axioms

MDPI

Article

New Definitions about $A^{\mathcal{I}}$-Statistical Convergence with Respect to a Sequence of Modulus Functions and Lacunary Sequences

Ömer Kişi [1,*], Hafize Gümüş [2] and Ekrem Savas [3]

1 Department of Mathematics, Faculty of Science, Bartin University, Bartin 74100, Turkey
2 Department of Mathematics, Faculty of Eregli Education, Necmettin Erbakan University, Eregli, Konya 42060, Turkey; hgumus@konya.edu.tr
3 Department of Mathematics, Istanbul Ticaret University, Üsküdar, Istanbul 34840, Turkey; ekremsavas@yahoo.com
* Correspondence: okisi@bartin.edu.tr; Tel.: +90-378-501-1000

Received: 19 February 2018; Accepted: 10 April 2018; Published: 13 April 2018

check for updates

Abstract: In this paper, using an infinite matrix of complex numbers, a modulus function and a lacunary sequence, we generalize the concept of $\mathcal{I}$-statistical convergence, which is a recently introduced summability method. The names of our new methods are $A^{\mathcal{I}}$-lacunary statistical convergence and strongly $A^{\mathcal{I}}$-lacunary convergence with respect to a sequence of modulus functions. These spaces are denoted by $S_\theta^A(\mathcal{I},F)$ and $N_\theta^A(\mathcal{I},F)$, respectively. We give some inclusion relations between $S^A(\mathcal{I},F)$, $S_\theta^A(\mathcal{I},F)$ and $N_\theta^A(\mathcal{I},F)$. We also investigate Cesáro summability for $A^{\mathcal{I}}$ and we obtain some basic results between $A^{\mathcal{I}}$-Cesáro summability, strongly $A^{\mathcal{I}}$-Cesáro summability and the spaces mentioned above.

Keywords: lacunary sequence; statistical convergence; ideal convergence; modulus function; $\mathcal{I}$-statistical convergence

MSC: 40A35, 40A05

1. Introduction

As is known, convergence is one of the most important notions in mathematics. Statistical convergence extends the notion. After giving the definition of statistical convergence, we can easily show that any convergent sequence is statistically convergent, but not conversely. Let E be a subset of $\mathbb{N}$, and the set of all natural numbers $d(E) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_E(j)$ is said to be a natural density of E whenever the limit exists. Here, χ_E is the characteristic function of E.

In 1935, statistical convergence was given by Zygmund in the first edition of his monograph [1]. It was formally introduced by Fast [2], Fridy [3], Salat [4], Steinhaus [5] and later was reintroduced by Schoenberg [6]. It has become an active research area in recent years. This concept has applications in different fields of mathematics such as number theory [7], measure theory [8], trigonometric series [1], summability theory [9], etc.

Following this very important definition, the concept of lacunary statistical convergence was defined by Fridy and Orhan [10]. In addition, Fridy and Orhan gave the relationships between the lacunary statistical convergence and the Cesàro summability. Freedman and Sember [9] established the connection between the strongly Cesàro summable sequences space $|\sigma_1|$ and the strongly lacunary summable sequence space N_θ.

$\mathcal{I}$-convergence has emerged as a generalized form of many types of convergences. This means that, if we choose different ideals, we will have different convergences. Koystro et al. [11] introduced this concept in a metric space. Also, Das et al. [12], Koystro et al. [13], Savaş and Das [14] studied ideal convergence. We will explain this situation with two examples later. Before defining $\mathcal{I}$-convergence, the definitions of ideal and filter will be needed.

An ideal is a family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ such that (i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii), and, for each $A \in \mathcal{I}$, each $B \subseteq A$ implies $B \in \mathcal{I}$. An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A filter is a family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ such that (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$, each $A \subseteq B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is an ideal in $\mathbb{N}$, then the collection

$$F(\mathcal{I}) = \{A \subset \mathbb{N} : \mathbb{N} \backslash A \in \mathcal{I}\}$$

forms a filter in $\mathbb{N}$ that is called the filter associated with $\mathcal{I}$.

