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# A Note on the Topological Group $c_0$

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**Abstract:** A well-known result of Ferri and Galindo asserts that the topological group $c_0$ is not reflexively representable and the algebra WAP$(c_0)$ of weakly almost periodic functions does not separate points and closed subsets. However, it is unknown if the same remains true for a larger important algebra Tame$(c_0)$ of tame functions. Respectively, it is an open question if $c_0$ is representable on a Rosenthal Banach space. In the present work we show that Tame$(c_0)$ is small in a sense that the unit sphere $S$ and $2S$ cannot be separated by a tame function $f \in$ Tame$(c_0)$. As an application we show that the Gromov's compactification of $c_0$ is not a semigroup compactification. We discuss some questions.

**Keywords:** Gromov's compactification; group representation; matrix coefficient; semigroup compactification; tame function

## 1. Introduction

Recall that for every Hausdorff topological group $G$ the algebra WAP$(G)$ of all weakly almost periodic functions on $G$ determines the universal semitopological semigroup compactification $u_w : G \to G^w$ of $G$. This map is a topological embedding for many groups including the locally compact case. For some basic material about WAP$(G)$ we refer to [1,2].

The question if $u_w$ always is a topological embedding (i.e., if WAP$(G)$ determines the topology of $G$) was raised by Ruppert [2]. This question was negatively answered in [1] by showing that the Polish topological group $G := H_+[0,1]$ of orientation preserving homeomorphisms of the closed unit interval has only constant WAP functions and that every continuous representation $h : G \to Is(V)$ (by linear isometries) on a reflexive Banach space $V$ is trivial. The WAP triviality of $H_+[0,1]$ was conjectured by Pestov.

Recall also that for $G := H_+[0,1]$ every Asplund (hence also every WAP) function is constant and every continuous representation $G \to \mathrm{Iso}(V)$ on an Asplund (hence also reflexive) space $V$ must be trivial [3]. In contrast one may show (see [4,5]) that $H_+[0,1]$ is representable on a (separable) Rosenthal space (a Banach space is *Rosenthal* if it does not contain a subspace topologically isomorphic to $l_1$).

We have the inclusions of topological $G$-algebras

$$\mathrm{WAP}(G) \subset \mathrm{Asp}(G) \subset \mathrm{Tame}(G) \subset \mathrm{RUC}(G).$$

For details about Tame$(G)$ and definition of Asp$(G)$ see [5–7]. We only remark that $f \in$ Tame$(G)$ if and only if $f$ is a matrix coefficient of a Rosenthal representation. That is, there exist: a Rosenthal Banach space $V$; a continuous homomorphism $h : G \to Is(V)$ into the topological group of all linear isometries $V \to V$ with strong operator topology; two vectors $v \in V$; $\psi \in V^*$ (the dual of $V$) such that $f(g) = \psi(h(g)v)$ for every $g \in G$.

Similarly, it can be characterized $f \in$ Asp$(G)$ replacing Rosenthal spaces by the larger class of Asplund spaces. A Banach space is *Asplund* if the dual of every separable subspace is separable. Every reflexive space is Asplund and every Asplund is Rosenthal. A standard example of an Asplund but nonreflexive space is just $c_0$.

Recall that $c_0$, as an additive abelian topological group, is not representable on a reflexive Banach space by a well-known result of Ferri and Galindo [8]. In fact, $\mathrm{WAP}(c_0)$ separates the points but not points and closed subsets. The group $c_0$ admits an injective continuous homomorphism $h : c_0 \to Is(V)$ with some reflexive $V$ but such $h$ cannot be a topological embedding.

Presently it is an open question if every topological group (abelian, or not) $G$ is Rosenthal representable and if $\mathrm{Tame}(G)$ determines the topology of $G$. Note that the algebra $\mathrm{Tame}(G)$ appears as an important modern tool in some new research lines in topological dynamics motivating its detailed study [5,7].

One of the good reasons to study $\mathrm{Tame}(G)$ is a special role of tameness in the dynamical Berglund-Fremlin-Talagrand dichotomy [5]; as well as direct links to Rosenthal's $l_1$-dychotomy. In a sense $\mathrm{Tame}(G)$ is a set of all functions which are not dynamically massive.

