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Review

Selective Survey on Spaces of Closed Subgroups of Topological Groups

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Abstract: We survey different topologizations of the set $\mathcal{S}(G)$ of closed subgroups of a topological group G and demonstrate some applications using Topological Groups, Model Theory, Geometric Group Theory, and Topological Dynamics.

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For a topological group G, $\mathcal{S}(G)$ denotes the set of all closed subgroups of G. There are many ways to endow $\mathcal{S}(G)$ with a topology related to the topology of G. Among these methods, the most intensively studied are the Chabauty topology, rooted in *Geometry of Numbers*, and the Vietoris topology, based on *General Topology*; both coincide if G is compact. The spaces of closed subgroups are interesting by their own merits, but they also have some deep applications in *Topological Groups and Model Theory, Geometric Group Theory, and Dynamical Systems.* This survey is my subjective look at this area.

1. Chabauty Spaces

1.1. From Minkowski to Chabauty

We recall that a *lattice* L in $\mathbb{R}^n$ is a discrete subgroup of rank n. We define *min* L as the length of the shortest nonzero vector from L, and we define *vol* $(\mathbb{R}^n/L)$ as the volume of a basic parallelepiped of L.

A sequence $(L_m)_{m\in\omega}$ of lattices in $\mathbb{R}^n$ converges to the lattice L if, for each $m \in \omega$, one can choose a basis $a_1(m), \dots, a_n(m)$ of L_m and a basis $a_1, \dots, a_n$ of L such that the sequence $(a_i(m))_{m\in\omega}$ converges to a_i for each $i \in \{1, \dots, n\}$. This convergence of lattices was introduced by H. Minkowski [1], and its usage in *Geometry of Numbers* (see [2]) is based on the following theorem from K. Mahler [3].

Theorem 1. *Let $\mathcal{M}$ be a set of lattices in $\mathbb{R}^n$. Every sequence in $\mathcal{M}$ has a convergent subsequence if and only if there exist two constants, $C > 0$ and $c > 0$, such that min $L > c$, vol $(\mathbb{R}^n \setminus L) < C$ for each $L \in \mathcal{M}$.*

What we know now is that Chabauty topology was invented by C. Chabauty in [4] in order to extend Theorem 1 to lattices in connected Lie groups.A discrete subgroup L of a connected Lie group G is called a *lattice* if the quotient space G/L is compact.

Let X be a Hausdorff locally compact space, and let *exp* X denote the set of all closed subsets of X. The sets

$$\{F \in exp\ X : F \cap K = \varnothing\},\ \{F \in exp\ X : F \cap U \neq \varnothing\},$$

where K runs over all compact subsets of X and U runs over all open subsets of X, form the subbase of the *Chabauty topology* on $exp\ X$. The space $exp\ X$ is compact and Hausdorff. If X is discrete, then $exp\ X$ is homeomorphic to the Cantor cube $\{0,1\}^{|X|}$.

We note also that a net $(F_\alpha)_{\alpha \in \mathcal{I}}$ converges in $exp\ X$ to F if and only if

- for every compact K of X such that $K \cap F = \emptyset$, there exists $\beta \in \mathcal{I}$ such that $F_\alpha \cap K = \emptyset$ for each $\alpha > \beta$;
- for every $x \in F$ and every neighborhood U of x, there exists $\gamma \in \mathcal{I}$ such that $F_\alpha \cap U \neq \emptyset$ for each $\alpha > \gamma$.

If G is a locally compact group, then $\mathcal{S}(G)$ is a closed subspace of $exp\ G$ (so, $\mathcal{S}(G)$ is compact); $\mathcal{S}(G)$ is called the *Chabauty space* of G.

Theorem 2. *[4]. Let G be a connected unimodular Lie group. A set $\mathcal{M}$ of lattices in G is relatively compact in $\mathcal{M}$ if and only if there exists a constant $C > 0$ and a neighborhood U with the identity e of G such that $L \cap U = \{e\}$ and vol $(G/L) < C$ for each $L \in \mathcal{M}$.*

There was some technical improvement made in [5] and the paper [4], which is included in [6], Chapter 8.

