

Article

Neutrosophic Incidence Graphs With Application

Muhammad Akram ^{1,*} , Sidra Sayed ¹ and Florentin Smarandache ² 

¹ Department of Mathematics, University of the Punjab, New Campus, Lahore 54590, Pakistan; sidratulmuntha228@yahoo.com

² Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; fsmarandache@gmail.com

* Correspondence: m.akram@pucit.edu.pk; Tel.: +92-429-9231241

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Abstract: In this research study, we introduce the notion of single-valued neutrosophic incidence graphs. We describe certain concepts, including bridges, cut vertex and blocks in single-valued neutrosophic incidence graphs. We present some properties of single-valued neutrosophic incidence graphs. We discuss the edge-connectivity, vertex-connectivity and pair-connectivity in neutrosophic incidence graphs. We also deal with a mathematical model of the situation of illegal migration from Pakistan to Europe.

Keywords: single-valued neutrosophic incidence graphs; edge-connectivity; vertex-connectivity and pair-connectivity; application

1. Introduction

The concept of graph theory was introduced by Euler. A crisp graph shows the relations between the elements of the vertex set. A weighted graph gives the extent of these relations. Many problems can be solved if proper weights are given. However, in many situations, the weights may not known, and the relationship is uncertain. Hence, a fuzzy relation can be used to handle such situations. Rosenfeld [1] developed the concept of a fuzzy graph. He also discussed several concepts like edges, paths, bridges and connectedness in a fuzzy graph. Most of the theoretical development of fuzzy graph theory is based on Rosenfeld's initial work. Bhutani et al. [2,3] introduced the advance concepts in fuzzy graphs.

Sometimes when the relationship between the elements of the vertex set is indeterminate, the fuzzy graph and its extension fails. This indeterminacy can be overcome by using single-valued neutrosophic graphs [4].

Dinesh, in [5], introduced the concept of unordered pairs of vertices which are not incident with end vertices. The fuzzy incidence graph not just shows the relations between vertices, but also provides information about the influence of a vertex on an edge. Dinesh extended the idea of the fuzzy incidence graph in [6] by introducing new concepts in this regard. Later, Methew et al. [7] discussed the connectivity concepts in fuzzy incidence graphs. Malik et al. [8] applied the notion of the fuzzy incidence graph in problems involving human trafficking. They discussed the role played by the vulnerability of countries and their government's response to human trafficking. Methew et al. [9] studied fuzzy incidence blocks and their applications in illegal migration problems. They used fuzzy incidence graphs as a model for a nondeterministic network with supporting links. They used fuzzy incidence blocks to avoid the vulnerable links in the network.

The paper is organized as follows: In Section 1, we give some preliminary notions and terminologies of fuzzy incidence graphs which are needed to understand the extended concept of the single-valued neutrosophic incidence graph. In Section 2, we present the definition of a single-valued neutrosophic incidence graph. We also discuss the edge-connectivity, vertex-connectivity and

pair-connectivity in neutrosophic incidence graphs. In Section 3, we give a mathematical model of the situation of illegal migration from Pakistan to Europe. Finally, the paper is concluded by some remarks in Section 4. Below we present some preliminary definitions from [6] and [4]. For further study on these topics, the readers are referred to references [1,7–16].

Let $G = (V, E)$ be a graph on a nonempty set, V . Then, $G' = (V, E, I)$ is called an incidence graph, where $I \subseteq V \times E$. The elements of I are called incidence pairs or simply, pairs.

A fuzzy incidence graph of an incidence graph, $G' = (V, E, I)$, is an ordered-triplet, $\tilde{G} = (\mu, \lambda, \psi)$, where μ is a fuzzy subset of V , λ is a fuzzy relation of V , and ψ is a fuzzy subset of I such that

$$\psi(x, xy) \leq \min\{\mu(x), \lambda(xy)\}, \forall x \in V, xy \in E.$$

We may compare elements of two neutrosophic sets A and B , that is

$$\begin{aligned} (T_A(x), I_A(x), F_A(x)) &< (T_B(x), I_B(x), F_B(x)) \\ \Rightarrow T_A(x) < T_B(x), I_A(x) < I_B(x), F_A(x) > F_B(x). \end{aligned}$$

2. Single-Valued Neutrosophic Incidence Graphs

Definition 1. A single-valued neutrosophic incidence graph of an incidence graph, $G' = (V, E, I)$, is an ordered-triplet, $\tilde{G} = (A, B, C)$, such that

1. A is a single-valued neutrosophic set on V .
2. B is a single-valued neutrosophic relation on V .
3. C is a single-valued neutrosophic subset of $V \times E$ such that

$$\begin{aligned} T_C(x, xy) &\leq \min\{T_A(x), T_B(xy)\}, \\ I_C(x, xy) &\leq \min\{I_A(x), I_B(xy)\}, \\ F_C(x, xy) &\leq \max\{F_A(x), F_B(xy)\}, \forall x \in V, xy \in E. \end{aligned}$$

Here, we discuss an example of a single-valued neutrosophic incidence graph (SVNIG).

Example 1. Consider an incidence graph, $G = (V, E, I)$, such that $V = \{a, b, c, d\}$, $E = \{ab, bc, bd, cd, ad\}$ and $I = \{(a, ab), (b, ab), (b, bc), (c, bc), (b, bd), (d, bd), (c, cd), (d, cd), (d, ad), (a, ad)\}$, as shown in Figure 1.

