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Article

Fractional Integration and Differentiation of the Generalized Mathieu Series

Ram K. Saxena [1] and Rakesh K. Parmar [2,*]

[1] Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur-342004, Rajasthan, India; ram.saxena@yahoo.com

[2] Department of Mathematics, Government College of Engineering and Technology, Bikaner-334004, Rajasthan, India

* Correspondence: rakeshparmar27@gmail.com

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Abstract: We aim to present some formulas for the Saigo hypergeometric fractional integral and differential operators involving the generalized Mathieu series $S_{\mu}(r)$, which are expressed in terms of the Hadamard product of the generalized Mathieu series $S_{\mu}(r)$ and the Fox–Wright function ${}_p\Psi_q(z)$. Corresponding assertions for the classical Riemann–Liouville and Erdélyi–Kober fractional integral and differential operators are deduced. Further, it is emphasized that the results presented here, which are for a seemingly complicated series, can reveal their involved properties via the series of the two known functions.

Keywords: Mathieu series; generalized Mathieuseries; fractional calculus operators

1. Introduction and Preliminaries

Fractional calculus, which has a long history, is an important branch of mathematical analysis (calculus) where differentiations and integrations can be of arbitrary non-integer order. During the past four decades or so, fractional calculus has been widely and extensively investigated and has gained importance and popularity due mainly to its demonstrated applications in numerous and diverse fields of science and engineering such as turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, and astrophysics (see, for detail, [1–5]).

We recall Saigo fractional integral and differential operators involving Gauss's hypergeometric function ${}_2F_1$ as a kernel. Let $\alpha,\ \beta,\ \eta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $x > 0$, then Saigo's fractional integral and differential operators $\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x)$, $\left(I_{-}^{\alpha,\beta,\eta} f\right)(x)$ and $\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x)$, $\left(D_{-}^{\alpha,\beta,\eta} f\right)(x)$ are defined as (see, for example, [1,2,4–6]):

$$\left(I_{0+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\, \mathrm{d}t, \tag{1}$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t, \tag{2}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) &= \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x)\\ &= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \quad (n = [\Re(\alpha)]+1),\end{aligned} \tag{3}$$

$$\left(D_{-}^{\alpha,\beta,\eta}f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta}f\right)(x)$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f\right)(x) \quad (n = [\Re(\alpha)] + 1). \tag{4}$$

Here and in what follows, $[x]$ denotes the greatest integer less than or equal to the real number x. When $\beta = -\alpha$, the operators in (1)–(4) coincide with the classical Riemann–Liouville fractional integrals and derivatives of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $x > 0$ (see, e.g., [1,4]):

$$\left(I_{0+}^{\alpha,-\alpha,\eta}f\right)(x) = \left(I_{0+}^{\alpha}f\right)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t, \tag{5}$$

$$\left(I_{-}^{\alpha,-\alpha,\eta}f\right)(x) = \left(I_{-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, \mathrm{d}t, \tag{6}$$

and

$$\left(D_{0+}^{\alpha,-\alpha,\eta}f\right)(x) = \left(D_{0+}^{\alpha}f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f(t)\, \mathrm{d}t$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha}f\right)(x) \qquad (n = [\Re(\alpha)] + 1), \tag{7}$$

$$\left(D_{-}^{\alpha,-\alpha,\eta}f\right)(x) = \left(D_{-}^{\alpha}f\right)(x) = (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^\infty (t-y)^{n-\alpha-1} f(t)\, \mathrm{d}t$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{n-\alpha}f\right)(x) \qquad (n = [\Re(\alpha)] + 1). \tag{8}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If $\beta = 0$ in (1)–(4) yields the so-called Erdélyi–Kober fractional integrals and derivatives of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $x > 0$ (see, e.g., [1,4]):

$$\left(I_{0+}^{\alpha,0,\eta}f\right)(x) = \left(I_{\eta,\alpha}^{+}f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, \mathrm{d}t, \tag{9}$$

