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Type I Almost-Homogeneous Manifolds of Cohomogeneity One—IV

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Abstract: This paper is one of a series in which we generalize our earlier results on the equivalence of existence of Calabi extremal metrics to the geodesic stability for any type I compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we actually carry all the earlier results to the type I cases. In Part II, we obtained a substantial amount of new Kähler–Einstein manifolds as well as Fano manifolds without Kähler–Einstein metrics. In particular, by applying Theorem 15 therein, we obtained complete results in the Theorems 3 and 4 in that paper. However, we only have partial results in Theorem 5. In this note, we provide a report of recent progress on the Fano manifolds $N_{n,m}$ when $n > 15$ and $N'_{n,m}$ when $n > 4$. We provide two pictures for these two classes of manifolds. See Theorems 1 and 2 in the last section. Moreover, we present two conjectures. Once we solve these two conjectures, the question for these two classes of manifolds will be completely solved. By applying our results to the canonical circle bundles, we also obtain Sasakian manifolds with or without Sasakian–Einstein metrics. These also provide open Calabi–Yau manifolds.

Keywords: Kähler manifolds; Einstein metrics; Ricci curvature; fibration; almost-homogeneous; cohomogeneity one; semisimple Lie group; Sasakian–Einstein; Calabi–Yau metrics

MSC: 53C10; 53C25; 53C55; 14J45

1. Introduction

This paper is the fourth part of [1]. In [1], we prove the following:

Proposition 1. *For any simply connected Type I, compact, Kähler, complex, almost-homogeneous manifold of cohomogeneity one with a hypersurface end, there is an extremal metric in a given Kähler class if and only if Condition (7) in [1] holds.*

Condition (7) therein was represented as a sign of a topological integral, as was shown in Section 6.1 of Part III [2]. See also our integral at the beginning of the next section.

We obtained similar results for the higher codimensional end case and the general case, in Sections 3 and 4 of Part II [3]. See also Theorem 15 in the last section for the Kähler–Einstein case therein.

As an application, considering the canonical circle bundle, we also obtained Sasakian manifolds with and without Sasaki–Einstein metrics (with the same Reeb vector field and CR structure, see [4] Theorem 2.4 (iv), also [5,6]).

We shall prove the converse for the Type II cases in [7].

We finished all cases in which the existence of the extremal metrics could be reduced to an ordinary differential equation problem. We also provided many examples for both the stable and unstable cases, as promised in [8]. It was difficult for us to find any example that is semistable but not stable. We note that, since the automorphism group was semisimple, the original Futaki invariants were zero for all manifolds considered in this paper. Therefore, we provide more classical examples than in [9].

The authors in [10] studied a few of the first cases in [1], i.e., the cases with $F = F(OP_n)$ in which $S = SO(4, \mathbf{C}), SO(6, \mathbf{C}), SO(8, \mathbf{C}), SO(10, \mathbf{C})$, where S is the induced group action on the fibers.

We classify the manifolds into three types: Types I, II, and III. This classification can be found in [8] Section 12.

Basically, Type III manifolds are completions of a C^* bundle over a homogeneous manifold. They were dealt with in [11,12]. See also [13]. Once the automorphism group has a nontrivial solvable radical, it has a C^* action. It then has a Type III structure. See [8,14,15].

In the case in which the automorphism group is semisimple, let K be the isometric group, which is compact, and let L be the isotropic subgroup of K at a point of a generic real hypersurface orbit. Let $N = Norn_K(L)$ be the normalizer. Then on the Lie algebra level N is a direct product of the Lie algebra of L and an algebra A. A is either of real dimension one or three. When $\dim A = 1$, its Lie group is a torus. When $\dim A = 3$, it is $so(3)$. We call the corresponding manifolds Types I and II. One can look at [2] for more information. See [16–19] for examples.

In [1], we used very explicit and elementary calculations to avoid the Cauchy–Riemann structure and other very abstract tools in [10,20] that also cumulating results from other papers of Spiro.

We dealt with the uniqueness in the second section therein, where we proved the existence of smooth geodesics in the Mabuchi moduli spaces of the Kähler metrics.

Proposition 2. *For any two smooth Kähler metrics, which are equivariant under the maximal compact subgroup in a given Kähler class on a Type I compact complex almost-homogeneous manifold of cohomogeneity one, there is a smooth geodesic connecting them in the Mabuchi moduli space of Kähler metrics.*

The same result was proven for the toric manifolds and the Type III manifolds in [21], for some Type II manifolds in [8], and for all Type II manifolds in [7].