The notion of a modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$; (ii) $f(x + y) = f(x) + f(y)$ for $x, y \geq 0$; (iii) f is increasing; and (iv) f is continuous from the right at 0. It follows that f must be continuous on $[0, \infty)$. Connor [16], Bilgin [17], Maddox [18], Kolk [19], Pehlivan and Fisher [20] and Ruckle [21] have used a modulus function to construct sequence spaces. Now, let S be the space of sequences of modulus functions $F = (f_k)$ such that $\lim_{x \to 0^+} \sup_k f_k(x) = 0$. Throughout this paper, the set of all modulus functions determined by F is denoted by $F = (f_k) \in S$ for every $k \in \mathbb{N}$.

In this paper, we aim to unify these approaches and use ideals to introduce the notion of $A^{\mathcal{I}}$-lacunary statistically convergence with respect to a sequence of modulus functions.

2. Definitions and Notations

First, we recall some of the basic concepts that will be used in this paper.

Let $A = (a_{ki})$ be an infinite matrix of complex numbers. We write $Ax = (A_k(x))$, if $A_k(x) = \sum_{i=1}^{\infty} a_{ki} x_k$ converges for each k.

Definition 1. *A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim x_k = L$. As we said before, statistical convergence is a natural generalization of ordinary convergence i.e., if $\lim x_k = L$, then $st - \lim x_k = L$ (Fast, [2]).

By a lacunary sequence, we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 2. *A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if, for every $\varepsilon > 0$,*

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$ (Fridy and Orhan, [10]).

Definition 3. *The sequence space N_θ is defined by*

$$N_\theta = \left\{ (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \right\}$$

(Fridy and Orhan, [10]).

Definition 4. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set*

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$$

(Kostyrko et al. [11]).

Example 1. *Define the set of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_f$. Then, $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence.*

Example 2. *Define the set $\mathcal{I}_d$ by $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then, $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence gives the statistical convergence.*

Following the line of Savas et al. [22], some authors obtained more general results about statistical convergence by using A matrix and they called this new method $\mathrm{A}^{\mathcal{I}}$-statistical convergence (see, e.g., [17,23]).

Definition 5. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-statistically convergent to $L \in X$ with respect to a sequence of modulus functions, for each $\varepsilon > 0$, for every $x \in X$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : f_k (|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left(S^A(\mathcal{I}, F)\right)$ (Yamancı et al. [23]).

3. Inclusions between $S^A(\mathcal{I}, F)$, $S_\theta^A(\mathcal{I}, F)$ and $N_\theta^A(\mathcal{I}, F)$ Spaces

We now consider our main results. We begin with the following definitions.

Definition 6. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers, $\theta = \{k_r\}$ be a lacunary sequence and $F = (f_k)$ be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-lacunary statistically convergent to $L \in X$ with respect to a sequence of modulus functions, for each $\varepsilon > 0$, for each $x \in X$ and $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : f_k (|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

Definition 7. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers, $\theta = \{k_r\}$ be a lacunary sequence and $F = (f_k)$ be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be strongly $A^{\mathcal{I}}$-lacunary convergent to $L \in X$ with respect to a sequence of modulus functions, if, for each $\varepsilon > 0$, for each $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} f_k (|A_k(x) - L|) \geq \varepsilon \right\} \in \mathcal{I}.$$

We shall denote by $S_\theta^A(\mathcal{I}, F)$, $N_\theta^A(\mathcal{I}, F)$ the collections of all $A^{\mathcal{I}}$-lacunary statistically convergent and strongly $A^{\mathcal{I}}$-lacunary convergent sequences, respectively.