$$\left(I_{-}^{\alpha,0,\eta}f\right)(x) = \left(K_{\eta,\alpha}^{-}f\right)(x) \equiv \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\, \mathrm{d}t, \tag{10}$$

and

$$\left(D_{0+}^{\alpha,0,\eta}f\right)(x) = \left(D_{\eta,\alpha}^{+}f\right)(x)$$
$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\alpha,\alpha+\eta-n}f\right)(x) \quad (n = [\Re(\alpha)] + 1), \tag{11}$$

$$\left(D_{-}^{\alpha,0,\eta}f\right)(x) = \left(D_{\eta,\alpha}^{-}f\right)(x)$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\alpha,\alpha+\eta}f\right)(x) \quad (n = [\Re(\alpha)] + 1), \tag{12}$$

$$\left(D_{\eta,\alpha}^{+}f\right)(x) = x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x t^{\alpha+\eta} (x-t)^{n-\alpha-1} f(t)\, \mathrm{d}t \quad (n = [\Re(\alpha)] + 1), \tag{13}$$

$$\left(D_{\eta,\alpha}^{-}f\right)(x) = x^{\eta+\alpha}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty} t^{-\eta}(t-x)^{n-\alpha-1}f(t)\,\mathrm{d}t \quad (n = [\Re(\alpha)]+1). \tag{14}$$

A detailed account of such operators along with their properties and applications has been considered by several authors (see, for details, [1–5]).

The following familiar infinite series of the form

$$S(r) = \sum_{n\geq 1}\frac{2n}{(n^2+r^2)^2}, \qquad r>0 \tag{15}$$

is known in literature as the Mathieu series. Émile Leonard Mathieu was the first to investigate such a series in 1890 in his book elasticity of solid bodies [7]. An alternative version of (15)

$$\widetilde{S}(r) = \sum_{n\geq 1}(-1)^{n-1}\frac{2n}{(n^2+r^2)^2}, \qquad r>0 \tag{16}$$

was introduced by Pogány et al. [8]. Closed form integral representations for $S(r)$ and $\widetilde{S}(r)$ are given by (see e.g., [8,9])

$$S(r) = \frac{1}{r}\int_0^{\infty}\frac{x\sin(rx)}{\mathrm{e}^x-1}\,\mathrm{d}x \tag{17}$$

and

$$\widetilde{S}(r) = \frac{1}{r}\int_0^{\infty}\frac{x\sin(rx)}{\mathrm{e}^x+1}\,\mathrm{d}x, \tag{18}$$

respectively. Several interesting problems and solutions deal with integral representations and bounds for the following mild generalization of the Mathieu series and its alternative version with a fractional power defined by ([10], p. 2, Equation (16)) (see also, [11], p. 181)

$$S_{\mu}(r) = \sum_{n\geq 1}\frac{2n}{(n^2+r^2)^{\mu+1}} \qquad (r>0,\ \mu>0) \tag{19}$$

and

$$\widetilde{S}_{\mu}(r) = \sum_{n\geq 1}(-1)^{n-1}\frac{2n}{(n^2+r^2)^{\mu+1}} \qquad (r>0,\ \mu>0), \tag{20}$$

respectively. Such a series has been widely considered in mathematical literature (see, e.g., papers of Cerone and Lenard [10], Diananda [12] and Pogány et al. [8]). Various applications of the familiar Mathieu series and its generalizations in probability theory with other variants such as trigonometric Mathieu series, harmonic Mathieu series, Fourier–Mathieu series and some other particular forms of the Mathieu series can be found in a recent paper [13].