1.2. Pontryagin–Chabauty Duality

This duality was established in [7] and detailed in [8]. We use the standard abbreviation LCA for a locally compact Abelian group. Let G be an LCA-group $G^\wedge$, let denote its dual group $G^\wedge = Hom\ (G, \mathbb{R}/\mathbb{Z})$, and let φ denote the canonical bijection $\mathcal{S}(G) \longrightarrow \mathcal{S}(G^\wedge)$, $\varphi(X) = \{f \in G^\wedge : X \subseteq ker\ f\}$.

Theorem 3. *For every LCA-group G, the bijection $\varphi : \mathcal{S}(G) \longrightarrow \mathcal{S}(G^\wedge)$ is a homeomorphism.*

Typically, Theorem 3 is applied to replace $\mathcal{S}(G)$ by $\mathcal{S}(G^\wedge)$ in the case of a compact Abelian group G.

In what follows, we use the following notations: $\mathbb{C}_n$ is a cyclic group of order n, $\mathbb{C}_{p^\infty}$ is a quasi-cyclic (or Prüfer) p-group, $\mathbb{Z}$ is a discrete group of integers, $\mathbb{Z}_p$ is the group of p-adic integers, and $\mathbb{Q}_p$ is an additive group of a field of p-adic numbers.

1.3. $S(G)$ for Compact G

The following two lemmas from [9] are the basic technical tools in this area.

Lemma 1. *If G, H are compact groups and $\varphi : G \longrightarrow H$ is a continuous surjective homomorphism, then the mapping $\mathcal{S}(G) \longrightarrow \mathcal{S}(H)$, $X \longmapsto \varphi(X)$ is continuous and open.*

The continuity is easily deduced, but to prove the openness, we need

Lemma 2. *Let G be a compact group, $X \in \mathcal{S}(G)$. Then, the following subsets form a base of the neighborhoods of X is $\mathcal{S}(G)$:*

$$\mathcal{N}_X(U, N, x_1, \dots, x_n) = \{u^{-1}Yu : u \in U,\ Y \in \mathcal{S}(G),\ Y \subseteq XN,\ Y \cap x_1U \neq \emptyset, \dots, Y \cap x_nU \neq \emptyset, \}$$

where U is a neighborhood of the identity of G, N is closed normal subgroup such that G/N is a Lie group, and $x_1, \dots, x_n$ are arbitrary elements of X, $n \in \mathbb{N}$.

In particular, if G is a compact Lie group, then Lemma 2 states that there is a neighborhood $\mathcal{N}$ of X such that each subgroup $Y \in \mathcal{N}$ is conjugated to some subgroup of X. The Montgomery–Yang theorem on tubes [10] (see also ([11], Theorem 5.4, Chapter 2)) plays the key role in the proof of Lemma 2.

We recall that the *cellularity* (or Souslin number) $c(X)$ of a topological space X is the supremum of cardinalities of disjoint families of open subsets of X. A topological space X is called *dyadic* if X is a continuous image of some Cantor cube $\{0,1\}^{\kappa}$.

The *weight* $w(X)$ of a topological space X is the minimal cardinality of the open bases of X.

Theorem 4. *[9]. For every compact group G, we have $c(\mathcal{S}(G)) \leq \aleph_0$. In addition, if $w(G) \leq \aleph_1$, then $\mathcal{S}(G)$ is dyadic.*

Theorem 5. *[12]. Let a group G be either profinite or compact and Abelian. If $w(G) > \aleph_2$, then the space $\mathcal{S}(G)$ is not dyadic.*

Theorem 6. *[12]. Let G be an infinite compact Abelian group such that $w(G) \leq \aleph_1$. Then, the space $\mathcal{S}(G)$ is homeomorphic to the Cantor cube $\{0,1\}^{w(G)}$ if and only if $\mathcal{S}(G)$ has no isolated points.*

An Abelian group G is called *Artinian* if every increasing chain of subgroups of G is finite; every such group is isomorphic to the direct *sum* $\oplus_{p\in F} \mathbb{C}_{p^\infty} \oplus K$, where F is a finite set of primes, and K is a finite subgroup. An Abelian group G is called *minimax* if G has a finitely generated subgroup N such that G/N is Artinian.

