coincide at all points except at $s' = 0$ and hence the pole $s' = 0$ is a pole of order 1, $s' = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ are each of order 1 and $s' = -1, -2, -3, \ldots$ are each of order 2. The sum of residues corresponding to the pole $s' = 0$ is given by

$$R_1 = \sqrt{\pi}\Gamma\left(\frac{1}{\alpha-1} - 1\right) \tag{23}$$

The sum of the residues corresponding to the poles $s' = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ is

$$\begin{aligned} R_2 &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\Gamma(-\frac{1}{2}-r)\Gamma(\frac{1}{2}-r)\Gamma(\frac{1}{\alpha-1}-\frac{1}{2}+r)\left[\frac{(\alpha-1)x^2}{4}\right]^{-\frac{1}{2}+r} \\ &= -2\pi\Gamma\left(\frac{1}{\alpha-1}-\frac{1}{2}\right)\left[\frac{(\alpha-1)x^2}{4}\right]^{\frac{1}{2}} {}_1F_2\left(\frac{1}{\alpha-1}-\frac{1}{2};\ \frac{3}{2},\frac{1}{2};\ -\frac{(\alpha-1)x^2}{4}\right) \end{aligned} \tag{24}$$

where ${}_1F_2$ is the hypergeometric function defined by

$${}_1F_2(a;\ b,c;\ x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r(c)_r}\frac{x^r}{r!}$$

where

$$(a)_r = \begin{cases} a(a+1)\cdots(a+r-1) & \text{if } r \geq 1,\ a \neq 0 \\ 1 & \text{if } r = 0 \end{cases}$$

The sum of the residues corresponding to poles $s' = -1, -2, -3, \ldots.$ of order 2 can be obtained as follows:

$$\begin{aligned} R_3 &= \sum_{r=0}^{\infty} \lim_{s'\to -1-r} \frac{\partial}{\partial s'}\Bigg[(s'+1+r)^2\Gamma(1+s')\Gamma(s')\Gamma\left(\frac{1}{2}+s'\right) \\ &\quad \times \Gamma\left(\frac{1}{\alpha-1}-1-s'\right)\left(\frac{(\alpha-1)x^2}{4}\right)^{-s'}\Bigg] \\ &= \sum_{r=0}^{\infty} \lim_{s'\to -1-r} \frac{\partial}{\partial s'}\left[\frac{\Gamma^2(2+s'+r)\Gamma\left(\frac{1}{2}+s'\right)\Gamma\left(\frac{1}{\alpha-1}-1-s'\right)}{(s'+r)^2(s'+r-1)^2\cdots(s'+1)^2 s'}\left(\frac{(\alpha-1)x^2}{4}\right)^{-s'}\right] \\ &= \sum_{r=0}^{\infty} \lim_{s'\to -1-r} \frac{\partial}{\partial s'}\Phi(s') \end{aligned} \tag{25}$$

where

$$\Phi(s') = \frac{\Gamma^2(2+s'+r)\Gamma\left(\frac{1}{2}+s'\right)\Gamma\left(\frac{1}{\alpha-1}-1-s'\right)}{(s'+r)^2(s'+r-1)^2\cdots(s'+1)^2 s'}\left(\frac{(\alpha-1)x^2}{4}\right)^{-s'}$$

We have

$$\frac{\partial}{\partial s'}\Phi(s') = \Phi(s')\frac{\partial}{\partial s'}[\ln(\Phi(s')]$$

$$\begin{aligned}\ln\Phi(s') &= 2\ln\left[\Gamma(2+s'+r)\right] + \ln\left[\Gamma\left(\frac{1}{2}+s'\right)\right] + \ln\left[\Gamma\left(\frac{1}{\alpha-1}-1-s'\right)\right]\\ &\quad - s'\ln\left(\frac{(\alpha-1)x^2}{4}\right) - 2\ln(s'+r) - \cdots - 2\ln(s'+1) - \ln(s')\\ \frac{\partial}{\partial s'}[\ln(\Phi(s')] &= 2\Psi(2+s'+r) + \Psi\left(\frac{1}{2}+s'\right) + \Psi\left(\frac{1}{\alpha-1}-1-s'\right)\\ &\quad - \ln\left(\frac{(\alpha-1)x^2}{4}\right) - \frac{2}{s'+r} - \frac{2}{s'+r-1} - \cdots - \frac{2}{s'+1} - \frac{1}{s'}\end{aligned}$$