Recently, Tomovski and Pogány [14] studied the several integral representations of the generalized fractional order Mathieu-type power series

$$S_{\mu}(r;z) = \sum_{n\geq 1}\frac{2n\,z^n}{(n^2+r^2)^{\mu+1}}, \qquad (\mu>0,\ r\in\mathbb{R},\ |z|<1). \tag{21}$$

Obviously, we have

$$S_{\mu}(r;1) = S_{\mu}(r) \quad \text{and} \quad S_{\mu}(r;-1) = -\widetilde{S}_{\mu}(r).$$

Various other investigations and generalizations of the Mathieu series with its alternative variants can also be found in [11,14–24], and the references cited therein.

The concept of the Hadamard product (or the convolution) of two analytic functions is useful in our present investigation. It can help us to decompose a newly emerged function into two known functions. If, in particular, one of the power series defines an entire function, then the Hadamard product series defines an entire function, too. Let

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \;\; (|z| < R_f) \;\; \text{and} \;\; g(z) := \sum_{n=0}^{\infty} b_n z^n \;\; (|z| < R_g)$$

be two power series whose radii of convergence are denoted by R_f and R_g, respectively. Then, their Hadamard product is the power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n \, b_n z^n = (g * f)(z) \;\; (|z| < R) \tag{22}$$

where

$$R = \lim_{n\to\infty} \left| \frac{a_n \, b_n}{a_{n+1} \, b_{n+1}} \right| = \left(\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \right) \cdot \left(\lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \right) = R_f \cdot R_g,$$

therefore, in general, we have $R \geq R_f \cdot R_g$ [25,26]. For various other investigations involving the Hadamard product (or the convolution), the interested reader may be referred to several recent papers on the subject (see, for example, [27,28] and the references cited in each of these papers).

In this paper, our aim is to study the compositions of the generalized fractional integration and differentiation operators (1)–(4) with the generalized Mathieu series (21) in terms of the Hadamard product (22) of the generalized Mathieu series and the Fox–Wright function. Further, corresponding assertions for the classical Riemann–Liouville and Erdélyi–Kober fractional integral and differential operators are deduced. The results presented in Theorems together with Corollaries are sure to be new and potentially useful, mainly because they are expressed in terms of the Hadamard product with two known functions. At least, a seemingly complicated resulting series expressed in terms of two known functions *means* that certain properties involved in the complicated resulting series can be revealed via the series of the known functions.

2. Fractional Integration of the Mathieu Series

We first recall the Fox–Wright function ${}_p\Psi_q(z)$ $(p, q \in \mathbb{N}_0)$ with p numerator and q denominator parameters defined for $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ and $\beta_1, \ldots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by (see, for details, [1,3]; see also [4,29]):

$${}_p\Psi_q \left[\begin{array}{l} (\alpha_1, A_1), \cdots, (\alpha_p, A_p); \\ (\beta_1, B_1), \cdots, (\beta_q, B_q); \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \, \frac{z^n}{n!} \tag{23}$$

$$\left(A_j \in \mathbb{R}^+ \; (j = 1, \ldots, p); \; B_j \in \mathbb{R}^+ \; (j = 1, \ldots, q); \; 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0 \right),$$

where the equality in the convergence condition holds true for

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

In particular, when $A_j = B_k = 1$ $(j = 1, \ldots, p; \;\; k = 1, \ldots, q)$, (23) reduces immediately to the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ (see, e.g., [29]):

$${}_pF_q \left[\begin{array}{l} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} z \right] = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)} \, {}_p\Psi_q \left[\begin{array}{l} (\alpha_1, 1), \cdots, (\alpha_p, 1); \\ (\beta_1, 1), \cdots, (\beta_q, 1); \end{array} z \right]. \tag{24}$$

Lemma 1. *Let α, β, $\eta \in \mathbb{C}$. Then, there exists the relation*

(a) *If $\Re(\alpha) > 0$ and $\Re(\sigma) > \max[0, \Re(\beta - \eta)]$, then*

$$(I_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1} \tag{25}$$

In particular, for $x > 0$, we have

$$(I_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)} x^{\sigma+\alpha-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > 0), \tag{26}$$

$$(I_{\eta,\alpha}^{+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)} x^{\sigma-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\eta)). \tag{27}$$