Let $\tilde{G} = (A, B, C)$ be a single-valued neutrosophic incidence graph associated with G , as shown in Figure 2, where

$$\begin{aligned} A &= \{(a, 0.2, 0.5, 0.8), (b, 0.3, 0.5, 0.1), (c, 0.9, 0.9, 0.1), (d, 0.8, 0.1, 0.2)\}, \\ B &= \{(ab, 0.2, 0.4, 0.7), (bc, 0.3, 0.4, 0.1), (bd, 0.1, 0.1, 0.1), (cd, 0.7, 0.1, 0.2), (ad, 0.1, 0.1, 0.5)\}, \\ C &= \{((a, ab), 0.2, 0.3, 0.7), ((b, ab), 0.1, 0.4, 0.6), ((b, bc), 0.3, 0.3, 0.1), ((c, bc), 0.2, 0.3, 0.1), \\ &\quad ((b, bd), 0.1, 0.1, 0.1), ((d, bd), 0.1, 0.1, 0.2), ((c, cd), 0.7, 0.1, 0.2), ((d, cd), 0.7, 0.1, 0.2) \\ &\quad ((d, ad), 0.1, 0.1, 0.4), ((a, ad), 0.1, 0.1, 0.5)\}. \end{aligned}$$

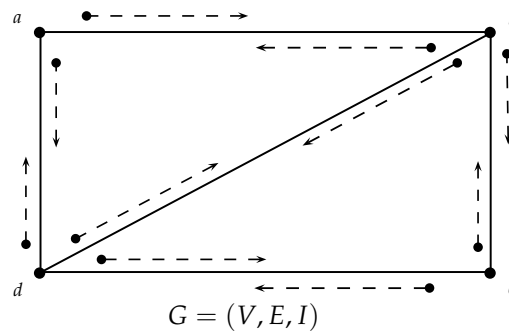


Figure 1. Incidence graph.

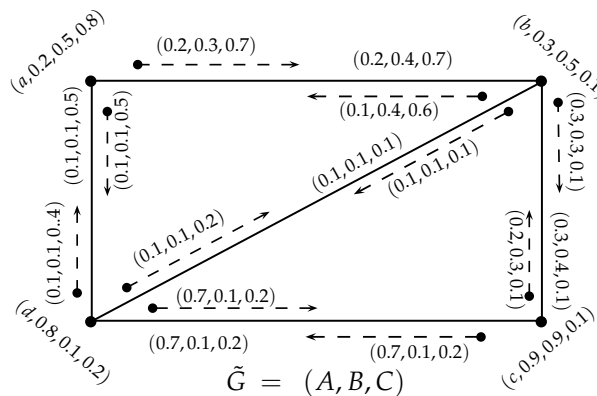


Figure 2. Single-valued neutrosophic incidence graph.

Definition 2. The support of an SVNIG $\tilde{G} = (A, B, C)$ is denoted by $G^* = (A^*, B^*, C^*)$ where

$$\begin{aligned}
 A^* &= \text{support of } A = \{x \in V : T_A(x) > 0, I_A(x) > 0, F_A(x) > 0\}, \\
 B^* &= \text{support of } B = \{xy \in E : T_B(xy) > 0, I_B(xy) > 0, F_B(xy) > 0\}, \\
 C^* &= \text{support of } C = \{(x, xy) \in I : T_C(x, xy) > 0, I_C(x, xy) > 0, F_C(x, xy) > 0\}.
 \end{aligned}$$

Now we introduce the concepts of edge, pair, walk, trail, path and connectedness in an SVNIG.

Definition 3. If $xy \in B^*$, then xy is an edge of the SVNIG $\tilde{G} = (A, B, C)$ and if $(x, xy), (y, xy) \in C^*$, then (x, xy) and (y, xy) are called pairs of \tilde{G} .

Definition 4. A sequence

$$\begin{aligned}
 P : u_0, (u_0, u_0u_1), u_0u_1, (u_1, u_0u_1), u_1, (u_1, u_1u_2), u_1u_2, (u_2, u_1u_2), u_2, \dots, \\
 u_{n-1}, (u_{n-1}, u_{n-1}u_n), u_{n-1}u_n, (u_n, u_{n-1}u_n), u_n
 \end{aligned}$$

of vertices, edges and pairs in \tilde{G} is a walk. It is a closed walk if $u_0 = u_n$.

In the above sequence, if all edges are distinct, then it is a trail, and if the pairs are distinct, then it is an incidence trail. P is called a path if the vertices are distinct. A path is called a cycle if the initial and end vertices of the path are same. Any two vertices of \tilde{G} are said to be connected if they are joined by a path.

Example 2. In the example presented earlier

$$P_1 : a, (a, ab), ab, (b, ab), b, (b, bd), bd, (d, bd), d, (d, da), da, (a, da), a$$

is a walk. It is a closed walk since the initial and final vertices are same, i.e., it is not a path, but it is a trail and an incidence trail.

$$P_2 : a, (a, ab), ab, (b, ab), b, (b, bd), bd, (d, bd), d$$

P_2 is a walk, path, trail and an incidence trail.

Definition 5. Let $\tilde{G} = (A, B, C)$ be a nSVNIG. Then, $\tilde{H} = (L, M, N)$ is a single-valued neutrosophic incidence subgraph of \tilde{G} if $L \subseteq A, M \subseteq B$ and $N \subseteq C$. \tilde{H} is a single-valued neutrosophic incidence spanning subgraph of \tilde{G} if $L^* = A^*$.