Theorem 7. *[12]. For a compact Abelian group G, the space $\mathcal{S}(G)$ has an isolated point if and only if the dual group $G^\wedge$ is minimax.*

1.4. $S(G)$ for LCA G

The space $\mathcal{S}(\mathbb{R})$ is homeomorphic to the segment $[0,1]$. By [13], $\mathcal{S}(\mathbb{R}^2)$ is homeomorphic to the sphere $\mathbf{S}^4$. For $n \geq 3$, $\mathcal{S}(\mathbb{R}^n)$ is not a topological manifold and its structure is far from understood (see [14]).

Theorem 8. *[15]. The space $\mathcal{S}(G)$ of an LCA-group G is connected if and only if G has a subgroup topologically isomorphic to $\mathbb{R}$.*

If F is a non-solvable finite group, then $\mathcal{S}(\mathbb{R} \times F)$ is not connected ([8], Proposition 8.6).

Theorem 9. *[8]. The space $\mathcal{S}(G)$ of an LCA-group G is totally disconnected if and only if G is either totally disconnected or each element of G belongs to a compact subgroup.*

Some more information on $\mathcal{S}(G)$ for LCA G can be found in [8] and the references therein, particularly on the topological dimension of $\mathcal{S}(G)$.

By Theorems 3 and 4, $c(\mathcal{S}(G)) \leq \aleph_0$ for every discrete Abelian group. We say that a topological space X has the *Shanin number* ω if any uncountable family $\mathcal{F}$ of the non-empty open subsets of X has an uncountable subfamily $\mathcal{F}'$ such that $\cap\mathcal{F}' \neq \emptyset$. Evidently, if a space X has the Shanin number ω, then $c(X) \leq \aleph_0$. By [16], Theorem 1, for every discrete Abelian group G, the space $\mathcal{S}(G)$ has the Shanin number ω. By [16], Theorem 3, for every infinite cardinal τ, there exists a solvable discrete group G such that $c(\mathcal{S}(G)) = |G| = \tau$.

1.5. $S(G)$ as a Lattice

The set $S(G)$ has the natural structure of a lattice with the operations $\vee$ and $\wedge$, where $A \wedge B = A \cap B$, and $A \vee B$ is the smallest closed subgroup of G containing A and B. In this subsection, we formulate some results from [17] about the interrelations between the topological and lattice structures on $S(G)$.

For $g \in G$, $\overline{< g >}$ denotes the subgroup of G topologically generated by g. A totally disconnected locally compact group G is called *periodic* if $\overline{< g >}$ is compact for each $g \in G$. In this case, $\pi(G)$ denotes the set of all prime numbers such that $p \in \pi(G)$ if and only if $g \in G$ such that $\overline{< g >}$ is topologically isomorphic either to $\mathbb{C}_{p^n}$ or to $\mathbb{Z}_p$; this g is called a *topological p-element*.

Theorem 10. *For a compact group G, the following statements are equivalent:*

(i) $\wedge$ *is continuous;*
(ii) $\wedge$ *and* $\vee$ *are continuous;*
(iii) *G is the semidirect product* $K \leftthreetimes P$*, where K is profinite with finite Sylow p-subgroups, P is Abelian profinite and each Sylow p-subgroup of G is* $\mathbb{Z}_p$*,* $\pi(K) \cap \pi(P) = \emptyset$*, and the centralizer of each Sylow p-subgroup of G has a finite index in G.*

Theorem 11. *For a locally compact group G, the operation* $\wedge$ *is continuous if and only if the following conditions are satisfied:*

(i) *G is either discrete or periodic;*
(ii) $\wedge$ *is continuous in* $\mathcal{S}(H)$ *for each compact subgroup H of G;*
(iii) *the centralizer of each topological p-element of G is open.*

We recall that a torsion group G is *layer-finite* if the set $\{g \in G : g^n = e\}$ is finite for each $n \in \mathbb{N}$. A layer-finite group G is called *thin* if each Sylow p-subgroup of G is finite (equivalently, G has no subgroup isomorphic to $\mathbb{C}_{p^\infty}$).