(b) *If $\Re(\alpha) > 0$ and $\Re(\sigma) < 1 + \min[\Re(\beta), \Re(\eta)]$, then*

$$(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\beta)\Gamma(1-\sigma+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\alpha+\beta+\eta)} x^{\sigma-\beta-1}. \tag{28}$$

In particular, for $x > 0$, we have

$$(I_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)} x^{\sigma+\alpha-1} \qquad (0 < \Re(\alpha) < 1 - \Re(\sigma)), \tag{29}$$

$$(K_{\eta,\alpha}^{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\eta)}{\Gamma(1-\sigma+\alpha+\eta)} x^{\sigma-1} \qquad (\Re(\sigma) < 1 + \Re(\sigma)). \tag{30}$$

We begin the exposition of the main results by presenting the composition formulas of generalized fractional integrals, (1) and (2), involving the generalized Mathieu series in terms of the Hadamard product (22) of the generalized Mathieu series (21) and the Fox–Wright function (23). It is emphasized that the results presented here, which are for a seemingly complicated series, can reveal their involved properties via the series of the two known functions.

Theorem 1. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > \max[0, \Re(\beta - \eta)]$. Then, the following Saigo hypergeometric fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of $S_\mu(r, t^\rho)$ holds true:*

$$\begin{aligned} &\left(I_{0+}^{\alpha,\beta,\eta} \left\{t^{\sigma-1}\, S_\mu(r, t^\rho)\right\}\right)(x) \\ &\quad = x^{\sigma-\beta+\rho-1}\, S_\mu(r, x^\rho) \, * \, {}_3\Psi_2 \left[\begin{matrix} (1,1), (\sigma+\rho,\rho), (\sigma+\eta-\beta+\rho,\rho); \\ (\sigma-\beta+\rho,\rho), (\sigma+\alpha+\eta+\rho,\rho); \end{matrix} \; x^\rho \right]. \end{aligned} \tag{31}$$

Proof. Using the definitions (1) and (21), by changing the order of integration and applying the relation (25), we find that $x > 0$

$$\begin{aligned} &\left(I_{0+}^{\alpha,\beta,\eta} \left\{t^{\sigma-1} S_\mu(r, t^\rho)\right\}\right)(x) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^{\mu+1}} \left(I_{0+}^{\alpha,\beta,\eta} t^{\sigma+\rho k-1}\right)(x) \\ &= x^{\sigma-\beta-1} \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^{\mu+1}} \frac{\Gamma(\sigma+\rho k)\Gamma(\sigma+\eta-\beta+\rho k)}{\Gamma(\sigma-\beta+\rho k)\Gamma(\sigma+\alpha+\eta+\rho k)} x^{\rho k}. \end{aligned} \tag{32}$$

by applying the Hadamard product (22) in (32), which in the view of (21) and (23), yields the desired formula (31). □

Theorem 2. *Let α, β, η, $\sigma \in \mathbb{C}$ and $\rho > 0$, $\mu > 0$, $r \in \mathbb{R}$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) < 1 + \min[\Re(\beta), \Re(\eta)]$. Then, the following Saigo hypergeometric fractional integral $I_{-}^{\alpha,\beta,\eta}$ of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x)$$
$$= x^{\sigma-\rho-\beta-1}\, S_\mu\left(r, \frac{1}{x^\rho}\right) * {}_3\Psi_2\left[\begin{array}{c} (1,1), (1-\sigma+\beta+\rho,\rho), (1-\sigma+\eta+\rho,\rho); \\ (1-\sigma+\rho,\rho), (1-\sigma+\alpha+\beta+\eta+\rho,\rho); \end{array} \frac{1}{x^\rho}\right]. \tag{33}$$