Definition 6. In an SVNIG, the strength of a path, P , is an ordered triplet denoted by $S(P) = (s_1, s_2, s_3)$, where

$$\begin{aligned} s_1 &= \min\{T_B(uv) : uv \in P\}, \\ s_2 &= \min\{I_B(uv) : uv \in P\}, \\ s_3 &= \max\{F_B(uv) : uv \in P\}. \end{aligned}$$

Similarly, the incidence strength of a path, P , in an SVNIG is denoted by $IS(P) = (is_1, is_2, is_3)$, where

$$\begin{aligned} is_1 &= \min\{T_C(u, uv) : (u, uv) \in P\}, \\ is_2 &= \min\{I_C(u, uv) : (u, uv) \in P\}, \\ is_3 &= \max\{F_C(u, uv) : (u, uv) \in P\}. \end{aligned}$$

Example 3. Let $G = (V, E, I)$ be an incidence graph, as shown in Figure 3, and $\tilde{G} = (A, B, C)$ is an SVNIG associated with G , which is shown in Figure 4.

Clearly, $P : u, (u, uv), uv, (v, uv), v, (v, vx), vx, (x, vx), x$ is a path in \tilde{G} .

The strength of the path, P , is $S(P) = (0.2, 0.1, 0.5)$, and the incidence strength of P is $IS(P) = (0.1, 0.1, 0.6)$.

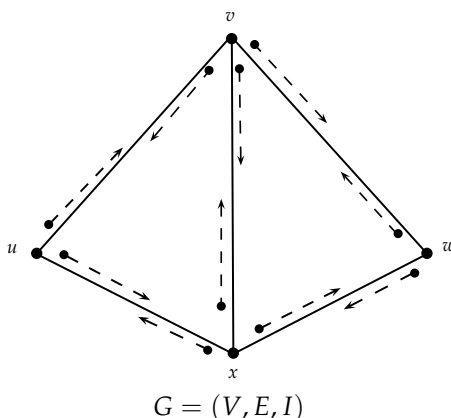


Figure 3. Incidence graph.

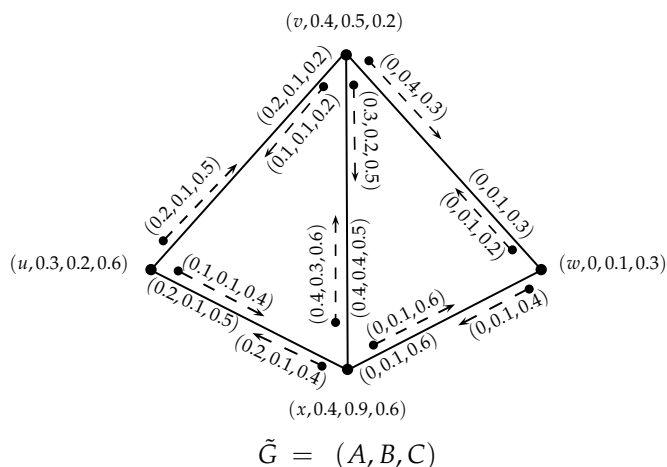


Figure 4. Single-valued neutrosophic incidence graph.

Definition 7. In an SVNIG, $\tilde{G} = (A, B, C)$ the greatest strength of the path from l to m , where $l, m \in A^* \cup B^*$ is the maximum of strength of all paths from l to m .

$$\begin{aligned}
 S^\infty(l, m) &= \max\{S(P_1), S(P_2), S(P_3), \dots\} \\
 &= (s_1^\infty, s_2^\infty, s_3^\infty) \\
 &= (\max(s_{11}, s_{12}, s_{13}, \dots), \max(s_{21}, s_{22}, s_{23}, \dots), \min(s_{31}, s_{32}, s_{33}, \dots)).
 \end{aligned}$$

$S^\infty(l, m)$ is sometimes called the connectedness between l and m .

Similarly, the greatest incidence strength of the path from l to m , where $l, m \in A^* \cup B^*$ is the maximum incidence strength of all paths from l to m .

$$\begin{aligned}
 IS^\infty(l, m) &= \max\{IS(P_1), IS(P_2), IS(P_3), \dots\} \\
 &= (is_1^\infty, is_2^\infty, is_3^\infty) \\
 &= (\max(is_{11}, is_{12}, is_{13}, \dots), \max(is_{21}, is_{22}, is_{23}, \dots), \min(is_{31}, is_{32}, is_{33}, \dots)),
 \end{aligned}$$

where $P_j, j = 1, 2, 3, \dots$ are different paths from l to m .

$IS^\infty(l, m)$ is sometimes referred as the incidence connectedness between l and m .

Example 4. In the SVNIG given in Figure 4, the total paths from vertex u to w are as follows:

- $P_1 : u, (u, ux), ux, (x, ux), x, (x, wx), wx, (w, wx), w.$
- $P_2 : u, (u, uv), uv, (v, uv), v, (v, vw), vw, (w, vw), w.$
- $P_3 : u, (u, uv), uv, (v, uv), v, (v, vx), vx, (x, vx), x, (x, wx), wx, (w, wx), w.$
- $P_4 : u, (u, ux), ux, (x, ux), x, (x, vx), vx, (v, vx), v, (v, vw), vw, (w, vw), w.$

The corresponding incidence strengths of each path are

$$\begin{aligned}
 IS(P_1) &= (s_{11}, s_{21}, s_{31}) = (0, 0.1, 0.6), \\
 IS(P_2) &= (s_{12}, s_{22}, s_{32}) = (0, 0.1, 0.5), \\
 IS(P_3) &= (s_{13}, s_{23}, s_{33}) = (0, 0.1, 0.6), \\
 IS(P_4) &= (s_{14}, s_{24}, s_{34}) = (0, 0.1, 0.6).
 \end{aligned}$$

Now, the greatest incidence strength of the path from u to w is calculated as follows:

$$\begin{aligned} IS^\infty(u, w) &= \max\{IS(P_1), IS(P_2), IS(P_3), IS(P_4)\} \\ &= (\max\{is_{11}, is_{12}, is_{13}, is_{14}\}, \max\{is_{21}, is_{22}, is_{23}, is_{24}\}, \min\{is_{31}, is_{32}, is_{33}, is_{34}\}) \\ &= (\max\{0, 0, 0, 0\}, \max\{0.1, 0.1, 0.1, 0.1\}, \min\{0.6, 0.5, 0.6, 0.6\}) \\ &= (0, 0.1, 0.5). \end{aligned}$$

Definition 8. An SVNIG, $\tilde{G} = (A, B, C)$, is a cycle if, and only if, the underlying graph, $G^* = (A^*, B^*, C^*)$, is a cycle.