Theorem 12. *Let G be a locally compact group. The operations* $\wedge$ *and* $\vee$ *are continuous if and only if G is periodic and topologically isomorphic to* $A \times B \times (C \leftthreetimes D)$*, where C has a dense thin layer-finite subgroup; A, B, D are Abelian with Sylow p-subgroups* $\mathbb{C}_{p^\infty}$*,* $\mathbb{Q}_p$*, or* $\mathbb{Z}_p$*; the sets* $\pi(A)$*,* $\pi(B)$*,* $\pi(G)$*,* $\pi(D)$ *are pairwise disjoint; and the centralizer of each Sylow p-subgroup of G is open.*

1.6. From Chabauty to Local Method

A topological group G is called *topologically simple* if each closed normal subgroup of G is either G or $\{e\}$. Every topologically simple LCA-group is discrete, and either $G = \{e\}$ or G is isomorphic to $\mathbb{C}_p$.

Following the algebraic tradition, we say that a group G is *locally nilpotent (solvable)* if every finitely generated subgroup is nilpotent (solvable).

In [18], Problem 1.76, V. Platonov asked whether there exists a non-Abelian, topologically simple, locally compact, locally nilpotent group. Here, we present the negative answer to this question for the locally solvable group obtained in [19].

Let G be a locally compact, locally solvable group. We take $g \in G \setminus \{e\}$, choose a compact neighborhood U of G, and denote by $\mathcal{F}$ the family of all topologically finitely generated subgroups of G containing g. We may assume that G is not topologically finitely generated, so $\mathcal{F}$ is directed by the inclusion $\subset$. For each $F \in \mathcal{F}$, we choose $A_F, B_F \in \mathcal{S}(F)$ such that $B_F \subset A_F$; A_F and B_F are normal in F, $A_F \cap U \neq \emptyset$, $B_F \cap U = \emptyset$, and A_F/B_F is Abelian. Since $\mathcal{S}(G)$ is compact, we can choose two subnets $(A_\alpha)_{\alpha \in \mathcal{I}}$, $(B_\alpha)_{\alpha \in \mathcal{I}}$ of the nets $(A_F)_{F \in \mathcal{F}}$, $(B_F)_{F \in \mathcal{I}}$ which converge to $A, B \in \mathcal{S}(G)$. Then A, B are normal in G, and A/B is Abelian. Moreover, $x \notin B$ and $A \cap U \neq \emptyset$. If $A \neq \{G\}$, then A is a proper normal subgroup of G; otherwise, G/B is Abelian.

In [20], the Chabauty topology was defined on some systems of closed subgroups of a locally compact group G. A system $\mathfrak{A}$ of closed subgroups of G is called *subnormal* if

- $\mathfrak{A}$ contains $\{e\}$ and G;
- $\mathfrak{A}$ is linearly ordered by the inclusion $\subset$;
- for any subset $\mathfrak{M}$ of $\mathfrak{A}$, the closure of $\bigcup_{F \in \mathfrak{M}} F \in \mathfrak{A}$ and $\bigcap_{F \in \mathfrak{M}} F \in \mathfrak{A}$;
- whenever A and B comprise a jump in $\mathfrak{A}$ (i.e., $B \subset A$, and no members of $\mathfrak{A}$ lie between B and A), B is a normal subgroup of A.

If the subgroups A, B form a jump, then A/B is called a factor of G. The system is called *normal* if each $A \in \mathfrak{A}$ is normal in G.

A group G is called an RN-group if G has a normal system with Abelian factors. Among the local theorems from [20], one can find the following: if every topologically finitely generated subgroup of a locally compact group G is an RN-group, then G is an RN-group. In particular, every locally compact, locally solvable group is an RN-group.

In 1941 (see ([21], pp. 78–83), A.I. Mal'tsev obtained local theorems for discrete groups as applications of the following general local theorem: if every finitely generated subsystem of an algebraic system A satisfies some property $\mathcal{P}$, which can be defined by some quasi-universal second-order formula, then A satisfies $\mathcal{P}$.

In [22], Mal'tsev's local theorem was generalized on a topological algebraic system. The part of the model-theoretical Compactness Theorem in Mal'tsev's arguments employs some convergents of closed subsets. A net $(F_\alpha)_{\alpha \in \mathcal{I}}$ of closed subsets of a topological space X S-converges to a closed subset F if

- for every $x \in F$ and every neighborhood U of x, there exists $\beta \in \mathcal{I}$ such that $F_\alpha \cap U \neq \emptyset$ for each $\alpha > \beta$;
- for every $y \in X \setminus F$, there exist a neighborhood $\mathcal{V}$ of y and a $\gamma \in \mathcal{I}$ such that $F_\alpha \cap \mathcal{V} = \emptyset$ for each $\alpha > \gamma$.

