Interior Regularity Estimates for a Degenerate Elliptic Equation with Mixed Boundary Conditions

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Abstract: The Marchaud fractional derivative can be obtained as a Dirichlet-to–Neumann map via an extension problem to the upper half space. In this paper we prove interior Schauder regularity estimates for a degenerate elliptic equation with mixed Dirichlet–Neumann boundary conditions. The degenerate elliptic equation arises from the Bernardis–Reyes–Stinga–Torrea extension of the Dirichlet problem for the Marchaud fractional derivative.

Keywords: marchaud fractional derivative; interior regularity; schauder estimate; extension problem; fractional order weighted Sobolev spaces

1. Introduction

In the last years there has been a growing interest in the study of fractional elliptic equations involving the right fractional Marchaud derivative $(\mathscr{D}_{right})^\alpha$, such as equations of the form

$$(\mathscr{D}_{right})^\alpha v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{in} \quad [b, \infty), \tag{1}$$

where without loss of generality $\Omega := [a, b] \subset \mathbb{R}$, with $a < b$ and $0 < \alpha < 1$.

Fractional diffusion problems of type (1) arise for example in the modelling of neuronal transmission in Purkinje cells, whose malfunctioning is known to be related to the lack of voluntary coordination and the appearance of tremors [1]. Further motivation comes from various experimental results which showed anomalous diffusion of fractional type, see for example [2,3] and references therein.

The right fractional Marchaud derivative of a function $w : \mathbb{R} \to \mathbb{R}$ is defined via Fourier transforms as

$$\widehat{(\mathscr{D}_{right})^\alpha w}(\xi) = (\pm i\xi)^\alpha \, \widehat{w}(\xi), \tag{2}$$

and it can also be expressed by the pointwise formula

$$(\mathscr{D}_{right})^\alpha v(x) = \frac{c_\alpha}{\Gamma(-\alpha)} \int_x^\infty \frac{v(y) - v(x)}{(y - x)^{1+\alpha}} \, dy, \tag{3}$$

where c_α is a positive normalization constant. We observe from (3) that the right fractional Marchaud derivative is a nonlocal operator. Nonlocal operators have the peculiarity of taking memory effects into account and capturing long-range interactions, i.e., events that happen far away in time or space. Further discussion of the difference between local integro-differential operators and nonlocal or fractional ones can be found in [4] and references therein. In this context, the nonlocality of the fractional Marchaud derivative prevents us from applying local PDE techniques to treat nonlinear

problems for $(\mathscr{D}_{right})^{\alpha}$. To overcome this difficulty, Bernardis, Reyes, Stinga and Torrea showed in [5] that the right fractional Marchaud derivative can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Similar extension properties have been found for the fractional Laplacian by Caffarelli and Silvestre in [6].

To be more precise, consider the function $\mathcal{U} : \mathbb{R}_+^2 := \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that solves the boundary value problem

$$\begin{cases} \mathcal{M}_\alpha \mathcal{U}(t, x) = 0 & \text{for } (t, x) \in \mathbb{R}_+^2, \\ \lim_{t \to 0} \mathcal{N}_\alpha \mathcal{U}(t, x) = f(x) & \text{for } x \in \Omega, \\ \mathcal{U}(t, x) = 0 & \text{for } x \in \mathbb{R} \setminus \Omega, \end{cases} \tag{4}$$

Then we have [5]:

$$\lim_{t \to 0} \mathcal{N}_\alpha \mathcal{U}(t, x) = c_\alpha (\mathscr{D}_{right})^{\alpha} v(x),$$

where $c_\alpha := \dfrac{4^{\alpha - 1/2} \Gamma(\alpha)}{\Gamma(1 - \alpha)}$ is a positive multiplicative constant depending only on $\alpha \in (0, 1)$. Here the differential operators $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are given respectively by:

$$\mathcal{M}_\alpha \mathcal{U} := -(\mathscr{D}_{right}) \mathcal{U} + \frac{1 - 2\alpha}{t} \mathcal{U}_t + \mathcal{U}_{tt}; \tag{5}$$

$$\mathcal{N}_\alpha \mathcal{U} := -t^{1-2\alpha} \mathcal{U}_t. \tag{6}$$

We use the notation $(\mathscr{D}_{right})$ for the derivative *from the right* at the point $x \in \mathbb{R}$, that is:

$$(\mathscr{D}_{right}) v(x) = \lim_{t \to 0^+} \frac{v(x) - v(x + t)}{t}, \tag{7}$$

for good enough functions v. Observe that $(\mathscr{D}_{right})$ equals the negative of the lateral derivative $\left(\frac{d}{dx^+} \right)$ as usually defined in calculus [5].

This characterization of $(\mathscr{D}_{right})^{\alpha} v$ via the local (degenerate) PDE (5) was used for the first time in [5] to get maximum principles. To solve (4), Stinga and Torrea noted that (5) can be thought of as the harmonic extension of v into $2 - 2\alpha$ extra dimensions (see [5]). From there, they established the fundamental solution and, using a conjugate equation, a Poisson formula for $\mathcal{U}$. Furthermore, taking advantage of the general theory of degenerate elliptic equations developed by Fabes, Jerison, Kenig and Serapioni in 1982–83, they proved comparison principles for $\mathcal{U}$ (and thus for v).

The aim of this paper is to prove an interior Schauder estimate for the problem (4), involving any fractional power of the derivative $(\mathscr{D}_{right})^{\alpha}$ as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem as in [5].

A significant contribution of the above extension problem is to provide a way of applying classical analysis methods to partial differential equations containing one-sided Marchaud derivative operators. By means of such extension techniques, a series of important results, such as comparison principles, Harnack inequalities, and regularity estimates for solutions to degenerate elliptic equations involving the fractional Laplacian, have been studied by many authors, for example [6–17]. The same analysis was done for the one-sided fractional derivative operator in the sense of Marchaud ([5] Theorem 1.1 and Corollary 1.2).

In view of these results, we immediately observe that interior regularity and boundary regularity for the degenerate elliptic equation with mixed boundary conditions involving the one-sided Marchaud derivative is missing in the literature. Indeed, from the pioneering work of [5,7,18] on the analogue extension problem for nonlocal operators that map Dirichlet to Neumann, one can reduce a nonlocal problem involving fractional derivatives to a local one by keeping their qualitative properties. Using this technique, one can study interior and boundary regularity. Hence the raison d'être for this work.

In order to underline what makes the difference between the extension problems introduced by Bernardis, Reyes, Stinga and Torrea [5], and the one introduced by Bucur and Ferrari [18], we point out that the extension problem established in [18] is based on a time-dependent initial condition, which leads to a heat conduction problem. Indeed, considering the function $\varphi : \mathbb{R} \to \mathbb{R}$ of one variable, formally representing the time variable, their approach relies on constructing a parabolic local operator by adding an extra variable, say the space variable, on the positive half-line, and working on the following problem in the half-plane $[0, \infty) \times \mathbb{R}$:

$$\begin{cases} \partial_t \mathcal{U} & = \Delta \mathcal{U} & (t, x) \in (0, \infty) \times \mathbb{R}, \\ \mathcal{U}(0, t) & = \varphi(t) & t \in \mathbb{R}. \end{cases} \tag{8}$$

The problem (8) is not the usual Cauchy problem for the heat operator, but a heat conduction problem.