Definition 9. The SVNIG $\tilde{G} = (A, B, C)$ is a neutrosophic cycle if, and only if, $G^* = (A^*, B^*, C^*)$ is a cycle and there exists no unique edge, $xy \in B^*$, such that

$$\begin{aligned} T_B(xy) &= \min\{T_B(uv) : uv \in B^*\}, \\ I_B(xy) &= \min\{I_B(uv) : uv \in B^*\}, \\ F_B(xy) &= \max\{F_B(uv) : uv \in B^*\}. \end{aligned}$$

Definition 10. The SVNIG $\tilde{G} = (A, B, C)$ is a neutrosophic incidence cycle if, and only if it is a neutrosophic cycle and there exists no unique pair, $(x, xy) \in C^*$, such that

$$\begin{aligned} T_C(x, xy) &= \min\{T_C(u, uv) : (u, uv) \in C^*\}, \\ I_C(x, xy) &= \min\{I_C(u, uv) : (u, uv) \in C^*\}, \\ F_C(x, xy) &= \max\{F_C(u, uv) : (u, uv) \in C^*\}. \end{aligned}$$

Example 5. Let $\tilde{G} = (A, B, C)$ be an SVNIG, as shown in Figure 5. \tilde{G} is a cycle, since $G = (A^*, B^*, C^*)$ (support of \tilde{G}) is clearly a cycle.

Also,

$$\begin{aligned} T_B(ab) &= 0.1 = \min\{T_B(ab), T_B(bc), T_B(cd), T_B(de), T_B(ea)\}, \\ I_B(ab) &= 0.1 = \min\{I_B(ab), I_B(bc), I_B(cd), I_B(de), I_B(ea)\}, \\ F_B(ab) &= 0.6 = \max\{F_B(ab), F_B(bc), F_B(cd), F_B(de), F_B(ea)\}, \end{aligned}$$

and

$$\begin{aligned} T_B(bc) &= 0.1 = \min\{T_B(ab), T_B(bc), T_B(cd), T_B(de), T_B(ea)\}, \\ I_B(bc) &= 0.1 = \min\{I_B(ab), I_B(bc), I_B(cd), I_B(de), I_B(ea)\}, \\ F_B(bc) &= 0.6 = \max\{F_B(ab), F_B(bc), F_B(cd), F_B(de), F_B(ea)\}. \end{aligned}$$

So, \tilde{G} is a neutrosophic cycle.

Furthermore, \tilde{G} is a neutrosophic incidence cycle since there is more than one pair, namely, (b, ab) and (d, de) , such that

$$\begin{aligned} T_C(b, ab) &= 0.1 = \min\{T_C(u, uv) : (u, uv) \in C^*\}, \\ I_C(b, ab) &= 0.1 = \min\{I_C(u, uv) : (u, uv) \in C^*\}, \\ F_C(b, ab) &= 0.7 = \max\{F_C(u, uv) : (u, uv) \in C^*\}, \end{aligned}$$

and

$$\begin{aligned} T_C(d, de) &= 0.1 = \min\{T_C(u, uv) : (u, uv) \in C^*\}, \\ I_C(d, de) &= 0.1 = \min\{I_C(u, uv) : (u, uv) \in C^*\}, \\ F_C(d, de) &= 0.7 = \max\{F_C(u, uv) : (u, uv) \in C^*\}. \end{aligned}$$

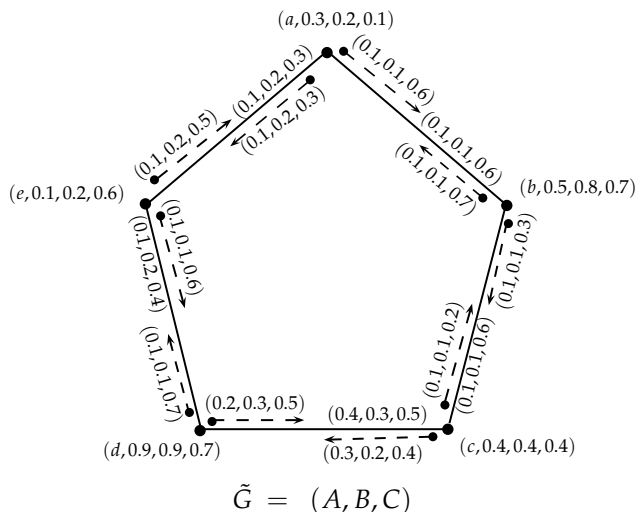


Figure 5. Single-valued neutrosophic incidence graph.

The concepts of bridges, cutvertices and cutpairs in SVNIG are defined as follows.

Definition 11. Let $\tilde{G} = (A, B, C)$ be an SVNIG. An edge uv , in \tilde{G} is called a bridge if, and only if, uv is a bridge in $G^* = (A^*, B^*, C^*)$ —that is, the removal of uv disconnects G^* .