Every net of closed subsets of an arbitrary (!) topological space has a convergent subnet. If X is a Hausdorff locally compact space, then the S-convergence coincides with the convergence in the Chabauty topology.

1.7. Spaces of Marked Groups

Let F_k be the free group of rank k, with the free generators $x_1, \ldots, x_k$, and let $\mathcal{G}_k$ denote the set of all normal subgroups of F_k. In the metric form, the Chabauty topology on $\mathcal{G}_k$ was introduced in [23] as a reply to Gromov's idea of the topologizations of some sets of groups [24].

Let G be a group generated by $g_1, \ldots, g_k$. The bijection $x_i \longmapsto g_i$ $g_1, \ldots, g_n$ can be extended to the homomorphism $f : F_k \longrightarrow G$. With the correspondence $G \longmapsto ker\ f$, $\mathcal{G}_k$ is called the *space marked k-generated groups*.

A couple of papers in development by [23] are aimed at understanding how large, in the topological sense, are the well-known classes of finitely generated groups, or how a given k-generated group is placed in $\mathcal{G}_k$ (see [25]). Among the applications of $\mathcal{G}_k$, we mention the construction of topologizable Tarski Monsters in [26].

1.8. Dynamical Development

Every locally compact group G acts on the Chabauty space $\mathcal{S}(G)$ by the rule: $(g, H) \longmapsto g^{-1}Hg$. Under this action, every minimal closed invariant subset of $\mathcal{S}(G)$ is called a *uniformly recurrent subgroup* (URS). The study of URSs was initiated by Glasner and Weiss [27] with the following observation.

Let G be a locally compact group G acting on a compact X so that is G minimal, i.e., the orbit of each point $x \in X$ is dense. We consider the mapping $Stab : X \longrightarrow \mathcal{S}(G)$, defined by $Stab(x) = \{g \in G : gx = x\}$. Then, there is the unique URS contained in the closure of $Stab(X)$. This URS is called the *stabilizer URS*. Glasner and Weiss asked whether every URS of a locally compact group G arises as the stabilizer URS of a minimal action of G on a compact space. This question was answered in the affirmative in [28].

2. Vietoris Spaces

For a topological space X, the Vietoris topology on the set $exp\ X$ of all closed subsets of X is defined by the subbase of the open sets

$$\{F \in exp\ X : F \subseteq U\},\ \{F \in exp\ X : F \cap V \neq \emptyset\},$$

where $U, \mathcal{V}$ run over all open subsets of X.

A net $(F_\alpha)_{\alpha \in \mathcal{I}}$ converges to F in $exp\ X$ if and only if

- for each open subset U of X such that $F \subseteq U$, there exists $\beta \in \mathcal{I}$ such that $F_\alpha \subseteq U$ for each $\alpha > \beta$;
- for each $x \in F$ and each neighborhood $\mathcal{V}$ of x, there exists $\gamma \in \mathcal{I}$ such that $F_\alpha \cap \mathcal{V} \neq \emptyset$ for each $\alpha > \gamma$.

If X is regular, then $\mathcal{S}(G)$ is closed in $exp\ G$. To my knowledge, the spaces $\mathcal{S}(G)$, where G needs not be compact, endowed with Vietoris topologies appeared in [29] with the characterization of LCA-groups G such that the canonical mapping $\varphi : \mathcal{S}(G) \longrightarrow \mathcal{S}(G^\wedge)$ is a homeomorphism.

2.1. Compactness

We cannot ask for a constructive description of arbitrary topological groups G with compact space $\mathcal{S}(G)$ because we know nothing about G with $S(G) = 2$.

Theorem 13. *[30]. Let G be a locally compact group. The space $\mathcal{S}(G)$ is compact if and only if G is one of the following groups:*

(i) *G is compact;*
(ii) *$\mathbb{C}_{p_1^\infty} \times \ldots \times \mathbb{C}_{p_n^\infty} \times K$, where $p_1, \ldots, p_n$ are distinct prime numbers, K is finite, and each p_i is not a divisor of $|K|$;*
(iii) *$Q_p \times K$, where K is finite and p does not divide $|K|$.*

A similar characterization of groups with compact $\mathcal{S}(G)$ is given in [31], provided that G has a base of neighborhoods at the identity consisting of subgroups.