By these reasons and since $H_+[0,1]$ is Rosenthal representable, it seems to be an attractive concrete question if $c_0$ is Rosenthal representable and it is worth studying how large is $\mathrm{Tame}(c_0)$. In the present work we show that $\mathrm{Tame}(c_0)$ is quite small (even for the discrete copy of $c_0$, see Theorem 3).

**Theorem 1.** *$\mathrm{Tame}(c_0)$ does not separate the unit sphere $S$ and $2S$.*

So, the closures of $S$ and $2S$ intersect in the universal tame compactification of $c_0$ (a fortiori, the same is true for the universal Asplund (HNS) semigroup compactification).

Another interesting question is if $c_0$ admits an embedding into a *metrizable* semigroup compactification. Note that any metrizable semigroup compactification of $H_+[0,1]$ is trivial.

In Section 3 we show that the Gromov's compactification $\gamma : c_0 \hookrightarrow P$, which is metrizable (and $\gamma$ is a $G$-embedding), is not a semigroup compactification.

**Theorem 2.** *Let $\gamma : c_0 \hookrightarrow P$ be the Gromov's compactification of the metric space $(c_0, \frac{d}{1+d})$, where $d(x,y) := ||x-y||$. Then $\gamma$ is not a semigroup compactification.*

This gives an example of a naturally defined separable unital (original topology determining) $G$-subalgebra of $\mathrm{RUC}(G)$ (for $G = c_0$) which is not left m-introverted in the sense of [9].

## 2. Tame Functions on $c_0$

Recall that a sequence $f_n$ of real-valued functions on a set $X$ is said to be *independent* if there exist real numbers $a < b$ such that

$$\bigcap_{n\in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n\in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets $P, M$ of $\mathbb{N}$. Every bounded independent sequence is an $l_1$-sequence [10].

As in [6,7] we say that a bounded family $F$ of real-valued (not necessarily continuous) functions on a set $X$ is a *tame family* if $F$ does not contain an independent sequence.

Let $G$ be a topological group, $f : G \to \mathbb{R}$ be a real-valued function. For every $g \in G$ define $fg : G \to \mathbb{R}$ as $(fg)(x) = f(gx)$ (for multiplicative $G$). Denote by $\mathrm{RUC}(G)$ the algebra of all bounded right uniformly continuous functions on $G$. So, $f \in \mathrm{RUC}(G)$ means that $f$ is bounded and for every $\epsilon > 0$ there exists a neighborhood $U$ of the identity $e$ (of the multiplicative group $G$) such that $|f(ux) - f(x)| < \epsilon$ for every $x \in G$ and $u \in U$. This algebra $\mathrm{RUC}(G)$ corresponds to the greatest $G$-compactification $G \to \beta_G G$ of $G$ (with respect to the left action), *greatest ambit* of $G$.

We say that $f \in \mathrm{RUC}(G)$ is a tame function if the orbit $fG := \{fg\}_{g\in G}$ is a tame family. That is, $fG$ does not contain an independent sequence; notation $f \in \mathrm{Tame}(G)$.

*2.1. Proof of Theorem 1*

We have to show that $\mathrm{Tame}(c_0)$ does not separate the spheres $S$ and $2S$ (where $S := \{x \in c_0 : ||x|| = 1\}$). In fact we show the following stronger result.

**Theorem 3.** *Let $G = c_0$ be the additive group of the classical Banach space $c_0$. Assume that $f : c_0 \to \mathbb{R}$ be any (not necessarily continuous) bounded function such that*

$$\begin{cases} f(x) \leq a & \forall \, ||x|| = 1 \\ b \leq f(x) & \forall \, ||x|| = 2 \end{cases}$$

*for some pair $a < b$ of real numbers. Then $f$ is not a tame function on the discrete copy of the group $c_0$.*

**Proof.** For every $n \in \mathbb{N}$ consider the function

$$f_n : c_0 \to \mathbb{R}, x \mapsto f(e_n + x),$$

where $e_n$ is a vector of $c_0$ having 1 as its $n$-th coordinate and all other coordinates are 0. Clearly, $f_n = fg_n$ where $g_n = e_n \in c_0$. We have to check that $fG$ is an untame family. It is enough to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ in $fG$ is an independent family of functions on $c_0$. We have to show that for every finite nonempty disjoint subsets $I, J$ in $\mathbb{N}$ the intersection

$$\bigcap_{n\in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n\in J} f_n^{-1}[b, \infty)$$

is nonempty.