An edge uv is called a neutrosophic bridge if

$$S'^{\infty}(x, y) < S^{\infty}(x, y), \text{ for some } x, y \in A^*,$$

$$(s_1'^{\infty}, s_2'^{\infty}, s_3'^{\infty}) < (s_1^{\infty}, s_2^{\infty}, s_3^{\infty})$$

$$\Rightarrow s_1'^{\infty} < s_1^{\infty}, s_2'^{\infty} < s_2^{\infty}, s_3'^{\infty} > s_3^{\infty}$$

where $S'^{\infty}(x, y)$ and $S^{\infty}(x, y)$ denote the connectedness between x and y in $G' = \tilde{G} - \{uv\}$ and \tilde{G} , respectively.

An edge, uv , is called a neutrosophic incidence bridge if

$$IS'^{\infty}(x, y) < IS^{\infty}(x, y), \text{ for some } x, y \in A^*,$$

$$(is_1'^{\infty}, is_2'^{\infty}, is_3'^{\infty}) < (is_1^{\infty}, is_2^{\infty}, is_3^{\infty})$$

$$\Rightarrow is_1'^{\infty} < is_1^{\infty}, is_2'^{\infty} < is_2^{\infty}, is_3'^{\infty} > is_3^{\infty}$$

where $IS'^{\infty}(x, y)$ and $IS^{\infty}(x, y)$ denote the incidence connectedness between x and y in $G' = \tilde{G} - \{uv\}$ and \tilde{G} , respectively.

Definition 12. Let $\tilde{G} = (A, B, C)$ be an SVNIG. A vertex, v , in \tilde{G} is a cutvertex if, and only if, it is a cutvertex in $G^* = (A^*, B^*, C^*)$ —that is $G^* - \{v\}$ is a disconnect graph.

A vertex, v , in an SVNIG is called a neutrosophic cutvertex if the connectedness between any two vertices in $G' = \tilde{G} - \{v\}$ is less than the connectedness between the same vertices in \tilde{G} —that is,

$$S'^{\infty}(x, y) < S^{\infty}(x, y), \text{ for some } x, y \in A^*.$$

A vertex, v , in SVNIG \tilde{G} is a neutrosophic incidence cutvertex if for any pair of vertices, x, y , other than v , the following condition holds:

$$IS'^{\infty}(x, y) < IS^{\infty}(x, y),$$

where $IS'^{\infty}(x, y)$ and $IS^{\infty}(x, y)$ denote the incidence connectedness between x and y in $G' = \tilde{G} - \{v\}$ and \tilde{G} , respectively.

Definition 13. Let $\tilde{G} = (A, B, C)$ be an SVNIG. A pair (u, uv) is called a cutpair if, and only if, (u, uv) is a cutpair in $G^* = (A^*, B^*, C^*)$ —that is, after removing the pair (u, uv) , there is no path between u and uv .

Let $\tilde{G} = (A, B, C)$ be an SVNIG. A pair (u, uv) is called a neutrosophic cutpair if deleting the pair (u, uv) reduces the connectedness between $u, uv \in A^* \cup B^*$, that is,

$$S'^{\infty}(u, uv) < S^{\infty}(u, uv),$$

where $S'^{\infty}(u, uv)$ and $S^{\infty}(u, uv)$ denote the connectedness between u and uv in $G' = \tilde{G} - \{(u, uv)\}$ and \tilde{G} , respectively.

A pair (u, uv) is called neutrosophic incidence cutpair if

$$IS'^{\infty}(u, uv) < IS^{\infty}(u, uv), \quad \text{for } u, uv \in A^* \cup B^*,$$

where $IS'^{\infty}(u, uv)$ and $IS^{\infty}(u, uv)$ denote the incidence connectedness between u and uv in $G' = \tilde{G} - \{(u, uv)\}$ and \tilde{G} , respectively.

Example 6. In the SVNIG, \tilde{G} , given in Figure 6, ab and bc are bridges, since their removal disconnects the underlying graph, G^* .

In \tilde{G} , ab, bc, cd and de are neutrosophic bridges, since, for $a, e \in A^*$,

$$S'^{\infty}(a, e) < S^{\infty}(a, e)$$

after the removal of each of the bridges. The edges— ab, bc, cd and de —are neutrosophic incidence bridges in \tilde{G} as well.

b and c are cutvertices. In addition, all the vertices of \tilde{G} are neutrosophic cutvertices, except for a , since the removal of a does not affect the connectedness of \tilde{G} . b, c, d and e are neutrosophic incidence cutvertices in \tilde{G} .

The pairs $(a, ab), (b, ab), (b, bc)$ and (c, bc) are the cutpairs, neutrosophic cutpairs and also neutrosophic incidence cutpairs in the given graph.

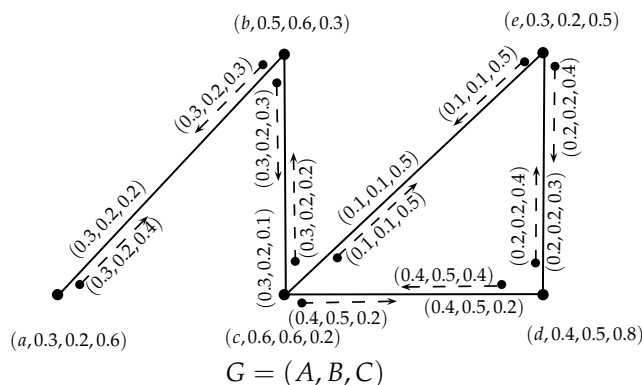


Figure 6. Single-valued neutrosophic incidence graph.

Theorem 1. Let $\tilde{G} = (A, B, C)$ be a SVNIG. If uv is a neutrosophic bridge, then uv is not a weakest edge in any cycle.