We also obtained many Kähler–Einstein manifolds as well as Fano manifolds without the Kähler–Einstein metric of Type I in the fifth section of Part II [3]. The Futaki invariants are zero in our case since the automorphism groups are semisimple. It turns out that our method is also easier than that in [22] for finding Kähler–Einstein metrics since they depended on the zeros of Futaki invariants, which might be very rare. Therefore, their method is more suitable for finding manifolds without any Kähler–Einstein metric. In our case, both the Kähler–Einstein manifolds and manifolds without Kähler–Einstein metric might be dense in a certain Zariski sense. It seems to us that it is hard to see any example with a vanishing generalized Futaki invariant. Furthermore, most of our manifolds are Fano, but it is not that easy in [22]. Moreover, it is very easy to check that [22] can be a Corollary of [23] and does not involve much stability.

In Part II [3], Theorems 3 and 4, we completely solved the existence of the Kähler–Einstein metric for the two classes of manifolds $M_{n,m}$ and $M'_{n.m}$ therein.

To understand better the M_n in our Section 5 in Part II [3], that is, the $M_{n,1}$ in our Theorem 3 there, it is not difficult to see that M_n is a Q^n bundle over

$$Q^{n+1} = \{[z_0, z_1, \cdots, z_n, z_{n+1}, z_{n+2}] \in \mathbf{C}P^{n+2}|_{z_0^2+\cdots+z_{n+2}^2=0}\}.$$

Let $N_n = \mathbf{P}(T^*_{Q^{n+1}})$. Then it is our manifold with $F = \mathbf{C}P^n$. Therefore, N_n is never Fano (see [24,25]). To construct M_n, we notice that by [26] (pp. 590–593) there is a holomorphic conformal structure on Q^{n+1} with respect to the line bundle $N = \mathcal{O}(2)$. Therefore, $N_n = \mathbf{P}(T^*_{Q^{n+1}}) = \mathbf{P}(N \otimes$

$T^*_{Q^{n+1}}) = \mathbf{P}(T_{Q^{n+1}})$. The exceptional divisor comes from the zero set of the corresponding holomorphic symmetric 2-tensor in N_n. Therefore, the vector bundle $\mathcal{O}(1) \oplus T_{Q^{n+1}}$ also has a conformal structure with respect to N. M_n is just the zero set of the corresponding holomorphic symmetric 2-tensor in $\mathbf{P}(\mathcal{O}(1) \oplus T_{Q^{n+1}})$. The branch double covering map $M_n \to N_n$ is introduced by the projection $\mathcal{O}(1) \oplus T_{Q^{n+1}} \to T_{Q^{n+1}}$.

As in the early parts, we shall deal with Lie algebras. A general reader might use [27] as a reference. In Part II [3], Theorem 5, we completely solved the integral problem for $N''_{n,m}$. They are all homogeneous and hence Kähler–Einstein. However, for $N_{n,m}$ and $N'_{n,m}$ therein, we only had partial results.

From Part II [3], the picture in the case of the $C_k = Sp(k, \mathbf{C})$ fiber structure (the paragraph after Theorem 9, compared with Theorems 3–5 and the results of this note, for example) is quite different from that in the case of $SO(n+1, \mathbf{C})$ or $Spin(7, \mathbf{C})$ structures. The results of Theorems 6–12 are quite complete compared with Theorem 5 in that paper.

In the next two sections, we shall provide more details. In Section 3, we provide two new theorems and some conjectures, which might lead to a complete solution of the two classes of manifolds $N_{n,m}$ and $N'_{n,m}$.

2. Preliminaries

Now, we apply our arguments in [1–3,8]. In this note, we only consider the case in which $S = SO(n+1, \mathbf{C})$. In Part II [3], we first consider the case in which $G = S = SO(n+1, \mathbf{C})$. Let m be the codimension. Then as before we have the integral in Theorem 15 of Part II [3]:

$$\int_0^{-K(F)+m-1} (-K(F) - D(F) - x)Q(x)dx.$$

Here, $K(F)$ is related to the canonical line bundle. We know that $K(F) = -n-1$ if $F = \mathbf{C}P^n$ and $-4k$ if $F = Gr(2k, 2)$, and $K(F) = -n$ if $F = Q^n$. See the paragraph right before Theorem 15 in Part II [3]. In the case of $F = \mathbf{C}P^n$, $D = Q^{n-1}$ and therefore $D(F) = 2$. In the case of $F = Q^n$, $D = Q^{n-1}$, which is the intersection of $Q^n \subset \mathbf{C}P^{n+1}$ with a hyperplane. Therefore, $D(Q^n) = 1$. We also have $D(Gr(2k,\ 2)) = 2$. See the paragraph after Theorem 1 in Part II [3]. In both cases of $N_{n,m}$ and $N'_{n,m}$, the divisors are of codimension 1. That is, $m - 1 = 0$. The upper limits of our integrals are $-K(F)$. In both cases, we have

$$-K(F) - D(F) = n - 1.$$

From the proof of Lemma 6 in [1], we see that the eigenvalues of the Ricci curvature of the exceptional divisor in the direction other than the 1-strings of roots with zero eigenvalues are represented as

$$Ric_s = n - 1 + m_2, c^{-1}a_{\rho,i} \pm (n - 1 + m_2)$$

where s are indices, similar to i. In the same way, the eigenvalues of the restriction of the Ricci curvature of the whole manifold in those direction are represented by

$$\tilde{Ric}_s = -l_\rho,\ c^{-1}a_{\rho,i} \mp l_\rho.$$

When $S = SO(n+1, \mathbf{C})$, by the fourth paragraph after Theorem 6 in Part I [1], we have $m_2 = 0$. By the paragraph after Theorem 4 in Part I [1], we have $c = 1$. By the second paragraph of Section 3 in Part II [3], we have $l_\rho = -(n + m + m_2)c = -n - 1$ if $F = \mathbf{C}P^n$ and $l_\rho = -(n - 1 + m)c = -n$ for $F = Q^n$.

Therefore, when $F = \mathbf{C}P^n$, we have

$$\tilde{Ric}_s = n + 1, a_{\rho,i} \mp (n + 1)$$

$$Ric_s = n - 1, a_{\rho,i} \mp (n-1).$$

When $F = Q^n$, Ric_s are the same and

$$\tilde{Ric}_s = n, a_{\rho,i} \mp n.$$

Therefore, according to the calculation in Section 6.1 of Part III [2] or Theorem 2 in Part I [1], we have

$$Q(x) = x^{n-1} \prod_i (a_{\rho,i}^2 - x^2).$$

Another way to get the $a_{\rho,i}$ is to look at the fiberation

$$D \to G/P.$$

We see that $a_{\rho,i}$ are just the eigenvalues of the Ricci curvatures of G/P. However, $a_{\rho,i}$ only occurs in our volume if e_1 acts non-trivially on the corresponding root. The concrete $a_{\rho,i}$ can be calculated by our Theorem 2 in Part I [1]. There, we have the Ricci curvature formula for G/P:

$$q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} H_\alpha.$$

There, the sum is over all the positive roots which contribute to the tangent space of G/P , i.e., the positive roots that are not in P.

For example, when $S = SO(8, \mathbf{C})$ and $G = SO(10, \mathbf{C})$ as in Item 4 in [10], the Ricci curvature of G/P is decided by $\sum_{\alpha \in \Delta^+ - \Delta_{P_{2,4}}} H_\alpha$. Here we assume that S is generated by e_i, $i > 1$. Therefore, the parabolic subgroup P is determined, which is why we put the first index 2 to imply e_2 as our e_1 earlier and the second index 4 as $S = D_4$. We see that $q_{G/P} = 8e_1$, and the only positive roots which are not in P with nontrivial e_2 actions are $e_1 \pm e_2$. Therefore, $a_{\rho,1} = (q_{G/P}, e_1 \pm e_2) = 8$. Therefore, it is not Fano for the $\mathbf{C}P^7$ case, and the coefficient in Item 4 of [10], was wrong (it was 16 instead of 8). Moreover, G/P has a complex dimension of 8. This comes from $e_1 \pm e_i$. However, only $e_1 \pm e_2$ are nontrivial by e_2. This is why we have a power of 1 in Part II [3] right before Theorem 3; however, in [10], they had a power of 4. The one with a $F = Q^n$ fiber is our $M_{7,1}$ in our Part II [3].

One series of examples that we already know are the product P_n of two copies of $\mathbf{C}P^n$ and M_n, N_n in [8,14,15]. Since M_n are Kähler–Einstein, so are P_n and N_n, whenever it is Fano.