Define $v = (v_k)_{k\in\mathbb{N}} \in c_0$ as follows: $v_j = 1$ for every $j \in J$ and $v_k = 0$ for every $k \notin J$. Then

(1) $v \in c_0$ and $||v|| = 1$.
(2) $||e_i + v|| = 1$, $f_i(v) = f(e_i + v) \leq a$ for every $i \in I$.
(3) $||e_j + v|| = 2$, $f_j(v) = f(e_j + v) \geq b$ for every $j \in J$.

So we found $v$ such that

$$v \in \bigcap_{n\in I} f_n^{-1}(-\infty, a] \cap \bigcap_{n\in J} f_n^{-1}[b, \infty).$$

□

**Corollary 1.** *The bounded RUC function*

$$f : c_0 \to [-1, 1], x \mapsto \frac{||x||}{1 + ||x||}$$

*is not tame on $c_0$ (even on the discrete copy of the group $c_0$).*

**Proof.** Observe that $f(S) = \frac{1}{2}, f(2S) = \frac{2}{3}$ and apply Theorem 3. □

Theorem 3 remains true for the spheres $rS$ and $2rS$ for every $r > 0$. In the case of Polish $c_0$ it is unclear if the same is true for any pair of different spheres around the zero. If, yes then this will imply that $\mathrm{Tame}(c_0)$ does not separate the zero and closed subsets. The following question remains open even for any topological group [5,7].

**Question 1.** *Is it true that $\mathrm{Tame}(c_0)$ separates the points and closed subsets ? Is it true that Polish group $c_0$ is Rosenthal representable ?*

## 3. Gromov's Compactification Need Not Be a Semigroup Compactification

Studying topological groups $G$ and their dynamics we need to deal with various natural closed unital $G$-subalgebras $\mathcal{A}$ of the algebra $\mathrm{RUC}(G)$. Such subalgebras lead to $G$-compactifications of $G$ (so-called *G-ambits*, [11]). That is we have compact $G$-spaces $K$ with a dense orbit $Gz \subset K$ such that the Gelfand algebra which corresponds to the compactification $G \to K, g \mapsto gz$ is just $\mathcal{A}$. Frequently but not always such compactifications are the so-called *semigroup compactifications*, which are very useful in topological dynamics and analysis. Compactifications of topological groups already is a fruitful research line. See among others [12–14] and references there. In our opinion semigroup compactifications deserve even much more attention and systematic study in the context of general topological group theory.

A semigroup compactification of $G$ is a pair $(\alpha, K)$ such that $K$ is a compact *right topological semigroup* (all right translations are continuous), and $\alpha$ is a continuous semigroup homomorphism from $G$ into $K$, where $\alpha(G)$ is dense in $K$ and the left translation $K \to K, x \mapsto \alpha(g)x$ is continuous for every $g \in G$.

One of the most useful references about semigroup compactifications is a book of Berglund, Junghenn and Milnes [9]. For some new directions (regarding topological groups) see also [3,4,15,16].

**Question 2.** *Which natural compactifications of topological groups $G$ are semigroup compactifications? Equivalently which Banach unital $G$-subalgebras of $RUC(G)$ are left m-introverted (in the sense of [9])?*

Recall that *left m-introversion* of a subalgebra $\mathcal{A}$ of $\mathrm{RUC}(G)$ means that for every $v \in \mathcal{A}$ and every $\psi \in MM(\mathcal{A})$ the matrix coefficient $m(v, \psi)$ belongs to $\mathcal{A}$, where

$$m(v, \psi) : G \to \mathbb{R}, g \mapsto \psi(g^{-1}v)$$

and $MM(\mathcal{A}) \subset \mathcal{A}^*$ denotes the spectrum (Gelfand space) of $\mathcal{A}$.