Theorem 14. *[32]. Let G be a locally compact group. A closed subset $\mathcal{F}$ of $\mathcal{S}(G)$ is compact if and only if the following conditions are satisfied:*

(i) *every descending chain of non-compact subgroups from F is finite;*

(ii) *every closed subset $\mathcal{F}'$ of $\mathcal{F}$ has only a finite number of non-compact subgroups maximal in $\mathcal{F}$;*
(iii) *if a closed subset $\mathcal{F}'$ of $\mathcal{F}$ has no non-compact subgroups, then $\cup\mathcal{F}'$ is compact.*

Two corollaries: Every compact subset of $\mathcal{S}(G)$ consisting of non-compact subgroups is scattered; a subset $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is countably compact.

For locally compact groups with the σ-compact space $\mathcal{S}(G)$ (see [33]), a description of the LCA-groups with locally compact space $\mathcal{S}(G)$ can be obtained in [34].

A topological group G is called *inductively compact* if every finite subset of G is contained in a compact subgroup. For a group G, $K(G)$ and $IK(G)$ denote the sets of all compact and closed inductively compact subgroups.

Theorem 15. *[35]. For every locally compact group G, $IK(G)$ is the closure of $K(G)$.*

Two corollaries: If G is a connected Lie group, then $K(G)$ is closed; $\mathcal{S}(G)$ is a k-space for each locally compact group G of countable weight, i.e., the topology of $\mathcal{S}(G)$ is uniquely determined by the family of all compact subsets of $\mathcal{S}(G)$.

2.2. Metrizability and Normality

LCA-groups G with metrizable and normal space $\mathcal{S}(G)$ were characterized by S. Panasyuk in the candidate thesis *Normality and metrizability of the space of closed subgroups*, Kyiv University, 1989.

Theorem 16. *For a discrete Abelian group G, the following statements are equivalent:*

(i) *$\mathcal{S}(G)$ is metrizable;*
(ii) *$\mathcal{S}(G)$ is normal;*
(iii) *G has a finitely generated subgroup H such that $G/H = \mathbb{C}_{p_1^\infty} \times \ldots \times \mathbb{C}_{p_n^\infty}$, where $p_1, \ldots, p_n$ are distinct primes.*

In the general case, metrizability and normality of $\mathcal{S}(G)$ are not equivalent, but if G is a connected semisimple Lie group, then $\mathcal{S}(G)$ is metrizable if and only if $\mathcal{S}(G)$ is normal if and only if G is compact (see [36,37]). The space $\mathcal{S}(G)$ for every connected solvable Lie group is metrizable [36].

2.3. Some Cardinal Invariants

We remind the reader that $c(X)$ denotes the cellularity of X.

Theorem 17. *[9]. For every infinite locally compact group G, we have $c(\mathcal{S}(G)) \leq c(G)$.*

Theorem 18. *[38]. For every locally compact group G, the following conditions are equivalent:*

(i) *$\mathcal{S}(G)$ is of countable pseudocharacter;*
(ii) *$\mathcal{S}(G)$ is of countable tightness;*
(iii) *$\mathcal{S}(G)$ is sequential;*
(iv) *$w(G) \leq \aleph_0$.*

3. Other Topologizations

3.1. Bourbaki Uniformities

Let $(X, \mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ induces the uniformity $\widetilde{\mathcal{U}}$ on the set $\mathcal{F}(X)$ of all non-empty closed subsets of X which have as a base the family of sets of the form

$$\{(A, B) \in \mathcal{F}(X) \times \mathcal{F}(X) : B \subseteq U(A),\ A \subseteq U(B)\},$$

whenever $U \in \mathcal{U}$. The uniformity $\widetilde{\mathcal{U}}$ was introduced in [39] (Chapter 2, § 1), and $\widetilde{\mathcal{U}}$ is called *the Bourbaki* (sometimes, Hausdorff–Bourbaki) *uniformity*.