In view of the type of problem we are interested in here, we choose to deal with the Bernardis–Reyes–Stinga–Torrea extension problem [5]. Our main result, which will be proved in Section 3 below, is as follows. We note that this result can be proved only using extension techniques.

Theorem 1. *Let $\alpha \in (0, 1)$ and let $\mathcal{U} \in L^\infty(\mathbb{R}_2^+) \cap H^1(t^\mu; \mathcal{B}_2^+)$ be a weak solution to*

$$\begin{cases} \mathcal{M}_\alpha\, \mathcal{U} = 0 & \text{in } \mathcal{B}_2^+, \\ \lim_{t \to 0} \mathcal{N}_\alpha\, \mathcal{U}(t, \cdot) = f & \text{on } B_2, \\ \mathcal{U} = 0 & \text{on } \mathbb{R} \setminus \Omega. \end{cases}$$

(a)   *For $1 < p < \infty$, if $f \in L^p(B_1, w)$ and $\gamma \in (0, \min(1, \alpha))$ is such that $0 < \alpha - \gamma - \frac{1}{p} < 1$, then $\mathcal{U} \in \mathcal{C}^{0, \alpha - \gamma - \frac{1}{p}}\left(\overline{\mathcal{B}_1^+}, w\right)$. Moreover,*

$$\| \mathcal{U} \|_{\mathcal{C}^{\alpha - \gamma - \frac{1}{p}}\left(\overline{\mathcal{B}_1^+}, w\right)} \leq C \left(\| \mathcal{U} \|_{L^p(\mathbb{R}_2^+, w)} + \| f \|_{L^p(\mathbb{R}, w)} \right),$$

   where C is a positive constant depending only on α, γ, and p.

(b)   *If $f \in L^\infty(B_1)$ and $\gamma \in (0, \min(1, \alpha))$, then $\mathcal{U} \in \mathcal{C}^{\alpha - \gamma}\left(\overline{\mathcal{B}_1^+}\right)$. Moreover,*

$$\|\mathcal{U}\|_{\mathcal{C}^{\alpha - \gamma}\left(\overline{\mathcal{B}_1^+}\right)} \leq C \left(\|\mathcal{U}\|_{\mathbb{R}_+^2} + \|f\|_{L^\infty(\mathbb{R})} \right),$$

   where C is a positive constant depending only on α and γ.

The paper is organised as follows. In Section 2, we give some notations and definitions of function spaces and their associated norms which will be needed in this work. We also provide some preliminary results and finally state our main result. In Section 3, we prove an intermediate result and provide the proof of the regularity estimate up to the boundary for the degenerate Equation (4) with the Neumann boundary condition stated in Theorem 1. Finally we end with the conclusion in Section 4.

2. Notations and Preliminary Results

In this section we introduce some notations, definitions, and preliminary results used throughout the paper.

Here and in the following, we consider $\alpha \in (0, 1)$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R} = \{z = (t, x) : t > 0\}$ and $\Omega \subset \mathbb{R}$ a bounded Lipschitz domain. For an open set Ω, an integer $k \geq 1$, and a real number $\lambda \in (0, 1]$, the Hölder spaces $\mathcal{C}^{k, \lambda}(\Omega)$ are defined as the subspaces of $\mathcal{C}^k(\Omega)$ consisting of functions whose k-th order derivatives are uniformly Hölder continuous with exponent λ in Ω.

Furthermore, we introduce the following notation for intervals, boxes, and balls:

$$
\begin{aligned}
B_r(x_0) &:= \{x \in \mathbb{R} : |x - x_0| < r\}, \\
\mathcal{B}_r^+(x_0) &:= [0, r) \times B_r(x_0), \\
\mathcal{B}_r(z_0) &:= \{z = (t, x) \in \mathbb{R} \times \mathbb{R} : |z - z_0| < r\},
\end{aligned}
\tag{9}
$$

We consider the function space

$$
\mathcal{L}_\alpha^1 := \left\{ \varphi : \mathbb{R} \to \mathbb{R} \quad : \quad \|\varphi\|_{\mathcal{L}_\alpha^1} := \int_\mathbb{R} \frac{|\varphi(x)|}{1 + |x|^{1+\alpha}} \, dx < \infty \right\}.
$$

For $\Omega \subset \mathbb{R}$ an open set, we say $v : \Omega \to \mathbb{R}$ is in $\mathcal{C}^{0,\gamma}(\Omega)$, i.e., Hölder continuous with exponent $\gamma \in (0, 1)$, if

$$
\|v\|_{\mathcal{C}^{0,\gamma}(\Omega)} := \sup_{x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\gamma} < +\infty.
$$

We recall the following definition of Sobolev spaces.

Definition 1 (Sobolev spaces). *For any real number α, the αth Sobolev space on $\mathbb{R}$ is defined to be*

$$
H^\alpha(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \hat{u} \in L_{loc}^2(\mathbb{R}), \|u\|_{H^\alpha} < \infty\},
$$

where the Sobolev norm $\| \cdot \|_{H^\alpha}$ is defined by

$$
\|u\|_{H^\alpha} := \left(\int_\mathbb{R} |\hat{u}(\lambda)|^2 \left(1 + |\lambda|^2 \right)^\alpha \, d\lambda \right)^{1/2}.
$$

For a general domain $X \subset \mathbb{R}$, the αth Sobolev space on X is defined to be

$$
H_{loc}^\alpha(X) := \{u \in \mathcal{D}'(X) : u\phi \in H^\alpha(\mathbb{R}) \text{ for all } \phi \in \mathcal{D}(X)\}.
$$

Let $a \in \mathbb{R}$ and $\alpha \in (0, 1)$ be two arbitrary parameters. We define the functional space

$$
C_a^{1,\alpha} := \left\{ f : \mathbb{R} \to \mathbb{R} \quad : \quad \text{for any } x > a, \ f \in AC([a, x]) \text{ and } f'(\cdot)(x - \cdot)^{-\alpha} \in L^1((a, x)) \right\}.
\tag{10}
$$

We denote here by $AC(I)$ the space of absolutely continuous functions on I.

Definition 2 (Caputo derivative). *The Caputo derivative of $v \in C_a^{1,\alpha}$ with initial point $a \in \mathbb{R}$ at the point $x > a$ is given by*

$$
D_a^\alpha v(x) := \frac{1}{\Gamma(1 - \alpha)} \int_a^x v'(y)(x - y)^{-\alpha} \, dy.
\tag{11}
$$

Definition 3. *The right Marchaud derivative of a well defined function v is given by*

$$
(\mathscr{D}_{right})^\alpha v(x) = \lim_{\delta \to 0^+} \frac{C}{\Gamma(-\alpha)} \int_{x+\delta}^\infty \frac{v(y) - v(x)}{(y - x)^{1+\alpha}} \, dy,
\tag{12}
$$

with C_α a positive normalisation constant.