3. New Kähler–Einstein Metrics

If the Kähler class is the Ricci class, we have, as in Part II [3], Section 5, $\alpha = \frac{u}{c}$, $l = l_\rho$, and

$$m(u) = 2Q_1(u).$$

Therefore,

$$f_l = [n - 1 + m_2 - c^{-1}\sqrt{U}]U^{\frac{n-2}{2}} Q_1(U).$$

In this section, we shall check case by case on the type of the groups (S, G).

We say that a manifold is *nef* if the anti-canonical line bundle is nef. We say that a manifold is *Fano* if the anti-canonical line bundle is positive.

First, if $S = G = B_k$ $k \geq 2$, we have $n = 2k$, Q_1 is a constant, $l_\rho = -(n+1)$ if $F = \mathbf{C}P^n$ or $-n$ if $F = Q^n$. Then,

$$\begin{aligned} C_n &= \int_0^{l_\rho^2} f_{l_\rho} dU = \int_0^{(n+1)^2} (n - 1 - \sqrt{U})U^{\frac{n-2}{2}} dU \\ &= \frac{2(n-1)}{n}(n+1)^n - 2(n+1)^n = -\frac{2(n+1)^n}{n} < 0 \end{aligned}$$

for the case in which $F = \mathbf{C}P^n$ and

$$C'_n = -\frac{n^{n-1}}{n+1} < 0$$

for the case in which $F = Q^n$. Therefore, there is a Kähler–Einstein metric. Again, this is known since the manifolds are homogeneous. The same formula is true for $G = S = D_k, n = 2k - 1$.

Now, we consider the situation in which $G = B_{k+1}$ and $S = B_k$. We have

$$a_{\rho,1} = 1 + 2k = n + 1 = -l_\rho$$

if $F = \mathbf{C}P^n$ or

$$a_{\rho,1} = n + 1 = -l_\rho + 1$$

if $F = Q^n$. The manifolds are nef but not Fano (or are Fano). The same is true for $G = D_{k+1}$ and $S = D_k$. We only need to consider the case in which $F = Q^n$. The integral is

$$\begin{aligned} I_n &= \int_0^n (n-1-v)v^{n-1}((n+1)^2 - v^2)dv \\ &= \frac{n^{n-1}}{(n+2)(n+3)}(-(n+1)(n+2)(n+3) + 3n^2) \\ &= \frac{n^{n-1}}{(n+2)(n+3)}(2n^3 - 6n^2 - 11n - 6) > 0 \end{aligned}$$

if $n \geq 5$. Otherwise, $I_n < 0$. Therefore, the corresponding manifolds M_n are Kähler–Einstein for $n \leq 4$. Others are non-Kähler–Einstein Fano manifolds. Each M_n is a Q^n bundle over Q^{n+1}.

Now, we consider the general situation in which $S = SO(n+1, \mathbf{C})$ and $G = SO(2m + n + 1, \mathbf{C})$, and P is the smallest parabolic subgroup of G containing S as a simple factor. In this case,

$$Q_1(v) = \prod_{j=0}^{m-1}((n + 2j + 1)^2 - v^2).$$

They are nef but not Fano (or are Fano). We have the integrals:

$$I_{n,m} = \int_0^n (n-1-v)v^{n-1} \prod_0^{m-1}((n + 2j + 1)^2 - v^2)dv.$$

For $m = 2$, we can use Mathematica to check $I_{n,2}$ with

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Integrate[(n-1-v)v^(n-1)((n + 1)^2 − v^2)((n + 3)^2 − v^2), {v, 0, n}]
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and have $I_{4,2} > 0$ but $I_{3,2} < 0$. In the same way, we can use Mathematica and check that $I_{3,m} < 0$ for $m \leq 7$ but $I_{3,8} > 0$.

We denote the corresponding manifolds by $M_{n,m}$. Therefore, we have in Part II [3] the following:

Proposition 3. *The manifolds $M_{n,0}$ are homogeneous Kähler–Einstein manifolds.*

$$M_{3,1}, M_{4,1}, M_{3,2}, M_{3,3}, M_{3,4}, M_{3,5}, M_{3,6}, M_{3,7}$$

are nonhomogeneous Kähler–Einstein manifolds. Other $M_{n,m}$ are Fano manifolds without any Kähler–Einstein metric.