Let G be a topological group. We endow G with the left uniformity L and $F(G)$ with the Bourbaki uniformity $\widetilde{L}$. We denote by $\mathcal{L}(G)$ and $\mathcal{B}(G)$ the subspaces of $\mathcal{F}(G)$ consisting of all subgroups and all totally bounded subsets of G.

Theorem 19. *[40]. Let G be a group with a base at the identity consisting of subgroups. The space $\mathcal{L}(G)$ is compact if and only if G is totally bounded and $K \bigcap G$ is dense in K for each closed subgroup K from the completion of G.*

In particular, if $\mathcal{L}(G)$ is compact, then G is totally minimal.

Theorem 20. *[40]. If a group G is complete in the left uniformity, then $\mathcal{B}(G)$ is complete.*

We recall that a topological group G is *almost metrizable* if each neighborhood of e contains a compact subgroup K such that the set of all open subsets containing K have a countable base. Every metrizable and every locally compact topological group is almost metrizable.

Theorem 21. *[40]. If an almost metrizable group G is complete in the left uniformity, then $\mathcal{F}(G)$ is complete.*

In [41], Theorem 21 is proved with the bilateral uniformity on G (and so on $\mathcal{F}(G)$) in place of the left uniformity.

3.2. Functionally Balanced Groups

For a topological group G, the set $\mathcal{F}(G)$ has the natural structure of a semigroup with the operation $(A, B) \longmapsto cl\ AB$.

Theorem 22. *[42]. For a topological group G, the following statements are equivalent:*

- *(i)* *$\mathcal{F}(G)$ is a topological semigroup;*
- *(ii)* *for every subset X of G and every neighborhood U of e, there exists a neighborhood V of e such that $VX \subseteq XU$;*
- *(iii)* *every bounded left uniformly continuous function on G is right uniformly continuous.*

A topological group G is called *balanced* (or a SIN-group) if the left and right uniformities of G coincide. A group G is called *functionally balanced* if G satisfies (iii) of Theorem 22. The study of functionally balanced groups was initiated by G. Itzkowitz [43].

The equivalence of (ii) and (iii) in Theorem 22 is a criterion for a topological group to be functionally balanced. In [44], this criterion was used to show that each almost-metrizable functionally balanced group is balanced.

3.3. Lattice Topologies

These topologies on a complete lattice $\mathcal{L}(G)$ of closed subgroups are algebraically defined by the lattice structure of $\mathcal{L}(G)$.

For example, a net $(A_\alpha)_{\alpha\in\mathcal{I}}$ in $\mathcal{L}(G)$ *order-converges* to $A \in \mathcal{L}(G)$ if there exist two nets $(B_\alpha)_{\alpha\in\mathcal{I}}$, $(C_\alpha)_{\alpha\in\mathcal{I}}$ in $\mathcal{L}(G)$ such that, for each $\alpha \in \mathcal{I}$, $B_\alpha \subseteq A_\alpha \subseteq C_\alpha$ and $\vee_{\alpha\in\mathcal{I}} B_\alpha = \wedge_{\alpha\in\mathcal{I}} C_\alpha = A$. By [45], for a compact group G, every net in $\mathcal{L}(G)$ has an order-convergent subset if and only if $\mathcal{L}(G)$ endowed with the Shabauty topology is a topological lattice (see Theorem 10).

More on the lattices' topologies on $\mathcal{L}(G)$ in the case of a compact G can be found in [46].

3.4. Segment Topologies

Let G be a topological group; $\mathcal{P}_G$ is the family of all subsets of G, and $[G]^{<\omega}$ is the family of all finite subsets of G. Each pair $\mathcal{A}, \mathcal{B}$ of subsets of $\mathcal{P}_G$ closed under finite unions defines the segment topology on $\mathcal{L}(G)$ with a base consisting of the segments

$$[A, G \setminus B] = \{X \in \mathcal{L}(G) : A \subseteq X \subseteq G \setminus B\},\ A \in \mathcal{A},\ B \in \mathcal{B}.$$

These topologies are described in [47] in the following three cases: $\mathcal{A} = \mathcal{B} = [G]^{<\omega}$; $\mathcal{A} = \mathcal{P}_G$ and $\mathcal{B} = [G]^{<\omega}$; $\mathcal{A} = [G]^{<\omega}$, $\mathcal{B} = \mathcal{P}_G$

3.5. (Σ, Θ)-Topologies

This general construction for topologizations of the set $\mathcal{L}(G)$ of closed subgroups of a topological group G from [48] produces Chabauty, Vietoris, and Bourbaki topologies, along with plenty of other topologies.