Remark 1. *Notice that the one-sided nonlocal derivative in the sense of Marchaud can also be obtained by extending the Caputo derivative. Indeed, by making an integration by parts of Equation (11), we obtain an equivalent definition [19,20] as follows:*

$$D^\alpha v(x) = C(\alpha) \int_x^\infty \frac{v(y) - v(x)}{(y - x)^{1+\alpha}} \, dy,$$

for all $x < 0$, so that $v(x) = v(0)$, where $C(\alpha)$ is a constant depending on α. Indeed, for sufficiently regular functions v, we have:

$$\begin{aligned}
D^\alpha v(x) &= \int_0^x \frac{v'(y)}{(x-y)^\alpha} \, dy = \int_0^x \left(\frac{d}{dy} \frac{(v(y) - v(x))}{(x-y)^\alpha} - \alpha \frac{(v(y) - v(x))}{(x-y)^{1+\alpha}} \right) dy \\
&= \frac{v(x) - v(0)}{x^\alpha} - \lim_{y \to x} \frac{v(x) - v(y)}{(x-y)^\alpha} - \alpha \int_0^x \frac{(v(y) - v(x))}{(x-y)^{1+\alpha}} \, dy \\
&= \frac{v(x) - u(0)}{x^\alpha} - v'(y) \lim_{y \to x} (x-y)^{1-\alpha} - \alpha \int_0^x \frac{v(y) - v(x)}{(x-y)^{1+\alpha}} \, dy \\
&= \frac{v(x) - v(0)}{x^\alpha} + \alpha \int_0^x \frac{v(x) - v(y)}{(x-y)^{1+\alpha}} \, dy.
\end{aligned} \tag{13}$$

Hence, we take the convention that $v(x) = v(0)$ for any $x \leq 0$. With this extension, one has that, for any $x > 0$,

$$\alpha \int_{-\infty}^0 \frac{v(x) - v(y)}{(x-y)^{1+\alpha}} \, dy = \alpha \int_{-\infty}^0 \frac{v(x) - u(0)}{(x-y)^{1+\alpha}} \, dy = \frac{v(x) - v(0)}{x^\alpha}.$$

So one can write (13) as

$$D^\alpha v(x) = C(\alpha) \int_{-\infty}^x \frac{v(x) - v(y)}{(x-y)^{1+\alpha}} \, dy.$$

This type of formula also relates the Caputo derivative to the so-called Marchaud derivative [20,21]. Therefore the results obtained in this paper could also be applied for the extended Caputo derivative.

Note that the integral in (12) is absolutely convergent for functions in the Schwartz class $\mathcal{S}$. Furthermore one should notice that the nonlocal operators $(\mathscr{D}_{right})^\alpha$ and $(\mathscr{D}_{right})^{-\alpha}$ depend on the values of v on the whole half line (x, ∞).

We recall that the inverse of the right fractional Marchaud derivative $(\mathscr{D}_{right})^{-\alpha}$ is defined as

$$(\mathscr{D}_{right})^{-\alpha} v(x) := \int_{\mathbb{R}} \frac{v(y)}{|x-y|^{1-\alpha}} \, dy = \mathcal{I}_\alpha * v(x) \tag{14}$$

where the Riesz potential (see [7,21]) is defined as

$$\mathcal{I}_\alpha = C_\alpha |x - y|^{\alpha - 1} \quad \text{for } \alpha < 1, \tag{15}$$

with the constant $C_\alpha = \frac{1}{\pi} \Gamma(1-\alpha) \sin \frac{\pi\alpha}{2}$.

From [5], we have that for $u \in \mathcal{S}$, $(\mathscr{D}_{right})^\alpha u \in \mathcal{S}_\alpha$, where

$$\mathcal{S}_\alpha := \left\{ f \in C^\infty(\mathbb{R}) : (1 + |x|^{1+\alpha}) f^k(x) \in L^\infty(\mathbb{R}), \text{ for each } k \geq 0 \right\}.$$

The topology in $\mathcal{S}_\alpha$ is given by the family of seminorms $[f]_k := \sup_{x \in \mathbb{R}} \left| (1 + |x|^{1+\alpha}) f^{(k)}(x) \right|$, for $k \geq 0$. Let $\mathcal{S}_\alpha'$ be the dual space of $\mathcal{S}_\alpha$; then $(\mathscr{D}_{right})^\alpha$ defines a continuous operator from $\mathcal{S}_\alpha'$ into $\mathcal{S}'$.

2.1. Weighted Spaces

Weighted spaces of smooth functions play an important role in the context of partial differential equations (PDEs). They are widely used, for instance, to treat PDEs with degenerate coefficients or domains with a nonsmooth geometry (see e.g., [22–25]), as is the case here. For evolution equations,

power weights in time play an important role in order to obtain results for rough initial data (see [26,27]). This subsection dedicated to weighted spaces is motivated by the appearance of the Muckenhoupt weight $w := t^{1-2\alpha}$ which appears in (5) and (6). For general literature on weighted function spaces we refer to [23–25,28–31] and references therein.

In a general framework, a function $w : \mathbb{R}^d \to [0, \infty)$, for an integer $d \geq 1$, is called a *weight* if w is locally integrable and the zero set $\{x : w(x) = 0\}$ has Lebesgue measure zero. For $p \in [1, \infty]$ we denote by A_p the *Muckenhoupt class* of weights. In the case $p \in (1, \infty)$, we say that $w \in A_p$ if

$$\sup_{\mathcal{B} \text{ cubes in } \mathbb{R}^d} \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} w(x)\, dx \right) \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty.$$

In the case $p = 1$, we say that $w : \mathbb{R} \to [0, \infty)$ belongs to A_1 if there exists some constant C such that

$$\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} w(y)\, dy \leq C w(x)$$

for all $x \in \mathcal{B}$ and all balls $\mathcal{B} \subset \mathbb{R}^d$. In the case $p = \infty$, we define $A_\infty = \bigcup_{1 \leq p < \infty} A_p$. Note that, for functions with support contained in $(-\infty, 0)$ or $(0, \infty)$, the class of weights is denoted by A_p^+ or A_p^- respectively. We refer to [27,30,32] for the general properties of these classes.

Example 1. *Problem (4) is a weighted—singular or degenerate, depending on the value of $\alpha \in (0, 1)$—elliptic equation on $\mathbb{R}_+^2$ with mixed boundary conditions. The weight $w := t^{1-2\alpha}$ belongs to the Muckenhoupt class A_2^+, i.e., there exists a constant C such that for any $\mathcal{B} \subset \mathbb{R}_+^2$,*

$$\left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |t|^{1-2\alpha} dt\, dx \right) \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |t|^{2\alpha-1} dt\, dx \right) \leq C.$$

For this reason, when working with one-sided weights, we can assume without loss of generality that $\Omega := [a, b] = \mathbb{R}$ (see e.g., [30] for more details).