$$\begin{aligned}\lim_{s'\to-1-r}\{\frac{\partial}{\partial s'}\ln[\Phi(s')]\} &= \Psi\left(-\frac{1}{2}-r\right) + \Psi\left(\frac{1}{\alpha-1}+r\right) + \Psi(1+r)\\ &\quad + \Psi(2+r) - \ln\left(\frac{(\alpha-1)x^2}{4}\right)\end{aligned} \tag{26}$$

where $\Psi(z)$ is a Psi function or digamma function (see Mathai [5]) and $\Psi(1) = -\gamma$, $\gamma = 0.5772156649\ldots$ is Euler's constant. Now

$$\lim_{s'\to-1-r}\Phi(s) = \frac{(-1)^{1+r}2\sqrt{\pi}\Gamma\left(\frac{1}{\alpha-1}+r\right)}{\left(\frac{3}{2}\right)_r r!(1+r)!}\left(\frac{(\alpha-1)x^2}{4}\right)^{1+r} \tag{27}$$

Then by using Equations (25)–(27) we get,

$$\begin{aligned}R_3 &= \sum_{r=0}^{\infty}\frac{(-1)^{1+r}2\sqrt{\pi}\Gamma\left(\frac{1}{\alpha-1}+r\right)}{\left(\frac{3}{2}\right)_r r!(1+r)!}\left(\frac{(\alpha-1)x^2}{4}\right)^{1+r}\\ &\quad\times\left[\Psi\left(-\frac{1}{2}-r\right) + \Psi\left(\frac{1}{\alpha-1}+r\right) + \Psi(1+r) + \Psi(2+r) - \ln\left(\frac{(\alpha-1)x^2}{4}\right)\right]\\ &= \left(\frac{2\sqrt{\pi}(\alpha-1)x^2}{4}\right)\sum_{r=0}^{\infty}\left(\frac{(\alpha-1)x^2}{4}\right)^r\left[A_r - \ln\left(\frac{(\alpha-1)x^2}{4}\right)\right]B_r\end{aligned} \tag{28}$$

where

$$A_r = \Psi\left(-\frac{1}{2}-r\right) + \Psi\left(\frac{1}{\alpha-1}+r\right) + \Psi(1+r) + \Psi(2+r) \tag{29}$$

and

$$B_r = \frac{(-1)^r\Gamma\left(\frac{1}{\alpha-1}+r\right)}{\left(\frac{3}{2}\right)_r r!(1+r)!} \tag{30}$$

Thus the series representation for the reaction probability is

$$\begin{aligned}\langle\sigma v\rangle_{ij} &= \left[\frac{8(\alpha-1)}{\mu kT}\right]^{\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{\alpha-1}-\frac{3}{2}\right)}S(0)\left\{\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{\alpha-1}-1\right)\right.\\ &\quad - 2\Gamma\left(\frac{1}{\alpha-1}-\frac{1}{2}\right)\left[\frac{(\alpha-1)x^2}{4}\right]^{\frac{1}{2}}{}_1F_2\left(\frac{1}{\alpha-1}-\frac{1}{2};\ \frac{3}{2},\frac{1}{2};\ -\frac{(\alpha-1)x^2}{4}\right)\\ &\quad\left. + \frac{2(\alpha-1)x^2}{4\sqrt{\pi}}\sum_{r=0}^{\infty}\left(\frac{(\alpha-1)x^2}{4}\right)^r\left[A_r - \ln\left(\frac{(\alpha-1)x^2}{4}\right)\right]B_r\right\}\end{aligned} \tag{31}$$

where A_r and B_r are as defined in Equations (29) and (30). For detailed theory of extended reaction rates and its series representations see Haubold and Kumar [10,11], Kumar and Haubold [12].