Proof. Let uv be a neutrosophic bridge and suppose, on the contrary, that uv is the weakest edge of a cycle. Then, in this cycle, we can find an alternative path, P_1 , from u to v that does not contain the edge uv , and $S(P_1)$ is greater than or equal to $S(P_2)$, where P_2 is the path involving the edge uv . Thus, removal of the edge uv from \tilde{G} does not affect the connectedness between u and v —a contradiction to our assumption. Hence, uv is not the weakest edge in any cycle. \square

Theorem 2. *If (u, uv) is a neutrosophic incidence cutpair, then (u, uv) is not the weakest pair in any cycle.*

Proof. Let (u, uv) be a neutrosophic incidence cutpair in \tilde{G} . On contrary suppose that (u, uv) is a weakest pair of a cycle. Then we can find an alternative path from u to uv having incidence strength greater than or equal to that of the path involving the pair (u, uv) . Thus, removal of the pair (u, uv) does not affect the incidence connectedness between u and uv but this is a contradiction to our assumption that (u, uv) is a neutrosophic incidence cutpair. Hence (u, uv) is not a weakest pair in any cycle. \square

Theorem 3. *Let $\tilde{G} = (A, B, C)$ be a SVNIG. If uv is a neutrosophic bridge in \tilde{G} , then*

$$S^\infty(u, v) = (s_1^\infty, s_2^\infty, s_3^\infty) = (T_B(uv), I_B(uv), F_B(uv)).$$

Proof. Let \tilde{G} be an SVNIG, and uv is a neutrosophic bridge in \tilde{G} . On the contrary, suppose that

$$S^\infty(u, v) > (T_B(uv), I_B(uv), F_B(uv)).$$

Then, there exists a u - v path, P , with

$$S(P) > (T_B(uv), I_B(uv), F_B(uv))$$

and

$$(T_B(xy), I_B(xy), F_B(xy)) > (T_B(uv), I_B(uv), F_B(uv)),$$

for all edges on path P . Now, P , together with the edge, uv , forms a cycle in which uv is the weakest edge, but it is a contradiction to the fact that uv is a neutrosophic bridge. Hence,

$$S^\infty(u, v) = (s_1^\infty, s_2^\infty, s_3^\infty) = (T_B(uv), I_B(uv), F_B(uv)).$$

\square

Theorem 4. *If (u, uv) is a neutrosophic incidence cutpair in an SVNIG $\tilde{G} = (A, B, C)$, then*

$$IS^\infty(u, uv) = (is_1^\infty, is_2^\infty, is_3^\infty) = (T_C(u, uv), I_C(u, uv), F_C(u, uv)).$$

Proof. The proof is on the same line as Theorem 3. \square

Theorem 5. *Let $\tilde{G} = (A, B, C)$ be an SVNIG and $G^* = (A^*, B^*, C^*)$ is a cycle. Then, an edge, uv , is a neutrosophic bridge of \tilde{G} if, and only if, it is an edge common to two neutrosophic incidence cutpairs.*

Proof. Suppose that uv is a neutrosophic bridge of \tilde{G} . Then, there exist vertices u and v with the uv edge lying on every path with the greatest incidence strength between u and v . Consequently, there exists only one path, P , (say) between u and v which contains a uv edge and has the greatest incidence strength. Any pair on P will be a neutrosophic incidence cutpair, since the removal of any one of them will disconnect P and reduce the incidence strength.

Conversely, let uv be an edge common to two neutrosophic incidence cutpairs (u, uv) and (v, uv) . Thus both (u, uv) and (v, uv) are not the weakest cutpairs of \tilde{G} . Now, $G^* = (A^*, B^*, C^*)$ being a cycle, there exist only two paths between any two vertices. Also the path P_1 from the vertex u to v not containing the pairs (u, uv) and (v, uv) has less incidence strength than the path containing them. Thus, the path with the greatest incidence strength from u to v is

$$P_2 : u, (u, uv), uv, (v, uv), v.$$

Also,

$$S^\infty(u, v) = S(P_2) = (T_B(uv), I_B(uv), F_B(uv)).$$

Therefore, uv is a neutrosophic bridge. \square

Definition 14. Let $\tilde{G} = (A, B, C)$ be an SVNIG. An edge, uv , of \tilde{G} is called a strong edge if

$$S'^\infty(u, v) \leq (T_B(uv), I_B(uv), F_B(uv)),$$

where $S'^\infty(u, v)$ represents the connectedness between u and v in $G' = \tilde{G} - \{uv\}$.

In particular, an edge, uv , is said to be an α -strong edge if

$$S'^\infty(u, v) < (T_B(uv), I_B(uv), F_B(uv)),$$

and it is called a β -strong edge if

$$S'^\infty(u, v) = (T_B(uv), I_B(uv), F_B(uv)).$$

Definition 15. A pair (u, uv) in an SVNIG, \tilde{G} , is called a strong pair if

$$IS'^\infty(u, uv) \leq (T_C(u, uv), I_C(u, uv), F_C(u, uv)),$$

where $IS'^\infty(u, uv)$ represents the incidence connectedness between u and uv in $G' = \tilde{G} - \{(u, uv)\}$.

In particular, (u, uv) is called α -strong pair if

$$IS'^\infty(u, uv) < (T_C(u, uv), I_C(u, uv), F_C(u, uv)),$$

and it is called β -strong pair if

$$IS'^\infty(u, uv) = (T_C(u, uv), I_C(u, uv), F_C(u, uv)).$$

It is not necessary for all edges and pairs to be strong. Edges and pairs exist which are not strong in an SVNIG. Such edges and pairs are given in the following definition.