Proof. Using the definitions (2) and (21), by changing the order of integration and applying the relation (28)

$$\left(I_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1} S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^{\mu+1}} \left(I_{-}^{\alpha,\beta,\eta}\, t^{\sigma-\rho k-1}\right)(x)$$
$$= x^{\sigma-\beta-1} \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^{\mu+1}} \frac{\Gamma(1-\sigma+\beta+\rho k)\Gamma(1-\sigma+\eta+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1-\sigma+\alpha+\beta+\eta+\rho k)} x^{-\rho k}. \tag{34}$$

by applying the Hadamard product (22) in (34), which in the view of (21) and (23), yields the desired formula (33). □

Further, we deduce the fractional integral formulas for the classical Riemann–Liouville and Erdélyi–Kober fractional integral and differential operators by letting $\beta = -\alpha$ and $\beta = 0$ respectively, which are asserted by Corollaries 1–4 below.

Corollary 1. *Let $\alpha, \sigma \in \mathbb{C}$ and $\rho > 0$, $\mu > 0$, $r \in \mathbb{R}$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > 0$. Then, the following Riemann–Liouville fractional integral I_{0+}^{α} of $S_\mu(r, t^\rho)$ holds true:*

$$\left(I_{0+}^{\alpha}\left\{t^{\sigma-1}\, S_\mu(r, t^\rho)\right\}\right)(x) = x^{\sigma+\rho+\alpha-1}\, S_\mu(r, x^\rho) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (\sigma+\rho,\rho); \\ (\sigma+\alpha+\rho,\rho); \end{array} x^\rho\right]. \tag{35}$$

Corollary 2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $\rho > 0$, $\mu > 0$, $r \in \mathbb{R}$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > -\Re(\eta)$. Then, the following Erdélyi–Kober fractional integral $I_{\eta,\alpha}^{+}$ of $S_\mu(r, t^\rho)$ holds true:*

$$\left(I_{\eta,\alpha}^{+}\left\{t^{\sigma-1}\, S_\mu(r, t^\rho)\right\}\right)(x) = x^{\sigma+\rho-1}\, S_\mu(r, x^\rho) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (\sigma+\eta+\rho,\rho); \\ (\sigma+\alpha+\eta+\rho,\rho); \end{array} x^\rho\right]. \tag{36}$$

Corollary 3. *Let $\alpha, \sigma \in \mathbb{C}$ and $\rho > 0$, $\mu > 0$, $r \in \mathbb{R}$ be such that $0 < \Re(\alpha) < 1 - \Re(\sigma)$. Then, the following Riemann–Liouville fractional integral I_{-}^{α} of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\left(I_{-}^{\alpha}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x) = x^{\sigma+\alpha-\rho-1}\, S_\mu\left(r, \frac{1}{x^\rho}\right) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (1-\sigma-\alpha+\rho,\rho); \\ (1-\sigma+\rho,\rho); \end{array} \frac{1}{x^\rho}\right]. \tag{37}$$

Corollary 4. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $\rho > 0$, $\mu > 0$, $r \in \mathbb{R}$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) < 1 + \Re(\eta)$. Then, the following Erdélyi–Kober fractional integral $K_{\eta,\alpha}^{-}$ of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\left(K_{\eta,\alpha}^{-}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x) = x^{\sigma-\rho-1}\, S_\mu\left(r, \frac{1}{x^\rho}\right) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (1-\sigma+\eta+\rho,\rho); \\ (1-\sigma+\alpha+\eta+\rho,\rho); \end{array} \frac{1}{x^\rho}\right]. \tag{38}$$

The results obtained in this section can be presented in terms of Gauss's hypergeometric functions by taking $\rho = 1$. Here, we present results for the classical Riemann–Liouville fractional integral operators.