We assume that, for each $H \in \mathcal{L}(G)$, $\Sigma(H)$ is some family of open subsets of G, $\Sigma = \cup_{H\in\mathcal{L}(G)} \Sigma(H)$, and the following conditions are satisfied:

- if $U, \mathcal{V} \in \Sigma(H)$, then $U \cap \mathcal{V}$ contains some $W \in \Sigma(H)$;
- for every $U \in \Sigma(H)$, there exists $\mathcal{V} \in \Sigma(H)$ such that $U \in \Sigma(K)$ for each $K \in \mathcal{L}(G)$, $K \subseteq \mathcal{V}$;
- $\bigcap_{U\in\Sigma(H)} \overline{U} = H$ for each $H \in \mathcal{L}(G)$.

Then, the family $\{X \in \mathcal{L}(G) : X \subseteq U\}$, $U \in \Sigma$, is a base for the Σ-*topology* on $\mathcal{L}(G)$.

Let τ denote the topology of G, and let $\mathcal{P}_\tau$ denote the family of all subsets of τ. We assume that, for each $H \in \mathcal{L}(G)$, $\Theta(H)$ is some subset of $\mathcal{P}_\tau$ such that the following conditions are satisfied:

- for every $\alpha, \beta \in \Theta(H)$, there is a $\gamma \in \Theta(H)$ such that $\alpha < \gamma$, $\beta < \gamma$ ($\alpha < \beta$ means that, for every $U \in \alpha$, there exists $V \in \beta$ such that $V \subseteq U$);
- for every $\alpha \in \Theta(H)$, there exists $\beta \in \Theta(H)$ such that if $K \in \mathcal{L}(G)$ and $K \cap V \neq \emptyset$ for each $V \in \beta$, then $\alpha < \gamma$ for some $\gamma \in \Theta(K)$;
- for each $H \in \mathcal{L}(G)$ and every neighborhood V of x, there exists $\alpha \in \Theta(H)$ such that $x \in U$, $U \subseteq V$ for some $U \in \alpha$.

Then, the family $\{X \in \mathcal{L}(G) : X \cap U \neq \emptyset \text{ for each } U \in \alpha\}$, where $\alpha \in \Theta(H)$, $H \in \mathcal{L}(G)$, is a base for the Θ-*topology* on $\mathcal{L}(G)$.

The upper bound of Σ- and Θ-topologies is called the (Σ, Θ)-*topology*.

A net $(H_\alpha)_{\alpha\in\mathcal{I}}$ converges in (Σ, Θ)-topology to $H \in \mathcal{L}(G)$ if and only if

- for any $U \in \Sigma(H)$, there exists $\beta \in \mathcal{I}$ such that $H_\alpha \subseteq U$ for each $\alpha > \beta$;
- for any $\alpha \in \Theta(H)$, there exists $\gamma \in \mathcal{I}$ such that $H_\alpha \cap \mathcal{V} \neq \emptyset$ for each $\alpha > \gamma$.

In [48], one can find characterizations of G with a compact and discrete $\mathcal{L}(G)$ for some concrete (Σ, Θ)-topologies.

3.6. Hyperballeans of Groups

Let G be a discrete group. The set $\{Fg : g \in G, F \in [G]^{<\omega}\}$ is a family of balls in the finitary coarse structure on G. For definitions of coarse structures and balleans, see [49,50]. The finitary coarse structure on G induces the coarse structure on $\mathcal{L}(G)$ in which $\{X \in \mathcal{L}(G) : X \subseteq FA,\ A \in FX\}$, $F \in [G]^{<\omega}$ is the family of balls centered at $A \in \mathcal{L}(G)$. The set $\mathcal{L}(G)$ endowed with the finitary coarse structure is called a hyperballean of G. Hyperballeans of groups, carefully studied in [51], can be considered as asymptotic counterparts of Bourbaki uniformities.

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