Theorem 1. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. $\left(S_\theta^A(\mathcal{I}, F)\right) \cap m(X)$ is a closed subset of $m(X)$ if X is a Banach space where $m(X)$ is the space of all bounded sequences of X.*

Proof. Suppose that $(x^n) \subset \left(S_\theta^A(\mathcal{I}, F)\right) \cap m(X)$ is a convergent sequence and it converges to $x \in m(X)$. We need to show that $x \in \left(S_\theta^A(\mathcal{I}, F)\right) \cap m(X)$. Assume that $x^n \to L_n\left(S_\theta^A(\mathcal{I}, F)\right), \forall n \in \mathbb{N}$. Take a sequence $\{\varepsilon_r\}_{n\in\mathbb{N}}$ of strictly decreasing positive numbers converging to zero. We can find an $r \in \mathbb{N}$ such that $\left\|x - x^j\right\|_\infty < \frac{\varepsilon_r}{4}$ for all $j \geq r$. Choose $0 < \delta < \frac{1}{5}$.

Now,

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x^n\right) - L_n\right|\right) \geq \frac{\varepsilon_r}{4}\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I})$$

and

$$B = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x^{n+1}\right) - L_{n+1}\right|\right) \geq \frac{\varepsilon_r}{4}\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\emptyset \notin \mathcal{F}(\mathcal{I})$, we can choose $r \in A \cap B$. Then,

$$\frac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x^n\right) - L_n\right|\right) \geq \frac{\varepsilon_r}{4} \vee f_k\left(\left|A_k\left(x^{n+1}\right) - L_{n+1}\right|\right) \geq \frac{\varepsilon_r}{4}\right\}\right| \leq 2\delta < 1.$$

Since $h_r \to \infty$ and $A \cap B \in \mathcal{F}(\mathcal{I})$ is infinite, we can actually choose the above r so that $h_r > 5$. Hence, there must exist a $k \in I_r$ for which we have simultaneously, $\left|x_k^n - L_n\right| < \frac{\varepsilon_r}{4}$ and $\left|x_k^{n+1} - L_{n+1}\right| < \frac{\varepsilon_r}{4}$.

Then, it follows that

$$\begin{aligned} \left|L_n - L_{n+1}\right| &\leq \left|L_n - x_k^n\right| + \left|x_k^n - x_k^{n+1}\right| + \left|x_k^{n+1} - L_{n+1}\right| \\ &\leq \left|x_k^n - L_n\right| + \left|x_k^{n+1} - L_{n+1}\right| + \left\|x - x^n\right\|_\infty + \left\|x - x^{n+1}\right\|_\infty \\ &\leq \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} = \varepsilon_r. \end{aligned}$$

This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is a Banach space, we can write $L_n \to L \in X$ as $n \to \infty$. We shall prove that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$. Choose $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $\varepsilon_r < \frac{\varepsilon}{4}$, $\left\|x - x_n\right\|_\infty < \frac{\varepsilon}{4}$. Now, since

$$\begin{aligned} &\tfrac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k(x) - L\right|\right) \geq \varepsilon\right\}\right| \\ &\leq \tfrac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k\left(x - x_n\right)\right|\right) + f_k\left(\left|A_k(x^n) - L_n\right|\right) + f_k\left(\left|L_n - L\right|\right) \geq \varepsilon\right\}\right| \\ &\leq \tfrac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k(x^n) - L_n\right|\right) \geq \tfrac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

It follows that

$$\begin{aligned} &\left\{r \in \mathbb{N} : \tfrac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k(x) - L\right|\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subset \left\{r \in \mathbb{N} : \tfrac{1}{h_r}\left|\left\{k \in I_r : f_k\left(\left|A_k(x) - L\right|\right) \geq \tfrac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \end{aligned}$$

for given $\delta > 0$. This shows that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$ and this completes the proof of the theorem. □

Theorem 2. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers, $\theta = \{k_r\}$ be a lacunary sequence and (f_k) be a sequence of modulus functions in S. Then, we have* *(i)* *If* $x_k \to L\left(N_\theta^A(\mathcal{I},F)\right)$, *then* $x_k \to L\left(S_\theta^A(\mathcal{I},F)\right)$ *and* $N_\theta^A(\mathcal{I},F) \subset S_\theta^A(\mathcal{I},F)$ *is proper for every ideal* $\mathcal{I}$;