Next, we consider the case in which $S = SO(n+1, \mathbf{C}), G = SO(2m + n + 1, \mathbf{C})$, and S_1 in Section 3 of [1] is maximal. In this case, we have positive roots in $\Delta^+ - \Delta_{P_{m+1,k}}$, e.g., when $n = 2k - 1$, $e_i \pm e_j, i \leq m < j, e_i + e_l, \ i < l \leq m$. For each i, there are only $e_i \pm e_{m+1}$, which are nontrivial by e_{m+1}.

$(q_{G/P}, e_i \pm e_{m+1}) = 2k + m - i + i - 1 = 2k + m - 1 = n + 1 + m - 1 = n + m$. Altogether there are m choices of i. Therefore,

$$Q_1(v) = ((n+m)^2 - v^2)^m$$

with $m > 1$. When $m > 1$, they are all Fano. The integral is

$$J_{n,m} = \int_0^{n+1} (n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = \mathbf{C}P^n$ or

$$J'_{n,m} = \int_0^{n} (n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = Q^n$, and when m tends to $+\infty$

$$m^{-2m} J_{n,m} \to e^{2(n-1)} C_n < 0$$

or

$$m^{-2m} J'_{n,m} \to e^{2(n-1)} C'_n.$$

Therefore, $J_{n.m} < 0$ (or $J'_{n,m} < 0$) when m is high enough.

Additionally, we can compare the change rate of the factor

$$h(v) = ((n+m)^2 - v^2)^m$$

for different m and n values. We let

$$t(m) = (\log h)' = m(\frac{1}{n+m+v} - \frac{1}{n+m-v}) = \frac{-2mv}{(n+m)^2 - v^2}.$$

Then

$$t(m+1) - t(m) = \frac{-2v[n^2 - m(m+1) - v^2]}{((n+m)^2 - v^2)((n+m+1)^2 - v^2)} > 0$$

if $m \geq n$. Therefore, if $J_{n,m} \leq 0$ with $m \geq n$, then $J_{n,m+1} < 0$. The same thing is also true for $J'_{n,m}$.

Now, we can use Mathematica to check $J'_{n,n}$ with

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Integrate[(n-1-v) v^(n-1) ((2n)^2 -v^2 )^(n) , {v, 0, n}]
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and obtain $J'_{n,n} < 0$ when $n = 3, 4$ but $J'_{5,5} > 0$.

We then use Mathematica to check $J'_{5,10}$ with

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Integrate[(5-1-v)v^4 (225 -v^2 )^(10) , {v, 0, 5}]
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and have $J'_{5,10} > 0$.

Similarly, by using Mathematica we have $J'_{5,20} < 0$ and $J'_{5,m} > 0$ if $2 \leq m \leq 13$ and $J'_{5,14} < 0$. Therefore, when $m \geq 14$, $J'_{5,m} < 0$; otherwise, $J'_{5,m} > 0$. We can also check that $J_{5,m} < 0$ for $m > 1$.

Similarly, we use Mathematica to check $J'_{4,m}$ for $m = 2, 3$ and $J'_{3,m}$ for $m = 2$. We find that all of them < 0. Therefore, $J'_{3,k}, J'_{4,k} < 0$ if $2 \leq k$, as are $J_{3,k}, J_{4,k}$.

In general, we expect that, if

$$m > \frac{n^2(n-1)}{e},$$

then $J'_{n,m} < 0$. For example, if $n = 6$ we expect $J'_{6,60} < 0$. We check it with Mathematica and get

$$J'_{6,60}, J'_{6,30}, J'_{6,27} < 0, \quad J'_{6,20}, J'_{6,25}, J'_{6,26} > 0.$$

In the same way, we find $J'_{6,k} > 0$ for $2 \leq k \leq 5$. Therefore, $J'_{6,k} > 0$ if $k \leq 26$; otherwise, $J'_{6,k} < 0$. We can also check that $J_{6,k} < 0$ for all $k > 1$. One might expect that, in all cases, $J_{n,m} < 0$. However, we have $J_{n,n} > 0$ for $n = 101, 51, 26, 25$, $J_{n,n} < 0$ for $n = 11, 21, 24$. Thus, one might expect that $J_{n,m} < 0$ for $n \leq 24$. However, we have $J_{n,m} > 0$ for $(n,m) = (24,12), (18,9), (16,8)$, etc. One can check that $J_{n,m} < 0$ for $n \leq 15$. Moreover, $J_{16,m} > 0$ if $5 \leq m \leq 8$; otherwise, $J_{16,m} < 0$. We can also check that $J_{n,2} < 0$. Furthermore,

$$J_{n,3} = C(n)(8n^8 + 6n^7 - 1534n^6 - 16019n^5 - 75163n^4 - 194786n^3 - 263486n^2 - 216981n - 76545) < 0$$

if and only if $n \leq 17$. We can check that $J_{17,m} < 0$ if and only if $m \leq 3$ or ≥ 11.