Corollary 5. *Let the conditions of Corollary 1 be satisfied, and let $\Re(\sigma) > 0$ and $\Re(\sigma+\alpha+1) > 0$. Then, for $x > 0$, there holds the relation*

$$\left(I_{0+}^{\alpha}\left\{t^{\sigma-1}\,S_{\mu}(r,t)\right\}\right)(x) = x^{\sigma+\alpha}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\alpha+1)}\,S_{\mu}(r,x)\,*\,{}_2F_1\left[\begin{array}{c}1,\sigma+1;\\ \sigma+\alpha+1;\end{array}\,x\right]. \tag{39}$$

Corollary 6. *Let the conditions of Corollary 3 be satisfied, and let $\Re(1-\sigma) > 0$ and $\Re(2-\sigma-\alpha) > 0$. Then, for $x > 0$, there holds the relation*

$$\left(I_{-}^{\alpha}\left\{t^{\sigma-1}\,S_{\mu}\left(r,\frac{1}{t}\right)\right\}\right)(x) = x^{\sigma+\alpha-2}\frac{\Gamma(2-\sigma-\alpha)}{\Gamma(2-\sigma)}\,S_{\mu}\left(r,\frac{1}{x}\right)\,*\,{}_2F_1\left[\begin{array}{c}1,2-\sigma-\alpha;\\ 2-\sigma;\end{array}\,\frac{1}{x}\right]. \tag{40}$$

3. Fractional Differentiation of the Mathieu Series

In this section, we present the composition formulas of generalized fractional derivatives, (3) and (4), involving the generalized Mathieu series in terms of the Hadamard product (22) of the generalized Mathieu series (21) and the Fox–Wright function (23).

Lemma 2. *Let α, β, $\eta \in \mathbb{C}$. Then, there exists the relations*

(a) *If $\Re(\alpha) > 0$ and $\Re(\sigma) > -\min[0,\Re(\alpha+\beta+\eta)]$, then*

$$(D_{0+}^{\alpha,\beta,\eta}t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)}x^{\sigma+\beta-1} \tag{41}$$

In particular, for $x > 0$, we have

$$(D_{0+}^{\alpha}t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}x^{\sigma-\alpha-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > 0), \tag{42}$$

$$(D_{\eta,\alpha}^{+}t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)}x^{\sigma-1} \qquad (\Re(\alpha) > 0,\ \Re(\sigma) > -\Re(\alpha+\eta)). \tag{43}$$

(b) *If $\Re(\alpha) > 0, \Re(\sigma) < 1+\min[\Re(-\beta-n),\Re(\alpha+\eta)]$ and $n = [\Re(\alpha)]+1$, then*

$$(D_{-}^{\alpha,\beta,\eta}t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\beta)}x^{\sigma+\beta-1}. \tag{44}$$

In particular, for $x > 0$, we have

$$(D_{-}^{\alpha}t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)}x^{\sigma-\alpha-1} \qquad (\Re(\alpha) > 0, \Re(\sigma) < 1+\Re(\alpha)-n), \tag{45}$$

$$(D_{\eta,\alpha}^{-}t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma-\eta)}x^{\sigma-1} \qquad (\Re(\alpha) > 0, \Re(\sigma) < 1+\Re(\alpha+\eta)-n). \tag{46}$$

Theorem 3. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) > -\min[0,\Re(\alpha+\beta+\eta)]$. Then, the following Saigo hypergeometric fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of $S_{\mu}(r,t^{\rho})$ holds true:*

$$\begin{aligned}&\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\,S_{\mu}(r,t^{\rho})\right\}\right)(x)\\ &\quad= x^{\sigma+\beta+\rho-1}\,S_{\mu}(r,x^{\rho})\,*\,{}_3\Psi_2\left[\begin{array}{c}(1,1),(\sigma+\rho,\rho),(\sigma+\alpha+\beta+\eta+\rho,\rho);\\ (\sigma+\beta+\rho,\rho),(\sigma+\eta+\rho,\rho);\end{array}\,x^{\rho}\right].\end{aligned} \tag{47}$$