(ii) *If* $x \in m(X)$, *the space of all bounded sequences of* X *and* $x_k \to L\left(S_\theta^A(\mathcal{I},F)\right)$, *then* $x_k \to L\left(N_\theta^A(\mathcal{I},F)\right)$;

(iii) $S_\theta^A(\mathcal{I},F) \cap m(X) = N_\theta^A(\mathcal{I},F) \cap m(X)$.

Proof. (i) Let $\varepsilon > 0$ and $x_k \to L\left(N_\theta^A(\mathcal{I},F)\right)$. Then, we can write

$$\begin{aligned} \sum_{k\in I_r} f_k(|A_k(x)-L|) &\geq \sum_{\substack{k\in I_r \\ f_k(|A_k(x)-L|)\geq\varepsilon}} f_k(|A_k(x)-L|) \\ &\geq \varepsilon\left|\{k\in I_r : f_k(|A_k(x)-L|)\geq\varepsilon\}\right|. \end{aligned}$$

Thus, for given $\delta > 0$,

$$\frac{1}{h_r}\left|\{k\in I_r : f_k(|A_k(x)-L|)\geq\varepsilon\}\right| \geq \delta \Longrightarrow \frac{1}{h_r}\sum_{k\in I_r} f_k(|A_k(x)-L|) \geq \varepsilon\delta,$$

i.e.,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\{k\in I_r : f_k(|A_k(x)-L|)\geq\varepsilon\}\right| \geq \delta\right\} \subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k(|A_k(x)-L|) \geq \varepsilon\delta\right\}.$$

Since $x_k \to L\left(N_\theta^A(\mathcal{I},F)\right)$, the set on the right-hand side belongs to $\mathcal{I}$ and so it follows that $x_k \to L\left(S_\theta^A(\mathcal{I},F)\right)$.

To show that $\left(S_\theta^A(\mathcal{I},F)\right) \subsetneq \left(N_\theta^A(\mathcal{I},F)\right)$, take a fixed $K \in \mathcal{I}$. Define $x = (x_k)$ by

$$(x_k) = \begin{cases} ku, & \text{for } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right],\ r = 1,2,3...,\ r \notin K, \\ ku, & \text{for } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right],\ r = 1,2,3...,\ r \in K, \\ \theta, & \text{otherwise,} \end{cases}$$

where $u \in X$ is a fixed element with $\|u\| = 1$ and θ is the null element of X. Then, $x \notin m(X)$ and for every $0 < \varepsilon < 1$ since

$$\frac{1}{h_r}\left|\{k\in I_r : f_k(|A_k(x)-0|)\geq\varepsilon\}\right| = \frac{\left[\sqrt{h_r}\right]}{\sqrt{h_r}} \to 0.$$

As $r \to \infty$ and $r \notin K$, for every $\delta > 0$,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\,\{k\in I_r : f_k(|A_k(x)-0|)\geq\varepsilon\}\,\right| \geq \delta\right\} \subset M \cup \{1,2,...,m\}$$

for some $m \in \mathbb{N}$. Since $\mathcal{I}$ is admissible, it follows that $x_k \to \theta\left(S_\theta^A(\mathcal{I},F)\right)$. Obviously,

$$\frac{1}{h_r}\sum_{k\in I_r} f_k(|A_k(x)-\theta|) \to \infty,$$

i.e., $x_k \nrightarrow \theta\left(N_\theta^A(\mathcal{I},F)\right)$. Note that, if $K \in \mathcal{I}$ is finite, then $x_k \nrightarrow \theta\left(S_\theta^A\right)$. This example shows that $A^{\mathcal{I}}$-lacunary statistical convergence is more general than lacunary statistical convergence.