Next, for a strongly measurable function f and a number $p \in [1, \infty)$, we define the weighted L^p norm by

$$\|f\|_{L^p(\mathbb{R}^d, w)} := \left(\int_{\mathbb{R}^d} \|f(x)\|^p w(x)\, dx \right)^{1/p},$$

and we define the weighted L^p space to be the following Banach space:

$$L^p(\mathbb{R}^d, w) := \{f \text{ strongly measurable} : \|f\|_{L^p(\mathbb{R}^d, w)} < \infty\}.$$

Definition 4 (see [8]). *Given $\alpha \in (0, 1)$, $\mu = 1 - 2\alpha \in (-1, 1)$, and an open set $\mathcal{B} \subset \overline{\mathbb{R}_+^2}$, we denote*

$$L^2(t^\mu; \mathcal{B}) := \left\{ \mathcal{U} : \mathbb{R}_+^2 \to \mathbb{R}, \ \int_{\mathcal{B}} t^\mu |\mathcal{U}|^2 dt\, dx < +\infty \right\},$$

endowed with the norm

$$\|\mathcal{U}\|_{L^2(t^\mu; \mathcal{B})} := \left(\int_{\mathcal{B}} t^\mu |\mathcal{U}|^2 dt\, dx \right)^{1/2}.$$

We also denote

$$H^1(t^\mu; \mathcal{B}) := \left\{ \mathcal{U} \in L^2(t^\mu; \mathcal{B}) : \nabla \mathcal{U} \in L^2(t^\mu; \mathcal{B}) \right\},$$

with the induced norm

$$\|\mathcal{U}\|_{H^1(t^\mu; \mathcal{B})} := \left(\int_{\mathcal{B}} t^\mu \left(|\mathcal{U}|^2 + |\nabla \mathcal{U}|^2 \right) dt\, dx \right)^{1/2}.$$

Using the variable $(t, x) \in \mathbb{R}^2_+$, the space $H^\alpha(\mathbb{R})$ coincides with the trace on $\partial\mathbb{R}^2_+$ of

$$\dot{H}^1(t^\mu; \mathcal{B}) := \left\{ \mathcal{U} \in L^2_{loc}(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} t^\mu \left(\mathcal{U}^2 + |\nabla\mathcal{U}|^2 \right) dt\, dx < +\infty \right\}.$$

In other words [8,9], for any given function $\mathcal{U} \in \dot{H}^1(t^\mu; \mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$, we have $v := \mathcal{U}|_{\partial\mathbb{R}^2_+} \in H^\alpha(\mathbb{R})$, and there exists a constant $C = C(\alpha) > 0$ such that

$$\| v \|_{H^\alpha(\mathbb{R})} \le C \| \mathcal{U} \|_{\dot{H}^1(t^\mu; \mathcal{B})}.$$

So by a density argument, every $\mathcal{U} \in \dot{H}^1(t^\mu; \mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$ has a well defined trace $v \in H^\alpha(\mathbb{R})$. Conversely, any $v \in H^\alpha(\mathbb{R})$ is the trace (restriction to $t = 0$) of a function $\mathcal{U} \in \dot{H}^1(t^\mu; \mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$.

Definition 5. *We say that a function $\mathcal{U} \in \dot{H}^1(t^\mu; \mathcal{B})$ is a weak solution of (4) if*

$$\int_{\mathcal{B}} t^\mu \nabla\mathcal{U}(t, x) \nabla\Psi(t, x) dt\, dx - c_\alpha^{-1} \int_\Omega f(x) \mathrm{Tr}(\Psi)(x)\, dx = 0, \tag{16}$$

where f is as in (1), $\mathrm{Tr}(\Psi)$ denotes the trace $\Psi\big|_{\{0\}\times\mathbb{R}}$, and $\Psi \in \dot{H}^1(t^\mu; \mathcal{B}) \cap \mathcal{C}\left(\overline{\mathbb{R}^2_+}\right)$ is an arbitrary test function.

2.2. The Extension Problem

In the next statement we recall the results obtained from [5] which show that the fractional derivatives on the line are Dirichlet-to–Neumann operators for an extension degenerate PDE problem in $\mathbb{R} \times (0, \infty)$, where the data f have been taken in the more general setting: more precisely a weighted $L^p(w)$ space, where w satisfies the one-sided version A_p^+ (see [30]) of the familiar A_p condition of Muckenhoupt.

Fix $0 < \alpha < 1$. Given a semigroup $\{T_t\}_{t\ge0}$ acting on real functions, the *generalized Poisson integral* of f is given by

$$P_t^\alpha f(x) = \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^\infty e^{-t^2/(4s)} T_s f(x) \frac{ds}{s^{1+\alpha}}, \quad x \in \mathbb{R}; \tag{17}$$

see ([5] (1.9)) for more details.

By considering the semigroup of translations $T_s f(x) = f(x + s)$, $s \ge 0$, we find

$$P_t^\alpha f(x) = f * P_t^\alpha(x) := \int_\mathbb{R} f(s) P_t^\alpha(x - s)\, ds,$$

where

$$P_t^\alpha(x) := \frac{t^{2\alpha} e^{t^2/4x}}{4^\alpha \Gamma(\alpha)(-x)^{1+\alpha}} \chi_{(-\infty,0)}(x). \tag{18}$$

Since the kernel P_t^α is increasing and integrable in $(-\infty, 0)$, it is well known that the function

$$P_*^\alpha f(x) := \sup_{t>0} |f| * P_t^\alpha(x) = \int_\mathbb{R} |f(t)| P_t^\alpha(x - t)\, dt,$$

is pointwise controlled by the usual Hardy–Littlewood maximal operator. However, since the support of P_t^α is $(-\infty, 0)$, a sharper control can be obtained by using the one-sided Hardy–Littlewood maximal operator. This control and the behavior of P_*^α in weighted L^p-spaces will be used in the results of this paper. We revise briefly recall the two fundamental theorems from [5].

Theorem 2 ([5]). *Consider the semigroup of translations $T_t f(x) = f(x + t)$, $t \ge 0$, initially acting on functions $f \in \mathcal{S}$. Let $P_t^\alpha f$, $0 < \alpha < 1$, be as in (17). Then:*

1. For $1 \leq p \leq \infty$, P_t^α is a bounded linear operator from $L^p(\mathbb{R})$ into itself and $\|P_t^\alpha f\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}$.

2. When $f \in \mathcal{S}$, the Fourier transform of $P_t^\alpha f$ is given by

$$\widehat{P_t^\alpha f}(\xi) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} (-it\xi^{1/2})^\alpha \|_\alpha (-it\xi^{1/2}) \widehat{f}(\xi), \quad \xi \in \mathbb{R},$$

   where $\|_\nu(z)$ is the modified Bessel function of the third kind or Macdonald's function, which is defined for arbitrary ν and $z \in \mathbb{C}$, see ([33] Chapter 5). In particular,

$$\widehat{P_t f}(\xi) = e^{-t(-i\xi)^{1/2}} \widehat{f}(\xi).$$

3. The maximal operator P_*^α defined by $P_*^\alpha f(x) = \sup_{t>0} |P_t^\alpha f(x)|$ is bounded from $L^p(\mathbb{R}, w)$ into itself, for $w \in A_p^+, 1 < p < \infty$, and from $L^1(\mathbb{R}, w)$ into weak-$L^1(\mathbb{R}, w)$, for $w \in A_1^+$.