Proof. Using the definitions (3) and (21), by changing the order of integration and applying the relation (41), we find for $x > 0$

$$\begin{aligned}\left(D_{0+}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}S_\mu(r,t^\rho)\right\}\right)(x) &= \sum_{k=1}^{\infty}\frac{2k}{(k^2+r^2)^{\mu+1}}\left(D_{0+}^{\alpha,\beta,\eta}t^{\sigma+\rho k-1}\right)(x)\\ &= x^{\sigma+\beta-1}\sum_{k=1}^{\infty}\frac{2k}{(k^2+r^2)^{\mu+1}}\frac{\Gamma(\sigma+\rho k)\Gamma(\sigma+\alpha+\beta+\eta+\rho k)}{\Gamma(\sigma+\beta+\rho k)\Gamma(\sigma+\eta+\rho k)}x^{\rho k}. \quad (48)\end{aligned}$$

by applying the Hadamard product (22) in (48), which in the view of (21) and (23), yields the desired formula (47). □

Theorem 4. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) < 1 + \min[\Re(-\beta-n), \Re(\alpha+\eta)]$, $n = [\Re(\alpha)] + 1$. Then, the following Saigo hypergeometric fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\begin{aligned}&\left(D_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}S_\mu\left(r,\frac{1}{t^\rho}\right)\right\}\right)(x)\\ &\quad= x^{\sigma-\rho+\beta-1}S_\mu\left(r,\frac{1}{x^\rho}\right) * {}_3\Psi_2\left[\begin{array}{c}(1,1),(1-\sigma-\beta+\rho,\rho),(1-\sigma+\alpha+\eta+\rho,\rho);\\ (1-\sigma+\rho,\rho),(1-\sigma+\eta-\beta+\rho,\rho);\end{array}\ \frac{1}{x^\rho}\right]. \quad (49)\end{aligned}$$

Proof. Using the definitions (4) and (21), by changing the order of integration and applying the relation (44), we find for $x > 0$

$$\begin{aligned}\left(D_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}S_\mu\left(r,\frac{1}{t^\rho}\right)\right\}\right)(x) &= \sum_{k=1}^{\infty}\frac{2k}{(k^2+r^2)^{\mu+1}}\left(D_{-}^{\alpha,\beta,\eta}t^{\sigma-\rho k-1}\right)(x)\\ &= x^{\sigma+\rho+\beta-1}\sum_{k=1}^{\infty}\frac{2k}{(k^2+r^2)^{\mu+1}}\frac{\Gamma(1-\sigma-\beta+\rho k)\Gamma(1-\sigma+\alpha+\eta+\rho k)}{\Gamma(1-\sigma+\rho k)\Gamma(1-\sigma+\eta-\beta+\rho k)}x^{-\rho k}. \quad (50)\end{aligned}$$

by applying the Hadamard product (22) in (50), which in the view of (21) and (23), yields the desired formula (49). □

Now, we deduce fractional derivative formulas for the classical Riemann–Liouville and Erdélyi–Kober fractional integral and differential operators by letting $\beta = -\alpha$ and $\beta = 0$ respectively, which are asserted by Corollaries 7–10 below.

Corollary 7. *Let $\alpha, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) > 0$. Then, the following Riemann–Liouville fractional differentiation D_{0+}^{α} of $S_\mu(r, t^\rho)$ holds true:*

$$\left(D_{0+}^{\alpha}\left\{t^{\sigma-1}S_\mu(r,t^\rho)\right\}\right)(x) = x^{\sigma+\rho-\alpha-1}S_\mu(r,x^\rho) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma+\rho,\rho);\\ (\sigma-\alpha+\rho,\rho);\end{array}\ x^\rho\right]. \quad (51)$$

Corollary 8. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) > -\Re(\alpha+\eta)$. Then, the following Erdélyi–Kober fractional derivative $D_{\eta,\alpha}^{+}$ of $S_\mu(r, t^\rho)$ holds true:*

$$\left(D_{\eta,\alpha}^{+}\left\{t^{\sigma-1}S_\mu(r,t^\rho)\right\}\right)(x) = x^{\sigma+\rho-1}S_\mu(r,x^\rho) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\sigma+\alpha+\eta+\rho,\rho);\\ (\sigma+\eta+\rho,\rho);\end{array}\ x^\rho\right]. \quad (52)$$