(ii) Suppose that $x \in l_\infty$ and $x_k \to L\left(S_\theta^A(\mathcal{I},F)\right)$. Then, we can assume that

$$f_k(|A_k(x)-L|) \leq M$$

for each $x \in X$ and all k.

Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x)-L|\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ f_k(|A_kx-L|)\geq\varepsilon}} f_k\left(|A_k(x)-L|\right)\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ f_k(|A_kx-L|)<\varepsilon}} f_k\left(|A_k(x)-L|\right)\\ &\leq \frac{M}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|+\varepsilon.\end{aligned}$$

Note that

$$A\left(\varepsilon\right)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|\geq\frac{\varepsilon}{M}\right\}\in\mathcal{I}.$$

If $n\in\left(A\left(\varepsilon\right)\right)^c$, then

$$\frac{1}{h_r}\sum_{k\in I_r} f_k|A_k(x)-L|<2\varepsilon.$$

Hence,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r} f_k|A_k(x)-L|\geq2\varepsilon\right\}\subset A\left(\varepsilon\right)$$

and thus belongs to $\mathcal{I}$. This shows that $x_k\to L\left(N_\theta^A\left(\mathcal{I},F\right)\right)$.

(iii) This is an immediate consequence of (i) and (ii). □

Theorem 3. *Let $A=(a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta=\{k_r\}$ is a lacunary sequence with $\liminf_r q_r>1$, then*

$$x_k\to L\left(S^A\left(\mathcal{I},F\right)\right)\Rightarrow x_k\to L\left(S_\theta^A\left(\mathcal{I},F\right)\right).$$

Proof. Suppose first that $\liminf_r q_r>1$, then there exists $\delta>0$ such that $q_r\geq1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}.$$

If $x_k\to L\left(S_\theta^A\left(\mathcal{I},F\right)\right)$, then for every $\varepsilon>0$, for each $x\in X$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\{k\leq k_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right| &\geq \frac{1}{k_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|\\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|.\end{aligned}$$

Then, for any $\delta>0$, we get

$$\begin{aligned}&\left\{r\in\mathbb{N}:\tfrac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|\geq\delta\right\}\\ &\subseteq\left\{r\in\mathbb{N}:\tfrac{1}{k_r}\left|\{k\leq k_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|\geq\tfrac{\delta\alpha}{(\alpha+1)}\right\}\in\mathcal{I}.\end{aligned}$$

This completes the proof. □

For the next result, we assume that the lacunary sequence θ satisfies the condition that, for any set $C\in\mathcal{F}\left(\mathcal{I}\right)$, $\cup\left\{n : k_{r-1}<n\leq k_r,\ r\in C\right\}\in\mathcal{F}\left(\mathcal{I}\right)$.

Theorem 4. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta = \{k_r\}$ is a lacunary sequence with $\limsup_r q_r < \infty$, then*

$$x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right) \text{ implies } x_k \to L\left(S^A(\mathcal{I}, F)\right).$$

Proof. If $\limsup_r q_r < \infty$, then, without any loss of generality, we can assume that there exists a $0 < M < \infty$ such that $q_r < M$ for all $r \geq 1$. Suppose that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$, and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{k \in I_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in \mathcal{F}(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$K_j = \frac{1}{h_j}\left|\{k \in I_j : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Now,

$$\begin{aligned}
\frac{1}{n}\left|\{k \leq n : f_k|A_k(x) - L| \geq \varepsilon\}\right| &\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&= \frac{1}{k_{r-1}}\{|\{k \in I_1 : f_k(|A_k(x) - L|) \geq \varepsilon\}|\} \\
&\quad + \frac{1}{k_{r-1}}\{|\{k \in I_2 : f_k(|A_k x - L|) \geq \varepsilon\}|\} \\
&\quad + \ldots + \frac{1}{k_{r-1}}\{|\{k \in I_r : f_k|A_k(x) - L| \geq \varepsilon\}|\} \\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}\left|\{k \in I_1 : f_k|A_k(x) - L| \geq \varepsilon\}\right| \\
&\quad + \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}\left|\{k \in I_2 : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&\quad + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}\left|\{k \in I_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}K_1 + \frac{k_2 - k_1}{k_{r-1}}K_2 + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}K_r \\
&\leq \left\{\sup_{j \in C} K_j\right\}\frac{k_r}{k_{r-1}} < M\delta.
\end{aligned}$$