4. Let $f \in L^p(w)$, for $w \in A_p^+, 1 \leq p < \infty$. The function $\mathcal{U}(x,t) \equiv P_t^\alpha f(x)$ is a classical solution to the extension problem (4).

Theorem 3 (Extension problem). *Let $f \in L^p(w)$, $w \in A_p^+$, $1 < p < \infty$. Then the function*

$$\mathcal{U}(x,t) := \frac{t^{2\alpha}}{4^\alpha \Gamma(\alpha)} \int_0^\infty e^{-t^2/(4\tau)} T_\tau f(x) \frac{d\tau}{\tau^{1+\alpha}}, \quad x \in \mathbb{R}, \ t > 0,$$

is a classical solution to the extension problem

$$\begin{cases} -(\mathscr{D}_{right})\mathcal{U} + \frac{1-2\alpha}{t} \mathcal{U}_t + \mathcal{U}_{tt} = 0, & in \ \mathbb{R} \times (0,\infty), \\ \lim_{t \to 0^+} \mathcal{U}(x,t) = f(x), & a.e. \ and \ in \ L^p(w). \end{cases}$$

Moreover, for $c_\alpha := \dfrac{4^{\alpha-1/2}\Gamma(\alpha)}{\Gamma(1-\alpha)} > 0$, we have

$$-c_\alpha \lim_{t \to 0^+} t^{1-2\alpha} \mathcal{U}_t(x,t) = (\mathscr{D}_{right})^\alpha f(x) \quad in \ the \ distributional \ sense.$$

Remark 2. *This parallel result regarding the extension problem in the case of the Marchaud fractional time derivative has been derived as well in [18,20].*

3. Regularity Estimate up to the Boundary for the Degenerate Equation with the Neumann Boundary Condition

In this section, we prove the interior regularity estimate up to the boundary for the degenerate equation with the Neumann boundary condition associated to problem (4). Namely we provide the proof of Theorem 1. But before we get into that, it is necessary to explain the main ideas in the proof of interior regularity provided by Theorem 1. The proof of Theorem 1 is inspired by [5,7,8,34]. The method for this proof differs substantially from interior regularity methods for second-order equations, but is similar to the proof for the fractional Laplacian. Recall that for second-order equations, one first shows that $D^2 u$ is bounded, and then the estimate for equations with bounded measurable coefficients implies a $C^{2,\sigma}$ estimate for $\sigma \in (0, \min(1, \alpha))$. This is also true for the boundary regularity for solutions to fully nonlinear equations [35].

We shall start by the regularity property of the problem (1). We show in Proposition 1 that the solution of the problem (1) is of class $C^{0,\sigma}$. To the best of the authors' knowledge, the proofs available in the literature are those dealing with the case of the fractional Laplacian (see for instance [7,36] (Proposition 2.1.9)). With this result in hand, and by making an appropriate change of variables, we will use this result and estimate to prove our main theorem.

We start by recalling the following lemma from [37], which gives a Liouville-type theorem for (1) in the case $f = 0$.

Lemma 1. *Let $u \in C(\mathbb{R})$ be a function satisfying $(\mathscr{D}_{right})^{\alpha} u = 0$ in $\mathbb{R}_+$, $u = 0$ in $\mathbb{R}_-$, and $|u(x)| \leq C(1 + |x|^{\gamma})$ for some $\gamma < \alpha$. Then $u(x) = kx^{\alpha}$.*

The proof of this lemma relies on similar reasoning as the proof of ([37] (Theorem 2.2.3)) for the Caputo density function.

In the case where we have a non-vanishing right hand side ($f \neq 0$) as in (1), we state the following Liouville-type theorem for the one-sided Marchaud derivative.

Proposition 1. *Let $\alpha \in (0, 1)$ and let $u \in \mathcal{L}^1_{\alpha} \cap L^{\infty}_{loc}(B_1)$ be the solution to*

$$(\mathscr{D}_{right})^{\alpha} u = f \quad in \quad B_1.$$

(a)   *For $1 < p < \infty$, if $f \in L^p(B_1, w)$ and $r \in (0, 1)$ and $\gamma \in (0, \min(1, \alpha))$ is such that $0 < \gamma - \frac{1}{p} < 1$, then $u \in C^{0,\gamma - \frac{1}{p}}(B_1, w)$ and there exists a constant $C := C(\alpha, r, \gamma, p) > 0$ such that*

$$\|u\|_{C^{0,\gamma - \frac{1}{p}}(B_r, w)} \leq C \left(\|u\|_{L^p(B_r, w)} + \|f\|_{L^p(B_1, w)} \right). \tag{19}$$

(b)   *If $f \in L^{\infty}(B_1)$ and $r \in (0, 1)$ and $\gamma \in (0, \min(1, \alpha))$, then $u \in C^{0,\gamma}(B_1)$ and there exists a constant $C := C(\alpha, r, \gamma) > 0$ such that*

$$\|u\|_{C^{0,\gamma}(B_r)} \leq C \left(\|u\|_{L^{\infty}(B_r)} + \|f\|_{L^{\infty}(B_1)} \right). \tag{20}$$

Proof. We will show that u has the corresponding regularity in a neighbourhood of the origin. We split the proof into two parts, as follows.

Proof of (a): $f \in L^p(B_1, w)$. Let $\eta \in C_c^{\infty}(\mathbb{R})$ be a smooth cutoff function such that $\eta = 1$ on B_r, $\eta = 0$ on $\mathbb{R} \setminus B_1$, and $0 \leq \eta \leq 1$ on $\mathbb{R}$. Consider the Riesz potential as defined in (15). Then the function

$$v(x) := \int_{\mathbb{R}} \mathcal{I}_{\alpha}(x, y)(\eta f)(y) dy, \quad \text{for all} \quad x \in \mathbb{R},$$

satisfies

$$(\mathscr{D}_{right})^{\alpha} v(x) = \eta(x) f(x) \quad \text{for all } x \in \mathbb{R}_+. \tag{21}$$

We first estimate the L^p norm of v for $\alpha < 1$. Since the kernel $(\mathscr{D}_{right})^{-\alpha}$ is positive and $\eta \geq 0$ is a smooth function with compact support in B_r, we write $v = (\mathscr{D}_{right})^{-\alpha}(\eta f) = (\mathscr{D}_{right})^{1-\alpha} \circ (\mathscr{D}_{right})^{-1}(\eta f)$. We note that, by using a similar argument as for the Poisson equation for the fractional Laplacian, we find that $(\mathscr{D}_{right})^{-1}(\eta f)$ is an element of $C^{1,\gamma}$ with norm depending only on $\|f\|_{C^{0,\gamma}}$.