It is not always easy to verify left m-introversion directly. Many natural $G$-compactifications of $G$ are semigroup compactifications. For example, it is true for the compactifications defined by the algebras $\mathrm{RUC}(G)$, $\mathrm{Tame}(G)$, $\mathrm{Asp}(G)$, $\mathrm{WAP}(G)$. Of course, the 1-point compactification is a semitopological semigroup compactification for any locally compact group $G$.

As to the counterexamples. As it was proved in [3], the subalgebra $\mathrm{UC}(G) := \mathrm{RUC}(G) \cap \mathrm{LUC}(G)$ of all uniformly continuous functions is not left m-introverted for $G := H(C)$, the Polish group of homeomorphisms of the Cantor set.

In this section we show that the Gromov's compactification of a metrizable topological group $G$ need not be a semigroup compactification.

Let $\rho$ be a bounded metric on a set $X$. Then the Gromov's compactification of the metric space $(X, \rho)$ is a compactification $\gamma : X \to P$ induced by the algebra $\mathcal{A}$ which is generated by the bounded set of functions

$$\{\rho_z : X \to \mathbb{R}, \rho_z(x) = \rho(z, x)\}_{z \in X}.$$

Then $\gamma$ always is a topological embedding. If $X$ is separable then $P$ is metrizable. Moreover, if $(X, \rho)$ admits a continuous $\rho$-invariant action of a topological group $G$ then $\gamma$ is a $G$-compactification of $X$; see [17].

Here we examine the following particular case. Let $G$ be a metrizable topological group. Choose any left invariant metric $d$ on $G$. Denote by $\gamma : G \to P$ the Gromov's compactification of the bounded metric space $(G, \rho)$, where $\rho = \frac{d}{1+d}$.

Consider the following natural bounded RUC function

$$f : G \to \mathbb{R}, x \mapsto \frac{||x||}{1 + ||x||}$$

where $||x|| := d(e, x)$. By $\mathcal{A}_f$ we denote the smallest closed unital $G$-subalgebra of $\mathrm{RUC}(G)$ which contains $fG = \{fg : g \in G\}$. Then $\mathcal{A}_f$ is the algebra which corresponds to the compactification $\gamma$. Indeed, $\rho_{g^{-1}}(x) = \rho(g^{-1}, x) = (fg)(x)$ for every $g, x \in G$.

*Proof of Theorem 2*

We have to prove Theorem 2.

**Proof.** By the discussion above, the unital $G$-subalgebra $\mathcal{A}_f$ of $\mathrm{RUC}(G)$ associated with $\gamma$ is generated by the orbit $fG$, where $f : G \to \mathbb{R}, f(x) = \frac{||x||}{1+||x||}$. Since $c_0$ is separable the algebra $\mathcal{A}_f$ is separable. Hence, $P$ is metrizable. If we assume that $\gamma$ is a semigroup compactification then the separability of $\mathcal{A}_f$ guarantees by [4] ( Prop. 6.13) that $\mathcal{A}_f \subset \mathrm{Asp}(G)$. On the other hand, since $\mathrm{Asp}(G) \subset \mathrm{Tame}(G)$, and $f \in \mathcal{A}_f$ we have $f \in \mathrm{Tame}(G)$. Now observe that $f$ separates the spheres $S$ and $2S$ and we get a contradiction to Corollary 1. □

**Question 3.** *Is it true that the Polish group $c_0$ admits a semigroup compactification $\alpha : c_0 \hookrightarrow P$ such that $P$ is metrizable and $\alpha$ is an embedding? What if $P$ is first countable?*

This question is closely related to the setting of this work. Indeed, by [4] (Prop. 6.13) (resp., by [4] (Cor. 6.20)) the metrizability (first countability) of $P$ guarantees that the corresponding algebra is a subset of $\mathrm{Asp}(G)$ (resp. of $\mathrm{Tame}(G)$).

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