Definition 16. Let $\tilde{G} = (A, B, C)$ be an SVNIG. An edge, uv , is said to be a δ -edge if

$$(T_B(uv), I_B(uv), F_B(uv)) < S'^\infty(u, v).$$

Similarly, a pair (u, uv) in \tilde{G} is called a δ -pair if

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) < IS'^\infty(u, uv).$$

Theorem 6. In an SVNIG, every neutrosophic incidence cutpair is a strong pair.

Proof. Let $\tilde{G} = (A, B, C)$ be an SVNIG. Let $(u, uv) \in C^*$ be a neutrosophic incidence cutpair. Then, by Definition 13, we have

$$IS'^\infty(u, uv) < IS^\infty(u, uv).$$

On the contrary, suppose that (u, uv) is not a strong incidence pair. Then, it follows that

$$IS'^\infty(u, uv) > (T_C(u, uv), I_C(u, uv), F_C(u, uv)).$$

Let P be the path from u to uv in $G' = \tilde{G} - \{(u, uv)\}$ with the greatest incidence strength. Then, P together with (u, uv) , forms a cycle in \tilde{G} . Now, in this cycle, (u, uv) is the weakest pair, but, based on

Theorem 2, it is not possible, since (u, uv) is a neutrosophic incidence cutpair. Hence, our assumption is wrong, and (u, uv) is a strong incidence pair. \square

Theorem 7. In an SVNIG $\tilde{G} = (A, B, C)$. The pair (u, uv) is a neutrosophic incidence cutpair if, and only if, it is α -strong.

Proof. Let (u, uv) be a neutrosophic incidence cutpair in \tilde{G} . Then, according to the Definition 13 of cutpair,

$$IS^\infty(u, uv) > IS'^\infty(u, uv).$$

Then, based on Theorem 4, it follows that

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) > IS'^\infty(u, uv),$$

which is the definition of α -strong pair. Hence, (u, uv) is an α -strong pair in \tilde{G} .

Conversely, let (u, uv) be an α -strong pair in \tilde{G} . Then, by definition

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) > IS'^\infty(u, uv).$$

It follows that $P : u, (u, uv), uv$ is a unique path from u to uv which has the greatest incidence strength of all paths. Therefore, any other path from u to uv will have a lower incidence strength.

$$IS^\infty(u, uv) > IS'^\infty(u, uv).$$

Hence, (u, uv) is a neutrosophic incidence cutpair. \square

Definition 17. Let $\tilde{G} = (A, B, C)$ be an SVNIG.

- (i) \tilde{G} is called a block if $G^* = (A^*, B^*, C^*)$ is block. That is, there are no cutvertices in G^* .
- (ii) \tilde{G} is called a neutrosophic block if \tilde{G} has no neutrosophic cutvertices.
- (iii) \tilde{G} is called a neutrosophic incidence block if it has no neutrosophic incidence cutvertices.

Example 7. Consider the SVNIG $\tilde{G} = (A, B, C)$ shown in Figure 7 with $A^* = \{a, b, c\}$ and $B^* = \{ab, bc, ac\}$. \tilde{G} is a block, since the crisp graph, G^* , has no cutvertex and it is a neutrosophic incidence block. \tilde{G} is not a neutrosophic block, since it has a neutrosophic cutvertex, namely, a .

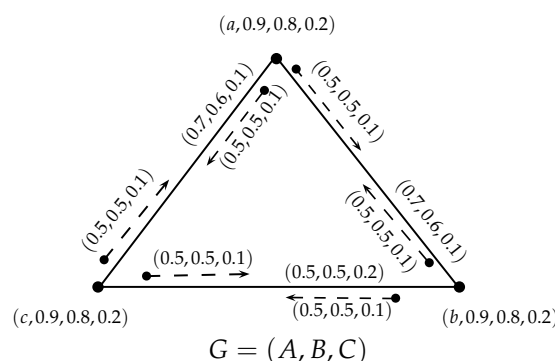


Figure 7. Single-valued neutrosophic incidence graph.

Theorem 8. Let $\tilde{G} = (A, B, C)$ be a neutrosophic incidence block. A pair, (u, uv) , in \tilde{G} , such that

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) = (\max T_C(x, xy), \max I_C(x, xy), \min F_C(x, xy)),$$

for all $(x, xy) \in C^*$, is a strong pair.

Proof. Let \tilde{G} be a neutrosophic incidence block. By definition, there are no neutrosophic incidence cutvertices in \tilde{G} . Let (u, uv) be a pair in \tilde{G} , such that

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) = (\max T_C(x, xy), \max I_C(x, xy), \min F_C(x, xy)).$$

We will prove that (u, uv) is a strong pair by showing that

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) \geq IS'^\infty(u, uv).$$

The incidence strength of any path, P , from u to uv will be less than or equal to $(T_C(u, uv), I_C(u, uv), F_C(u, uv))$. If (u, uv) is the only pair in \tilde{G} with

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) = (\max T_C(x, xy), \max I_C(x, xy), \min F_C(x, xy)),$$

then every other path from x to xy in \tilde{G} will have less incidence strength than $(T_C(u, uv), I_C(u, uv), F_C(u, uv))$, and hence,

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) > IS'^\infty(u, uv).$$

Thus, (u, uv) is an α -strong pair.

If (u, uv) is not unique, then the maximum possible value for the incidence strength of any path in $G' = \tilde{G} - \{(u, uv)\}$ will be equal to $(T_C(u, uv), I_C(u, uv), F_C(u, uv))$. Therefore, there exists a path from u to uv with an incidence strength equal to $(T_C(u, uv), I_C(u, uv), F_C(u, uv))$, that is

$$(T_C(u, uv), I_C(u, uv), F_C(u, uv)) = IS'^\infty(u, uv).$$

Then, (u, uv) is β -strong. \square

3. Application

According to the Federal Investigation Agency (FIA), Pakistan is among the fourth largest country in terms of its citizens who illegally enter Europe. There is no formally declared policy of the Government of Pakistan for Migration and Pakistani Migrants. Every year, thousands of Pakistanis fleeing poverty, unemployment, law and other problems attempt to illegally enter Europe. A lot of them even die before reaching the destination. These illegal immigrants use land routes featuring Pakistan, Iran, Turkey and Greece to enter Europe. Greece is a gateway to the west, and roughly nine out of ten people illegally entering Europe follow this route. Below, we present a mathematical model of this phenomenon.