Since ηf is compactly supported, we get

$$\|v\|_{C^{0,\gamma}(\mathbb{R})} \leq C_{\alpha,\gamma} \|\eta f\|_{L^p(B, w^p)} + C \|v\|_{L^p(\mathbb{R}, w^p)} \leq C_{\alpha,\gamma,B'} \|f\|_{L^p(B, w^p)}.$$

For $\alpha < 1$ and $\gamma \in (0, \min(\alpha, 1))$ and $x, y \in B_r$, we have

$$v(x) - v(y) = \int_{\mathbb{R}} (\mathcal{I}_{\alpha}(x, z) - \mathcal{I}_{\alpha}(y, z)) (\eta f)(z) dz$$

$$= C_{1,\alpha} \int_{\mathbb{R}} \left(|x - z|^{\alpha-1} - |y - z|^{\alpha-1} \right) (\eta f)(z) \, dz.$$

Next we consider the following inequalities [38], valid for $\gamma \in (0, \min(1, \alpha))$ and $m \in \mathbb{R}$ with $m + \gamma > 0$ and for every $x, y, z \in B_r$:

$$\left| \left(|x - z|^{-m} - |y - z|^{-m} \right) \right| \leq \frac{|m|}{m + \gamma} |x - y|^\gamma \left(|x - z|^{-(m+\gamma)} + |y - z|^{-(m+\gamma)} \right). \qquad (22)$$

For $m = 1 - \alpha$, and for $1 < p < \infty$, we can write

$$|v(x) - v(y)| < \frac{(1 - \alpha)}{(1 - \alpha + \gamma)} \int_{\mathbb{R}} |x - y|^\gamma \left(|x - z|^{\alpha - 1 - \gamma} + |y - z|^{\alpha - 1 - \gamma} \right) |(\eta f)(z)| dz$$

$$\leq C_{\alpha, \gamma} |x - y|^{\gamma - \frac{1}{p}} \int_{\mathbb{R}} |x - y|^{\frac{1}{p}} |x - z|^{\alpha - 1 - \gamma} |(\eta f)(z)| dz,$$

since $\dfrac{|y - z|}{|x - z|} > 1$. Using the fact that the support of η is always contained in the ball of radius 2 centred at $x \in B_r$, we have that

$$|v(x) - v(y)| \leq C(1, \alpha, \gamma) |x - y|^\gamma \int_{\mathbb{R}} |x - y|^{\alpha - 1 - \gamma + \frac{1}{p}} |(\eta f)(y)| dy,$$

$$\leq C(1, \alpha, \gamma) |x - y|^{\gamma - \frac{1}{p}} \left(\int_{\mathbb{R}} w^{-1} |x - y|^{p(\alpha - 1 - \gamma) + 1} w |(\eta f)(y)|^p dy \right)^{1/p}$$

$$\leq C(1, \alpha, \gamma) |x - y|^{\gamma - \frac{1}{p}} \left(\int_{B(x,2)} w |(\eta f)(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B(x,2)} \frac{w^{-\frac{1}{p-1}}}{|x - y|^{\frac{p(-\alpha + 1 + \gamma) - 1}{p-1}}} dy \right)^{\frac{p-1}{p}}$$

$$\leq C(1, \alpha, \gamma) |x - y|^{\gamma - \frac{1}{p}} \left(\int_{B_1} w |f(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B_1} w^{-\frac{1}{p-1}} |y|^{\frac{p(\alpha - 1 - \gamma) + 1}{p-1}} dy \right)^{\frac{p-1}{p}}$$

$$\leq 2C(1, \alpha, \gamma) |x - y|^{\gamma - \frac{1}{p}} \|f\|_{L^p(B_1, w)} \left(\int_{B_1} w^{-\frac{1}{p-1}} |y|^{\frac{p(\alpha - 1 - \gamma) + 1}{p-1}} dy \right)^{\frac{p-1}{p}},$$

up to relabelling of the positive constant $C(\alpha, \gamma)$ that depends on α and γ. Replacing $w(y)$ by its value $|y|^{1 - 2\alpha}$ and using the polar coordinates $y = rx$, $r > 0$, we get that

$$\int_{B_1} w(y)^{-\frac{1}{p-1}} |y|^{\frac{p(\alpha - 1 - \gamma) + 1}{p-1}} dy := \int_{S_1} \int_0^1 |r|^{\frac{1}{p-1}(2\alpha + p(\alpha - 1 - \gamma))} dr\, d\sigma(x) \leq \frac{(p - 1)}{p(\alpha - 1 - \gamma) + p - 1)} |S_1|$$

$$\leq C(p, \gamma, \alpha).$$

Then,

$$|v(x) - v(y)| \leq C(p, \gamma, \alpha) |x - y|^{\gamma - \frac{1}{p}} \|f\|_{L^p(B_1, w)}.$$

Hence, we conclude that

$$\|v\|_{C^{0, \gamma - \frac{1}{p}}(B_r)} \leq C_\alpha \|f\|_{L^p(B_1, w)}, \qquad (23)$$

for every $\gamma \in (0, \min(1, \alpha))$.

Next, by change of variables, the function $\xi := u - v$ satisfies $(\mathscr{D}_{right})^\alpha \xi = 0$ in B_r by (21). Then, thanks to the derivative estimate, for every $r' \in (0, r)$,

$$\|\nabla \xi\|_{L^p(B_{r'}, w)} \leq C_{\alpha, r'} \|\xi\|_{L^p(B_1, w)} \leq C_{\alpha, r'} \left(\|u\|_{L^p(B_r, w)} + \|v\|_{L^p(B_r, w)} \right).$$

The difference function $\xi = u - v$ is smooth in B_1 and is bounded. From this observation, together with (23), we have that

$$\|u\|_{C^{0, \gamma - \frac{1}{p}}(B_{r'})} = \|\xi + v\|_{C^{0, \gamma}(B_{r'})} \leq C_{\alpha, r'} \left(\|u\|_{L^p(B_r, w)} + \|f\|_{L^p(B_1, w)} \right),$$

for every $\gamma \in (0, \min(1, \alpha))$ with $0 < \gamma - \frac{1}{p} < 1$, as required.