Corollary 9. *Let $\alpha, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) < \Re(\alpha) - [\Re(\alpha)]$. Then, the following Riemann–Liouville fractional differentiation D_{-}^{α} of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\left(D_{-}^{\alpha}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x) = x^{\sigma-\rho-\alpha-1}\, S_\mu\left(r, \frac{1}{x^\rho}\right) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (1-\sigma+\alpha+\rho, \rho); \\ (1-\sigma+\rho, \rho); \end{array} \frac{1}{x^\rho}\right]. \quad (53)$$

Corollary 10. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $\rho > 0, \mu > 0, r \in \mathbb{R}$ be such that $\Re(\alpha) \geq 0$ and $\Re(\sigma) < \Re(\alpha+\eta) - [\Re(\alpha)]$. Then, the following Erdélyi–Kober fractional differentiation $D_{\eta,\alpha}^{-}$ of $S_\mu\left(r, \frac{1}{t^\rho}\right)$ holds true:*

$$\left(D_{\eta,\alpha}^{-}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t^\rho}\right)\right\}\right)(x) = x^{\sigma-\rho-1}\, S_\mu\left(r, \frac{1}{x^\rho}\right) * {}_2\Psi_1\left[\begin{array}{c} (1,1), (1-\sigma+\alpha+\eta+\rho, \rho); \\ (1-\sigma+\eta+\rho, \rho); \end{array} \frac{1}{x^\rho}\right]. \quad (54)$$

The results obtained in this section can be presented in terms of Gauss's hypergeometric functions by taking $\rho = 1$. Here, we present results for the classical Riemann–Liouville fractional derivative operators.

Corollary 11. *Let the conditions of Corollary 7 be satisfied, and let $\Re(\sigma+1) > 0$ and $\Re(\sigma-\alpha+1) > 0$. Then, for $x > 0$, there holds the relation*

$$\left(D_{0+}^{\alpha}\left\{t^{\sigma-1}\, S_\mu(r,t)\right\}\right)(x) = x^{\sigma-\alpha-2}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha+1)}\, S_\mu(r,x) * {}_2F_1\left[\begin{array}{c} 1, \sigma+1; \\ \sigma-\alpha+1; \end{array} x\right]. \quad (55)$$

Corollary 12. *Let the conditions of Corollary 9 be satisfied, and let $\Re(2-\sigma) > 0$ and $\Re(2-\sigma+\alpha) > 0$. Then, for $x > 0$, there holds the relation*

$$\left(D_{-}^{\alpha}\left\{t^{\sigma-1}\, S_\mu\left(r, \frac{1}{t}\right)\right\}\right)(x) = x^{\sigma-\alpha-2}\frac{\Gamma(2-\sigma+\alpha)}{\Gamma(2-\sigma)}\, S_\mu\left(r, \frac{1}{x}\right) * {}_2F_1\left[\begin{array}{c} 1, 2-\sigma+\alpha; \\ 2-\sigma; \end{array} \frac{1}{x}\right]. \quad (56)$$

4. Concluding Remarks and Observations

In our present investigation, with the help of the concept of the Hadamard product (or the convolution) of two analytic functions, we have obtained the composition formulas of the generalized fractional integrals, (1) and (2), involving the generalized Mathieu series in terms of the Hadamard product (22) of the generalized Mathieu series (21) and the Fox–Wright function (23). Further, we have also deduced the fractional integral formulas for the classical Riemann–Liouville and the Erdélyi–Kober fractional integral and differential operators by letting $\beta = -\alpha$ and $\beta = 0$, respectively. The results presented here, which are for a seemingly complicated series, can reveal their involved properties via the series of the two known functions.

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