The following discussion is adapted from [1]. The solution of the differential Equation (4) with initial condition $N_i(t) = 1$ when $t = 0$ is

$$N_i^{\alpha}(t)\mathrm{d}t = [1 + (\alpha - 1)c_i t]^{-\frac{1}{\alpha-1}}\mathrm{d}t \tag{32}$$

When c_i in Equation (32) is a constant, the total number of reactions in the time interval $0 \le t \le t_0$ is obtained as

$$\int_0^{t_0} N_i^{\alpha}(t)\mathrm{d}t = \int_0^{t_0} [1 + (\alpha - 1)c_i t]^{-\frac{1}{\alpha-1}}\mathrm{d}t = \frac{1}{c_i(2-\alpha)}\left\{[1 + (\alpha - 1)c_i t_0]^{-\frac{2-\alpha}{\alpha-1}} - 1\right\} \tag{33}$$

Now $[1 + (\alpha - 1)c_i t_0]^{-\frac{2-\alpha}{\alpha-1}} - 1$ is the probability that the lifetime of species i is $\le t_0$ when t follows a distribution with density function

$$f(t) = c_i(2-\alpha)[1 + (\alpha - 1)c_i t]^{-\frac{1}{\alpha-1}}, 0 \le t < \infty, c_i > 0, 1 < \alpha < 2 \tag{34}$$

or we have

$$N_i(t) = \frac{f(t)}{c_i(2-\alpha)} \tag{35}$$

Equation (34) is the Tsallis statistics for $\alpha > 1$, see [13,14] which can also be seen as a particular case of the pathway model Equation (6) for $\alpha > 1$. If c_i in Equation (32) is a function of time, say $c_i(t)$, then it should be replaced by $\int c_i(t)\mathrm{d}t$. When $c_i = c_i(t) = d_i t$ where $d_i > 0$ is independent of t, then in this case $\int c_i(t)\mathrm{d}t = \frac{d_i t^2}{2}$, then

$$N_i(t) = \frac{\Gamma\left(\frac{1}{\alpha-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)}\left[\frac{\pi}{2d_i(\alpha-1)}\right]^{\frac{1}{2}} g(t) \tag{36}$$

where

$$g(t) = \frac{\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{1}{\alpha-1} - \frac{1}{2}\right)}\left[\frac{2d_i(\alpha-1)}{\pi}\right]^{\frac{1}{2}} [1 + (\alpha - 1)\frac{d_i t^2}{2}]^{-\frac{1}{\alpha-1}}, 0 \le t < \infty, d_i > 0, 1 < \alpha < 2 \tag{37}$$

The density in Equation (34) is the lifetime density of the destruction of the species i, with the expected mean value

$$\mathbf{E}(t) = \frac{1}{c_i(3-2\alpha)} \tag{38}$$

where $\mathbf{E}(\cdot)$ is the expected value of $(\cdot)$. The mean value of the lifetime density function given in Equation (37) is

$$\mathbf{E}(t) = \frac{\Gamma\left(\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{1}{\alpha-1} - \frac{1}{2}\right)}\left[\frac{2(\alpha-1)}{\pi d_i}\right]^{\frac{1}{2}} \frac{1}{2-\alpha} \tag{39}$$

Now as $\alpha \to 1$ we get the expected mean lifetime

$$\mathbf{E}(t) = \left(\frac{2}{\pi d_i}\right)^{\frac{1}{2}} \tag{40}$$

of the lifetime density function

$$g^*(t) = \left(\frac{2d_i}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{d_i t^2}{2}}, 0 \le t < \infty, d_i > 0 \tag{41}$$

considered by Haubold and Mathai [1].

From the lifetime density function given in Equation (34) and the mean lifetime Equation (38) we can infer that

1. the expected lifetime of the species depends on the value of c_i and α. As $c_i(3-2\alpha)$ increases the expected lifetime decreases and vice versa.
2. $\frac{1}{c_i(2-\alpha)}f(t)\Delta(t)$ can be interpreted as the amount of the net destruction in a small time interval $\Delta(t)$. As the net destruction is faster the lifetime becomes smaller.