Proof of (b): $f \in L^\infty(B_1)$. The proof in this case is similar to the previous one. We consider as above a smooth cutoff function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta = 1$ on B_r, $\eta = 0$ on $\mathbb{R} \setminus B_1$, and $0 \leq \eta \leq 1$ on $\mathbb{R}$. Then we consider the Riesz potential as defined in (15), so that we can estimate the L^∞ norm of v for $\alpha < 1$. Since the kernel $\mathcal{I}_\alpha$ is positive and $\eta \geq 0$ is a smooth function with compact support in B_r, we get

$$\|v\|_{C^{0,\gamma}(\mathbb{R})} \leq C_{\alpha,\gamma} \|\eta f\|_{L^\infty(\mathbb{R})} + C\|v\|_{L^\infty(\mathbb{R})} \leq C_{\alpha,\gamma,B'} \|f\|_{L^\infty(B)}.$$

Next, by using the inequality stated in (22), we get that for $\gamma \in (0, \min(\alpha, 1))$ and $x, y \in B_r$,

$$
\begin{aligned}
|v(x) - v(y)| &\leq C_{\alpha,\gamma} |x - y|^\gamma \int_\mathbb{R} |x - y|^{\alpha - 1 - \gamma} \eta(y)|f(y)|dy \\
&\leq C_{\alpha,\gamma} |x - y|^\gamma \|f\|_{L^\infty(B_1)} \int_{B_2} |x - y|^{\alpha - 1 - \gamma} dy \\
&\leq C_{\alpha,\gamma} \|f\|_{L^\infty(B_1)} |x - y|^\gamma.
\end{aligned}
$$

Hence, we conclude that

$$\|v\|_{C^{0,\gamma}(B_r)} \leq C_\alpha \|f\|_{L^\infty(B_1)}, \tag{24}$$

for every $\gamma \in (0, \min(1, \alpha))$.

Next, by change of variables, the function $\xi := u - v$ satisfies $(\mathscr{D}_{right})^\alpha \xi = 0$ in B_r by (21). Therefore, thanks to ([7] (Corollary 1.13)), we have the derivative estimate for every $r' \in (0, r)$:

$$\|\nabla w\|_{L^\infty(B'_r)} \leq C_{\alpha,r'} \|w\|_{L^\infty(B_1)} \leq C_{\alpha,r'} (\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}).$$

The difference function $\xi = u - v$ is smooth in B_1 and is bounded. From this, together with (24), we conclude that

$$\|u\|_{C^{0,\gamma}(B_{r'})} = \|w + v\|_{C^{0,\gamma}(B_{r'})} \leq C_{\alpha,r'} \left(\|u\|_{L^\infty(B_r)} + \|f\|_{L^\infty(B_1)} \right),$$

for every $\gamma \in (0, \min(1, \alpha))$. $\square$

Now we are in a position to state and prove our main result on the interior Schauder estimate for the solution function $\mathcal{U}$ on the set $\overline{\mathcal{B}_1^+}$.

Proof of the Main Result: Theorem 1

Proof. Again, we present the two parts of the proof separately.

Proof of (a): $f \in L^p(\mathbb{R}, w)$. We choose a cut-off function $\eta \in \mathcal{C}_c^\infty(B_2)$ such that $\eta \equiv 1$ on B_2 and $0 \leq \eta \leq 1$ on $\mathbb{R}$. Let $\overline{v}$ be the unique solution to the equation

$$(\mathscr{D}_{right})^\alpha \overline{v} = \overline{f} \qquad \text{in} \quad \mathbb{R},$$

where $\overline{f} := \eta f$. Making use of the previous result Proposition 1, we know that $\overline{v} \in C^{\alpha - \gamma - \frac{1}{p}}(\mathbb{R}, w)$ and

$$\|\overline{v}\|_{C^{\alpha - \gamma - \frac{1}{p}}(\mathbb{R}, w)} \leq C \left(\|\overline{v}\|_{L^p(\mathbb{R}, w)} + \left\|\overline{f}\right\|_{L^p(\mathbb{R}, w)} \right),$$

where $C > 0$ is a constant that depends only on α, γ, and p.

The next step is to consider the Bernardis–Reyes–Stinga–Torrea extension $\overline{\mathcal{U}}$ of $\overline{v}$, i.e., the function

$$\overline{\mathcal{U}}(t, \cdot) = P_t^\alpha(t, \cdot) \star \overline{v},$$

which satisfies the equations

$$
\begin{cases}
\mathcal{M}_\alpha \, \overline{\mathcal{U}}(t,x) = 0 & \text{in } \mathbb{R}^2_+, \\
\lim_{t\to 0} \mathcal{N}_\alpha \, \overline{\mathcal{U}}(t,x) = (\mathscr{D}_{right})^\alpha \overline{v}(x) = \overline{f}(x) & \text{on } \mathbb{R}.
\end{cases}
$$

By a change of variables, we have

$$
\overline{\mathcal{U}}(t,x) = \left(P_t^\alpha(t,\cdot) \star \overline{v}\right)(x) = \int_{\mathbb{R}} \overline{v}(x - ty) H_\alpha(y) dy, \tag{25}
$$

where

$$
H_\alpha(y) = P_t^\alpha(1,y) = \frac{C_{1,\alpha} e^{-1/(4(-y))}}{(-y)^{1+\alpha}} \chi_{(-\infty,0)}(y). \tag{26}
$$

Then, if we set $z_1 = (t_1,x_1), z_2 = (t_2,x_2) \in \overline{\mathbb{R}^2_+}$, we have the estimate

$$
|\overline{\mathcal{U}}(z_1) - \overline{\mathcal{U}}(z_2)| \leq |z_1 - z_2|^{\alpha - \gamma - \frac{1}{p}} \, \| \, \overline{v} \, \|_{C^{\alpha - \gamma - \frac{1}{p}}(\mathbb{R},w)} \int_{\mathbb{R}} \max\{|y|^{\alpha - \gamma - \frac{1}{p}}, 1\} H_\alpha(y) dy,
$$

$$
\leq C|z_1 - z_2|^{\alpha - \gamma - \frac{1}{p}} \left(\|\overline{v}\|_{L^p(\mathbb{R},w)} + \left\| \overline{f} \right\|_{L^p(\mathbb{R},w)} \right).
$$

By direct computation from (25), and using Theorem 2, we have:

$$
\|\overline{\mathcal{U}}\|_{L^p(\mathbb{R}^2_+,w)} \leq \|\overline{v}\|_{L^p(\mathbb{R},w)} \leq C \left\| \overline{f} \right\|_{L^p(\mathbb{R},w)}.
$$

Therefore,

$$
\|\overline{\mathcal{U}}\|_{C^{\alpha - \gamma - \frac{1}{p}}\left(\mathbb{R}^2_+\right)} \leq C \left(\|\overline{V}\|_{L^\infty(\mathbb{R}^2_+)} + \left\| \overline{f} \right\|_{L^\infty(\mathbb{R})} \right), \tag{27}
$$

for a positive constant $C > 0$ depending only on α, p and γ.