Consider SVNIG $\tilde{G} = (A, B, C)$ as shown in Figure 8, a mathematical model of the situation of illegal migration from Pakistan to European, where

$$A = \{(Pakistan, 0.9, 0.8, 0.7), (Iran, 0.8, 0.6, 0.8), (Turkey, 0.9, 0.8, 0.7), (Greece, 0.9, 0.8, 0.6)\}$$

is the set of countries under consideration,

$$B = \{((Pak, Iran), 0.7, 0.6, 0.4), ((Iran, Turkey), 0.5, 0.5, 0.5), ((Turkey, Greece), 0.6, 0.8, 0.5)\}$$

represents the flow of people traveling legally from country x to country y and

$$C = \{((Pak, (Pak, Iran)), 0.5, 0.6, 0.3), ((Iran, (Pak, Iran)), 0.4, 0.2, 0.8), ((Iran, (Iran, Turkey)), 0.5, 0.5, 0.4), ((Turkey, (Iran, Turkey)), 0.3, 0.5, 0.4), ((Turkey, (Turkey, Greece)), 0.6, 0.8, 0.2), ((Greece, (Turkey, Greece)), 0.2, 0.2, 0.6)\}$$

represents the flow of people traveling illegally from country x to the country y . It is clear that each pair in this model is a neutrosophic incidence cutpair. So, every government of the countries featuring in this path must make hard and fast rules to control illegal migration as it creates lot of problems for both sending and receiving countries. Policy makers and practitioners need to develop a comprehensive understanding of the phenomenon of illegal migration in order to manage it effectively. We present our proposed method in Algorithm 1.

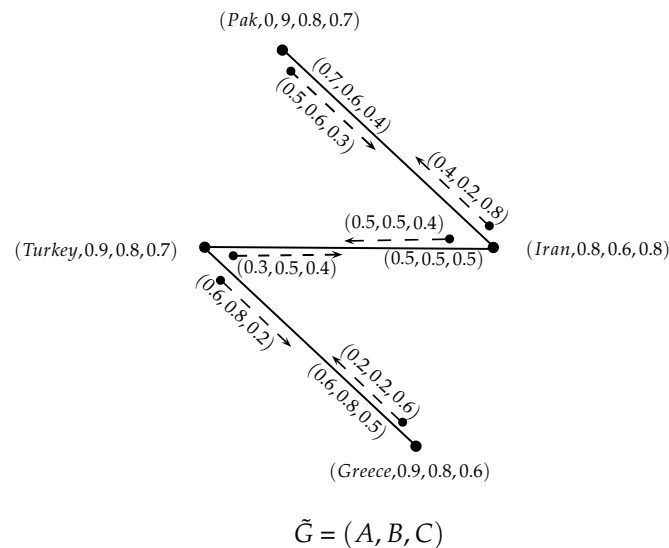


Figure 8. Model of the situation of illegal migration from Pakistan to Europe.

Algorithm 1 Method of Finding Neutrosophic Incidence Cutpair

1. Input the vertex set, V .
2. Input the edge set, $E \subseteq V \times V$.
3. Construct the single-valued neutrosophic set, A , on V .
4. Construct the single-valued neutrosophic relation, B , on E .
5. Construct the single-valued neutrosophic set, C , on $V \times E$.
6. Calculate the incidence strength, $IS(u_i, u_j)$, of all possible paths from u_i to u_j , such that

$$\begin{aligned}
 is_1 &= \min\{T_C(u_i, u_i u_{i+1}) : (u_i, u_i u_{i+1}) \in I\}, \\
 is_2 &= \min\{I_C(u_i, u_i u_{i+1}) : (u_i, u_i u_{i+1}) \in I\}, \\
 is_3 &= \max\{F_C(u_i, u_i u_{i+1}) : (u_i, u_i u_{i+1}) \in I\}.
 \end{aligned}$$

7. Calculate the greatest incidence strength, $IS^\infty(u_i, u_j)$, of paths from u_i to u_j .
 8. Remove the pair $(u_i, u_i u_{i+1})$ from I .
 9. Repeat step 6 and step 7 to calculate the incidence strength, $IS'^\infty(u_i, u_j)$ from u_i to u_j .
 10. Compare the two greatest incidence strengths.
 11. If $IS'^\infty(u_i, u_j) < IS^\infty(u_i, u_j)$, then $(u_i, u_i u_{i+1})$ is the required neutrosophic incidence cutpair.
-

4. Conclusions

Graph theory is a useful tool for analyzing and modeling different mathematical structures. However, its failure to determine relationships between vertices (nodes) and edge (arcs) led to the introduction of the fuzzy incidence graph. The single-valued neutrosophic incidence graph is an extension of fuzzy incidence graph, which can be used as a tool for constructing different mathematical models with indeterminate information and interconnected supporting links. In this

paper, we discussed different properties of single-valued neutrosophic incidence graphs. We studied the block structure of single-valued neutrosophic incidence graphs. We aim to extend the application of single-valued neutrosophic incidence graphs to human trafficking.

Author Contributions: M.A., S.S. and F.S. conceived and designed the experiments; S.S. and M.A. wrote the paper.

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