Choosing $\delta_1 = \dfrac{\delta}{M}$ and in view of the fact that $\cup\{n : k_{r-1} < n \leq k_r,\ r \in C\} \subset T$ where $C \in \mathcal{F}(\mathcal{I})$, it follows from our assumption on θ that the set T also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Combining Theorems 3 and 4, we get the following theorem.

Theorem 5. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta = \{k_r\}$ is a lacunary sequence with $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then*

$$x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right) = x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right).$$

4. Cesàro Summability for $A^{\mathcal{I}}$

Definition 8. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-Cesàro summable to L if, for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{n \in \mathbb{N} : \left|\frac{1}{n}\sum_{k=1}^{n} f_k\left(A_k(x) - L\right)\right| \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left((\sigma_1)_\theta^A(\mathcal{I}, F)\right)$.

Definition 9. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be strongly $A^{\mathcal{I}}$-Cesàro summable to L if, for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$.

Theorem 6. *Let θ be a lacunary sequence. If $\liminf_r q_r > 1$, then*

$$x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right) \Rightarrow x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right).$$

Proof. If $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r \geq 1 + \delta$ for all $r \geq 1$. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. Let $\varepsilon > 0$ and define the set

$$S = \left\{k_r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) < \varepsilon\right\}.$$

We can easily say that $S \in \mathcal{F}(\mathcal{I})$, which is a filter of the ideal $\mathcal{I}$,

$$\begin{aligned}
\frac{1}{h_r}\sum_{k \in I_r} f_k\left(|A_k(x) - L|\right) &= \frac{1}{h_r}\sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}} f_k\left(|A_k(x) - L|\right) \\
&= \frac{k_r}{h_r}\frac{1}{k_r}\sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) - \frac{k_{r-1}}{h_r}\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} f_k\left(|A_k(x) - L|\right) \\
&\leq \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'
\end{aligned}$$

for each $k_r \in S$. Choose $\eta = \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'$. Therefore,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r} f_k\left(|A_k(x) - L|\right) < \eta\right\} \in \mathcal{F}(\mathcal{I}),$$

and it completes the proof. □

Theorem 7. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $(x_k) \in m(X)$ and $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$, then $x_k \to L\left((\sigma_1)_\theta^A(\mathcal{I}, F)\right)$.*

Proof. Suppose that $(x_k) \in m(X)$ and $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$. Then, we can assume that

$$f_k(|A_k x - L|) \leq M$$

for all $k \in \mathbb{N}$. In addition, for each $\varepsilon > 0$, we can write

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} f_k(A_k(x) - L)\right| &\leq \frac{1}{n}\sum_{k=1}^{n} f_k(|A_k(x) - L|) \\
&\leq \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)\geq\frac{\varepsilon}{2}}}^{n} f_k(|A_k(x) - L|) \\
&\quad + \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)<\frac{\varepsilon}{2}}}^{n} f_k(|A_k(x) - L|) \\
&\leq M\frac{1}{n}|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| + \varepsilon.
\end{aligned}$$

Consequently, if $\delta > \varepsilon > 0$, δ and ε are independent, and, putting $\delta_1 = \delta - \varepsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \left|\frac{1}{n}\sum_{k,l=1}^{n} f_k(A_k(x) - L)\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n}|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \tfrac{\delta_1}{M}\right\} \in \mathcal{I}.$$