Therefore, we obtained in Part II [3] that, if we denote the corresponding manifolds by $N_{n,m}$ (or $N'_{n.m}$), then the following holds:

Proposition 4. *$N_{n,m}$ $3 \leq n \leq 15$, and $N'_{3,m}, N'_{4,m}$ admit a Kähler–Einstein metric for all $m > 1$. $N'_{5,m}$ admit a Kähler–Einstein metric if and only if $m > 13$. $N'_{6,m}$ admit a Kähler–Einstein metric if and only if $m > 26$. $N_{16,m}$ admit a Kähler–Einstein metric if and only if $m > 8$ or $2 \leq m < 5$. $N_{17,m}$ admit a Kähler–Einstein metric if and only if $m > 10$ or $2 \leq m < 4$. $N_{n,2}$ admit a Kähler–Einstein metric for any n. $N_{n,3}$ admit a Kähler–Einstein metric if and only if $n \leq 17$. In general, $N_{n,m}$ (or $N'_{n,m}$) admit a Kähler–Einstein metric when m is large enough, i.e., there is an integer $N(n)$ (or $N'(n)$) such that if $m > N(n)$ (or $> N'(n)$) then $N_{n,m}$ (or $N'_{n,m}$) admit a Kähler–Einstein metric. Moreover, if $m \geq n$ and $N_{n,m}$ (or $N'_{n,m}$) admit a Kähler–Einstein metric, so does $N_{n,m+1}$ (or so does $N'_{n,m+1}$).*

Now, what would happen if n is large? We used a computer to test the smallest possible $N(n)$ and $N'(n)$ in Proposition 4. We obtain the following new results:

Theorem 1. *$N(n)$ is equal to the following:*
1. *$2n - 24$ if $15 < n < 24$;*
2. *$2n - 25$ if $23 < n < 31$;*
3. *$2n - 26$ if $30 < n < 42$;*
4. *$2n - 27$ if $41 < n < 57$;*
5. *$2n - 28$ if $56 < n < 82$;*
6. *$2n - 29$ if $81 < n < 133$;*
7. *$2n - 30$ if $132 < n < 287$;*
8. *$2n - 31$ if $286 < n < 2601$ (at least).*

We also have the following:

1. *$M_{n,2}$ is Einstein iff $n < 27$;*
2. *$M_{n,3}$ is Einstein iff $n < 18$;*
3. *$M_{n,4}$ is Einstein iff $n < 17$.*

We conjecture that $N(n)$ has an asymptotic line $2n - a$. Since all function points and values are integers, if there is truly an asymptotic line $2n - a$, then we have $N(n) = 2n - a$ eventually. More precisely, one might have the following:

Conjecture 1. *The manifolds $M_{m,n}$ are Kähler–Einstein if and only if $m > N(n)$ or m is that in the last part of Theorem 1. Moreover, $N(n) = 2n - 31$ whenever $n > 286$.*

We obtain Theorem 1 by a direct calculation with the help of a computer. Using a different method, we are able to extend Item 8 in Theorem 1 to $n = 6000$. It seems to us that it is true at least up to $n = 200{,}000$.

This is an undergraduate project taken by the second author in the University of California at Riverside. We believe that it is worth publishing at this stage so far.

One might think that the same would be true for $N'(n)$. However, we have

Theorem 2. *$N'(n) = 2n^2 - 10n + b(n)$, where $b(n)$ is equal to the following:*

1. *13 if $n = 5$;*
2. *14 if $n = 6$;*
3. *15 if $6 < n < 9$;*
4. *16 if $8 < n < 11$;*
5. *17 if $10 < n < 17$;*
6. *18 if $16 < n < 32$;*
7. *19 if $31 < n < 101$ (at least).*

Although $N'(n)$ will not be asymptotic when n is large enough, if $b(n)$ has a limit, then again, because everything are integers, we would obtain $N'(n)$ when n is large enough.

We should get further calculation later on. However, we could have the following conjecture now:

Conjecture 2. *There is a high N and an integer B such that whenever $n > N$, we have $b(n) = B$.*

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