Next we put $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$, so that $\widetilde{\mathcal{U}}$ satisfies

$$
\begin{cases}
\mathcal{M}_\alpha \, \widetilde{\mathcal{U}} = 0 & \text{in } \mathcal{B}_2^+, \\
\lim_{t\to 0} \mathcal{N}_\alpha \, \widetilde{\mathcal{U}}(t,\cdot) = (1-\eta)f = 0 & \text{on } B_2.
\end{cases}
$$

Considering the even reflection $\widetilde{Z}$ of $\widetilde{\mathcal{U}}$ in the variable t, as described in ([37] (Lemma 4.1)), we have that

$$
\mathcal{M}_\alpha \, \widetilde{Z} = 0 \text{ in } \mathcal{B}_2.
$$

From the definition (7) of $\left(\mathscr{D}_{right}\right)$, and using ([5] (Corollary 1.13)) or ([39] (Corollary 1.5)), we have that for $x \in B_1$ and $t \in (-1,1)$ fixed,

$$
\left| (\mathscr{D}_{right})\widetilde{Z}(t,x) \right| \leq C \parallel \widetilde{Z}(t,\cdot) \parallel_{L^p(B_2,w)}. \tag{28}
$$

Next, from the fact that

$$
(\mathscr{D}_{right})\widetilde{Z} = \widetilde{Z}_{tt} + \frac{\mu}{|t|} \widetilde{Z}_t
$$

and from the inequality (28), we obtain

$$
\left| \widetilde{Z}_{tt} + \frac{\mu}{|t|} \widetilde{Z}_t \right| \leq C \parallel \widetilde{Z} \parallel_{L^p(B_2,w)}.
$$

Therefore

$$
\left| \left(|t|^\mu \widetilde{Z}_t \right)_t \right| \leq C|t|^\mu \parallel \widetilde{Z} \parallel_{L^p(B_2,w)}
$$

Hence

$$\left| \widetilde{Z}_{tt} \right| \leq C \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2, w)}.$$

For any point $(t, x) \in \overline{\mathcal{B}}_1$, we have, by (28), that

$$\left| (\mathscr{D}_{right}) \widetilde{Z}(t, x) \right| + \left| \widetilde{Z}_{tt}(t, x) \right| \leq C \parallel \widetilde{Z} \parallel_{L^p(\mathcal{B}_2, w)},$$

which implies that

$$\widetilde{Z} \in C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1, w).$$

Thus, we have that $\widetilde{\mathcal{U}} \in C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1^+, w)$ such that

$$\parallel \widetilde{\mathcal{U}} \parallel_{C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1^+, w)} \leq C \left(\parallel \mathcal{U} \parallel_{L^p(\mathcal{B}_2^+, w)} + \parallel \overline{v} \parallel_{L^p(\mathbb{R})} \right)$$

$$\leq C \left(\parallel \mathcal{U} \parallel_{L^p(\mathcal{B}_2^+, w)} + \parallel f \parallel_{L^p(\mathcal{B}_2, w)} \right).$$

We finally obtain

$$\parallel \mathcal{U} \parallel_{C^{\alpha-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1^+, w)} \leq C \left(\parallel \widetilde{\mathcal{U}} \parallel_{C^{1-\gamma-\frac{1}{p}}(\overline{\mathcal{B}}_1^+, w)} + \parallel \overline{\mathcal{U}} \parallel_{C^{\alpha-\gamma-\frac{1}{p}}(\overline{\mathbb{R}_+^2}, w)} \right)$$

$$\leq C \left(\parallel \mathcal{U} \parallel_{L^p(\mathcal{B}_2^+, w)} + \parallel f \parallel_{L^p(\mathcal{B}_2), w} \right),$$

since $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$. This ends the proof of the first case.

Proof of (b): $f \in L^\infty(\mathbb{R})$. The proof here is similar to the first case above. By considering the same cut-off function $\eta \in C_c^\infty(B_2)$ with $\eta \equiv 1$ on B_2 and $0 \leq \eta \leq 1$ on $\mathbb{R}$, we let $\overline{v}$ be the unique solution to the equation

$$(\mathscr{D}_{right})^\alpha \overline{v} = \overline{f} \qquad \text{in} \quad \mathbb{R},$$

where $\overline{f} := \eta f$. Making use of the previous result Proposition 1, we have that $\overline{v} \in C^{\alpha-\gamma}(\mathbb{R})$ and

$$\|\overline{v}\|_{C^{\alpha-\gamma}(\mathbb{R})} \leq C \left(\|\overline{v}\|_{L^\infty(\mathbb{R})} + \left\| \overline{f} \right\|_{L^\infty(\mathbb{R})} \right),$$

where $C > 0$ is a constant that depends only on α and γ.

The next step is to consider the Bernardis–Reyes–Stinga–Torrea extension $\overline{\mathcal{U}}$ of $\overline{v}$, i.e.,

$$\overline{\mathcal{U}}(t, \cdot) = P_t^\alpha(t, \cdot) \star \overline{v},$$

which satisfies the equation

$$\begin{cases} \mathcal{M}_\alpha \overline{\mathcal{U}} = 0 & \text{in } \mathbb{R}_+^2, \\ \lim_{t \to 0} \mathcal{N}_\alpha \overline{\mathcal{U}}(t, x) = (\mathscr{D}_{right})^\alpha \overline{v}(x) = \overline{f}(x) & \text{on } \mathbb{R}. \end{cases}$$

Proceeding as in the previous case, it follows that

$$\|\overline{\mathcal{U}}\|_{C^{\alpha-\gamma}(\overline{\mathbb{R}^2})} \leq C \left(\|\overline{\mathcal{U}}\|_{L^\infty(\mathbb{R}^2)} + \left\| \overline{f} \right\|_{L^\infty(\mathbb{R})} \right), \tag{29}$$

for a positive constant $C > 0$ depending only on α and γ.

Next we put $\widetilde{\mathcal{U}} = \mathcal{U} - \overline{\mathcal{U}}$, so that $\widetilde{\mathcal{U}}$ satisfies

$$\begin{cases} \mathcal{M}_\alpha \widetilde{\mathcal{U}} = 0 & \text{in } \mathcal{B}_2^+, \\ \lim_{t \to 0} \mathcal{N}_\alpha \widetilde{\mathcal{U}}(t, \cdot) = (1 - \eta)f = 0 & \text{on } B_2. \end{cases}$$

We finally obtain

$$\| \, \mathcal{U} \, \|_{C^{\alpha-\gamma}(\overline{\mathcal{B}_1^+})} \le C \left(\| \, \widetilde{\mathcal{U}} \, \|_{C^{1-\gamma}(\overline{\mathcal{B}_1^+})} + \| \, \overline{\mathcal{U}} \, \|_{C^{\alpha-\gamma}(\overline{\mathbb{R}_+^2})} \right) \le C \left(\| \, \mathcal{U} \, \|_{L^\infty(\mathcal{B}_2^+)} + \| \, f \, \|_{L^\infty(B_2)} \right),$$

which ends the proof. $\quad\square$

4. Conclusions

Regularity theorems are an important result in the theory of PDEs, and their fractional counterparts also play a significance role in the study of problems involving nonlocal behaviour. As already observed in various papers [6,8,9,34,40–42] in the theory of fractional nonlocal PDEs, it is possible to find the qualitative behaviour of a solution. In this paper we have shown that the degenerate elliptic equation with mixed boundary conditions for a problem with fractional Marchaud derivative admits an interior regularity estimate. The current work fits in with some results obtained in the case of fractional Laplacians with Caffareli–Silvestre extensions. We stress that the types of regularity results proved herein form only a small subset of many possible versions of regularity theorems which can only be obtained using extension techniques.

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