This shows that $x_k \to L\left((\sigma_1)_\theta^A(\mathcal{I}, F)\right)$. □

Theorem 8. *Let θ be a lacunary sequence. If $\limsup_r q_r < \infty$, then*

$$x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right) \Rightarrow x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right).$$

Proof. If $\limsup_r q_r < \infty$, then there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right)$ and define the sets T and R such that

$$T = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r} f_k(|A_k(x) - L|) < \varepsilon_1\right\}$$

and

$$R = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k(|A_k(x) - L|) < \varepsilon_2\right\}.$$

Let

$$A_j = \frac{1}{h_j}\sum_{k \in I_j} f_k(|A_k(x) - L|) < \varepsilon_1$$

for all $j \in T$. It is obvious that $T \in \mathcal{F}(\mathcal{I})$. Choose n as being any integer with $k_{r-1} < n < k_r$, where $r \in T$,

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) &\leq \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) \\
&= \frac{1}{k_{r-1}}\left(\sum_{k\in I_1} f_k\left(|A_k(x) - L|\right) + \sum_{k\in I_2} f_k\left(|A_k(x) - L|\right)\right. \\
&\quad \left. + \ldots + \sum_{k\in I_r} f_k\left(|A_k(x) - L|\right)\right) \\
&= \frac{k_1}{k_{r-1}}\left(\frac{1}{h_1}\sum_{k\in I_1} f_k\left(|A_k(x) - L|\right)\right) + \frac{k_2 - k_1}{k_{r-1}}\left(\frac{1}{h_2}\sum_{k\in I_2} f_k\left(|A_k(x) - L|\right)\right) \\
&\quad + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}\left(\frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x) - L|\right)\right) \\
&= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}} A_r \\
&\leq \left(\sup_{j\in T} A_j\right) \frac{k_1}{k_{r-1}} \\
&< \varepsilon_1 M.
\end{aligned}$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{M}$ and in view of the fact that $\cup\{n : k_{r-1} < n < k_r, r \in T\} \subset R$, where $T \in \mathcal{F}(\mathcal{I})$, it follows from our assumption on θ that the set R also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Theorem 9. *If $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$, then $x_k \to L\left(S^A(\mathcal{I}, F)\right)$.*

Proof. Let $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$ and $\varepsilon > 0$ is given. Then,

$$\begin{aligned}
\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) &\geq \sum_{\substack{k=1 \\ f_k(|A_k x - L|)\geq\varepsilon}}^{n} f_k\left(|A_k(x) - L|\right) \\
&\geq \varepsilon \left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right|
\end{aligned}$$

and so

$$\frac{1}{\varepsilon n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \geq \frac{1}{n}\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right|.$$

Thus, for a given $\delta > 0$,

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right| \geq \delta\right\} \\
&\qquad \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \geq \varepsilon\delta\right\} \in \mathcal{I}.
\end{aligned}$$

Therefore, $x_k \to L\left(S^A(\mathcal{I}, F)\right)$. □

Theorem 10. *Let* $(x_k) \in m(X)$. *If* $x_k \to L\left(S^A(\mathcal{I}, F)\right)$. *Then,* $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$.

Proof. Suppose that (x_k) is bounded and $x_k \to L\left(S^A(\mathcal{I}, F)\right)$. Then, there is an M such that $f_k(|A_k(x) - L|) \leq M$ for all k. Given $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f_k(|A_k(x) - L|) &= \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)\geq\varepsilon}}^{n} f_k(|A_k(x) - L|) \\
&+ \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)<\varepsilon}}^{n} f_k(|A_k(x) - L|) \\
&\leq \frac{1}{n}M\left|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&\quad + \frac{1}{n}\varepsilon\left|\{k \leq n : f_k(|A_k(x) - L|) < \varepsilon\}\right| \\
&\leq \frac{M}{n}\left|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| + \varepsilon.
\end{aligned}$$

Then, for any $\delta > 0$,

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k(|A_k(x) - L|) \geq \delta\right\} \\
&\qquad \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \geq \frac{\delta}{M}\right\} \in \mathcal{I}.
\end{aligned}$$

Therefore, $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$. $\square$

5. Conclusions

$\mathcal{I}$-statistical convergence gained a different perspective after identification of the $A^{\mathcal{I}}$-statistical convergence with an infinite matrix of complex numbers. Some authors have studied this new method with different sequences. Our results in this paper were developed with lacunary sequences. By also using a modulus function, we obtain more interesting and general results. These definitions can be adapted to many different concepts such as random variables in order to have different results.