3. Connection of Extended Kinetic Equation to Fractional Calculus

Let $\Phi_1(t_1)$ be an integrable function as defined in Equation (34) and $\Phi_2(t_2)=\theta(t)$ be any integrable function, then by the Mellin convolution property, we have

$$\Phi(u)=\int_t \frac{1}{t}\Phi_1(t)\Phi_2(\frac{u}{t})\mathrm{d}t=\int_t \frac{1}{t}\Phi_1(\frac{u}{t})\Phi_2(t)\mathrm{d}t \tag{42}$$

Then $\Phi(u)$, after substituting $\Phi_1(t_1)$ and $\Phi_2(t_2)$, takes the form

$$\Phi(u)=c_i(2-\alpha)\int_{t=0}^{\infty}\frac{1}{t}[1+(\alpha-1)c_i\frac{u}{t}]^{-\frac{1}{\alpha-1}}\theta(t)\mathrm{d}t \tag{43}$$

for $\alpha>1$ which can be considered as a generalized fractional Kober type operator of an integrable function $\theta(t)$. Here as $\alpha\to 1_-$ we have

$$\Phi(u)=c_i(2-\alpha)\int_{t=0}^{\infty}t^{-1}\mathrm{e}^{-c_i\frac{u}{t}}\theta(t)\mathrm{d}t \tag{44}$$

More general cases can be seen in the paper Mathai and Haubold [15].

4. Fractional Kinetic Equation and Its Solution

The following discussion is based on [16,17]. In Equation (4), if instead of an ordinary classical integral we use a fractional integral, we get the reaction equation as

$$N(t)-N_0=-c^{\nu}{}_0D_t{}^{-\nu}N(t),\nu>0 \tag{45}$$

where ${}_0D_t{}^{-\nu}N(t)$ represents the Riemann-Liouville fractional integral defined as

$${}_0D_t{}^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-u)^{\nu-1}f(u)\mathrm{d}u,\nu>0 \tag{46}$$

with ${}_0D_t^0f(t)=f(t)$. Taking Laplace transform, simplifying and then taking the inverse Laplace transform one gets

$$N(t)=N_0\sum_{k=0}^{\infty}\frac{(-1)^k(ct)^{k\nu}}{\Gamma(1+k\nu)}=N_0E_{\nu}(-c^{\nu}t^{\nu}) \tag{47}$$

where $E_\nu(-c^\nu t^\nu)$ is the Mittag-Leffler function, introduced by M. G. Mittag-Leffler in 1903 [18] as

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \beta \in \mathbf{C}, \Re(\beta) > 0, z \in \mathbf{C} \tag{48}$$

If we consider a generalization of the fractional kinetic equation considered by Mathai, Haubold and Saxena [16,17] in the form

$$N(t) - N_0 t^{\omega-1} E^{\gamma}_{\nu,\omega}(-c^\nu t^\nu) = -c^\nu{}_0 D_t^{-\nu} N(t), \nu > 0, \omega > 0, \gamma > 0 \tag{49}$$

where $E^{\gamma}_{\nu,\omega}(.)$ is the generalized three parameter Mittag-Leffler function introduced by Prabhakar [19] defined as

$$E^{\gamma}_{\beta,\rho}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\beta k + \rho) k!}, \beta, \rho, \gamma \in \mathbf{C}, \Re(\beta) > 0, \Re(\rho) > 0, z \in \mathbf{C} \tag{50}$$

then the solution of the fractional kinetic Equation (49) is

$$N(t) = N_0 t^{\omega-1} E^{\gamma+1}_{\nu,\omega}(-c^\nu t^\nu) \tag{51}$$

5. Conclusions

The linear and non-linear kinetic equations establish a connection between the Boltzmann-Gibbs statistical mechanics and Tsallis non-extensive statistical mechanics. The pathway parameter α plays a key role in switching between these two cases. The theory of extended reaction rates and its closed form solutions can be seen in Haubold and Kumar [10] and Kumar and Haubold [12]. Further, the fractional diffusion equation and its solution help us to understand the connection with the classical Laplace transform. In 2013, the author has solved the fractional kinetic equations discussed here by P_α-transform [20]. Various fractional differential equations and their solution by various transforms are studied by many authors, see [21,22]. It should be noted that the Mittag-Leffler function arises in the solution of a fractional diffusion equation whereas the exponential function arises in the solution of its classical counterpart. A possible connection of the extended kinetic equation to fractional calculus can be established through the procedure adopted here.

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Conflicts of Interest

The author declares no conflict of interest.

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