Acknowledgments: We would like to thanks the referees for their valuable comments.

Author Contributions: Both authors completed the paper together. Both authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zygmund, A. *Trigonometric Series*; Cambridge University Press: Cambridge, UK, 1979.
2. Fast, H. Sur la convergence statistique. *Colloq. Math.* **1951**, *2*, 241–244.
3. Fridy, J.A. On statistical convergence. *Analysis* **1985**, *5*, 301–313.
4. Salat, T. On statistically convergent sequences of real numbers. *Math. Slovaca* **1980**, *30*, 139–150.
5. Steinhaus, H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math.* **1951**, *2*, 73–74.
6. Schoenberg, I.J. The integrability of certain functions and related summability methods. *Am. Math. Mon.* **1959**, *66*, 361–375.
7. Erdös, P.; Tenenbaum, G. Sur les densities de certaines suites d'entiers. *Proc. Lond. Math. Soc.* **1989**, *3*, 417–438.

8. Miller, H.I. A measure theoretical subsequence characterization of statistical convergence. *Trans. Am. Math. Soc.* **1995**, *347*, 1811–1819.
9. Freedman, A.R.; Sember, J.J. Densities and summability. *Pac. J. Math.* **1981**, *95*, 10–11.
10. Fridy, J.A.; Orhan, C. lacunary statistical convergence. *Pac. J. Math.* **1993**, *160*, 43–51.
11. Kostyrko, P.; Macaj, M.; Salat, T. $\mathcal{I}$-convergence. *Real Anal. Exchang.* **2000**, *26*, 669–686.
12. Das, P.; Ghosal, S. Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP). *Comput. Math. Appl.* **2010**, *29*, 2597–2600.
13. Kostyrko, P.; Macaj, M.; Salat, T.; Sleziak, M. $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. *Math. Slovaca* **2005**, *55*, 443–464.
14. Savas, E.; Das, P. A generalized statistical convergence via ideals. *Appl. Math. Lett.* **2011**, *24*, 826–830.
15. Nakano, H. Concave modulars. *J. Math. Soc. Jpn.* **1953**, *5*, 29–49.
16. Connor, J.S. On strong matrix summability with respect to a modulus and statistical convergence. *Can. Math. Bull.* **1989**, *32*, 194–198.
17. Bilgin, T. Lacunary strong A-convergence with respect to a modulus. *Mathematica* **2001**, *4*, 39–46.
18. Maddox, I.J. Sequence spaces defined by a modulus. *Math. Proc. Camb. Philos. Soc.* **1986**, *100*, 161–166.
19. Kolk, E. On strong boundedness and summability with respect to a sequence moduli. *Tartu Ülikooli Toimetised* **1993**, *960*, 41–50.
20. Pehlivan, S.; Fisher, B. On some sequence spaces. *Indian J. Pure Appl. Math.* **1994**, *25*, 1067–1071.
21. Ruckle, W.H. FK spaces in which the sequence of coordinate vectors is bounded. *Can. J. Math.* **1973**, *25*, 973–978.
22. Das, P.; Savas, E.; Ghosal, S. On generalizations of certain summability methods using ideals. *Appl. Math. Lett.* **2011**, *24*, 1509–1514.
23. Yamancı, U.; Gürdal, M.; Saltan, S. $A^{\mathcal{I}}$-statistical convergence with respect to a sequence of modulus functions. *Contemp. Anal. Appl. Math.* **2014**, *2*, 136–145.