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Lipschitz Stability for Non-Instantaneous Impulsive Caputo Fractional Differential Equations with State Dependent Delays

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Abstract: In this paper, we study Lipschitz stability of Caputo fractional differential equations with non-instantaneous impulses and state dependent delays. The study is based on Lyapunov functions and the Razumikhin technique. Our equations in particular include constant delays, time variable delay, distributed delay, etc. We consider the case of impulses that start abruptly at some points and their actions continue on given finite intervals. The study of Lipschitz stability by Lyapunov functions requires appropriate derivatives among fractional differential equations. A brief overview of different types of derivative known in the literature is given. Some sufficient conditions for uniform Lipschitz stability and uniform global Lipschitz stability are obtained by an application of several types of derivatives of Lyapunov functions. Examples are given to illustrate the results.

Keywords: non-instantaneous impulses; Caputo fractional derivative; differential equations; state dependent delays; lipschitz stability

AMS Subject Classifications: 34A37, 34K20, 34K37

1. Introduction

Many papers in the literature study stability of solutions of differential equations via Lyapunov functions. One type of stability, useful in real world problems, is the so-called Lipschitz stability and Dannan and Elaydi [1] introduced the notion of Lipschitz stability for ordinary differential equations. As noted in [1], this type of stability is important only for nonlinear problems since it coincides with uniform stability in linear systems. Based on theoretical results for Lipschitz stability in [1], the dynamic behavior of a spacecraft when a single magnetic torque-rod is used for achieving a pure spin condition is studied in [2]. Recently, stability properties of delay fractional differential equations without any type of impulse are considered and we refer the reader to [3] and the references therein.

In this paper, we study the Lipschitz stability for a nonlinear system of non-instantaneous impulsive fractional differential equations with state dependent delay (NIFrDDE). The impulses start abruptly at some points and their actions continue on given finite intervals. Non-instantaneous impulsive differential equations were introduced by Hernandez and O'Regan in 2013 (see, for example, [4]). The systematic description of solutions of both ordinary and Caputo fractional differential equations with non-instantaneous impulse and without delays is given in the monograph [5]. In addition, some results for non-instantaneous fractional equations without any type of delay are presented in [6–8]. In [9], Caputo fractional differential equations with time varying delays is considered (we note that the model had no impulses). However, in this paper, for the first time, we consider together

1. Lipschitz stability;
2. state dependent delays (note a special case is time varying delays); and
3. models with non-instantaneous impulses.

There are two different approaches in the literature for the interpretation of the solution of fractional differential equations with impulses (for more details, see [6] and Chapter 2 of the book [5]). In the first interpretation, the lower limit of the fractional derivative is one and the same on the whole interval of study and at each point of jump we consider a boundary value problem defined by the impulsive function. In the second interpretation, the lower limit of the fractional derivative changes at each time of jump with the idea of considering an initial value problem at each jump point.

In this paper, we use the second approach to study Lipschitz stability properties of nonlinear non-instantaneous impulsive delay differential equations. The delays are bounded and depend on both the time and the state. Note several stability properties are studied in the literature for Caputo fractional differential equations (for example, see [10] (without delays), [3] (with delays and no impulses), and [11] (with multiple discrete delays without impulses)). Our study is based on Lyapunov functions and the Razumikhin technique. A brief overview in the literature of different types of derivatives of Lyapunov functions among the studied fractional differential equation is given. Several sufficient conditions for uniform Lipschitz stability and global uniform Lipschitz stability are obtained by an application of these derivatives. Some examples illustrating the results are given.

2. Notes on Fractional Calculus

We give the main definition of fractional derivatives used in the literature (see, for example, [12–14]). We give these definitions for scalar functions. Throughout the paper, we assume $q \in (0,1)$.

- *Riemann–Liouville (RL) fractional derivative* :

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \ t \geq t_0 $$

where $\Gamma(.)$ denotes the Gamma function.

- *Caputo fractional derivative*

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0. $$

Note that for a constant m the equality ${}_{t_0}^{C}D_t^q m = 0$ holds. However, for any given t^* ,we denote ${}_{t_0}^{C}D_t^q m(t^*) = {}_{t_0}^{C}D_t^q m(t)|_{t=t^*}$.

- *The Grünwald–Letnikov fractional derivative* is given by

$$ {}_{t_0}^{GL}D_t^q m(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \ {}_qC_r \ m(t-rh), \quad t \geq t_0 $$

and the Grünwald–Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \, {}_qC_r m(t-rh), \quad t \geq t_0, $$

where ${}_qC_r = \frac{q(q-1)...(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

From the relation between the Caputo fractional derivative and the Grünwald–Letnikov fractional derivative using Equation (1), we define the Caputo fractional Dini derivative of a function as

$$ {}_{t_0}^{C}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], $$

i.e.,

$$ {}_{t_0}^{C}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{q}{r} \Big(m(t-rh) - m(t_0) \Big) \Big]. $$

The fractional derivatives for scalar functions could be easily generalized to the vector case by taking fractional derivatives with the same fractional order for all components.

3. Statement of the Problem and Basic Definitions

Let the positive constant r be given and the points $\{t_i\}_1^\infty, \{s_i\}_1^\infty$ be such that $0 < s_i < t_i < s_{i+1}$, $i = 1, 2, \ldots$. Let $t_0 \geq 0$ be the given initial time. Without loss of generality, we can assume $t_0 \in [0, s_1)$.

Consider the space PC_0 of all functions $y : [-r, 0] \to \mathbb{R}^n$, which are piecewise continuous endowed with the norm $||y||_{PC_0} = \sup_{t\in[-r,0]}\{||y(t)|| : y \in PC_0\}$ where $||.||$ is a norm in $\mathbb{R}^n$.

The intervals $(t_i, s_{i+1}),\ i = 0, 1, 2, \ldots$ are the intervals on which the fractional differential equations are given and on the intervals $(s_i, t_i),\ i = 1, 2, \ldots$ the impulsive conditions are given.

The Caputo fractional derivative has a memory and it depends significantly on its lower derivative. This property as well as the meaning of impulses in the differential equation lead to two basic approaches to Caputo fractional differential equations with non-instantaneous impulses:

- Unchangeable lower limit of the Caputo fractional derivative: the lower limit of the fractional derivative is equal to the initial time t_0 on the whole interval of consideration.
- Changeable lower limit of the Caputo fractional derivative: the lower limit of the fractional derivative is equal to the left end t_i on the interval $(t_i, s_{i+1}), i = 0, 1, 2, \ldots$ without impulses.

In this paper, we study the case of changeable lower limit of the Caputo fractional derivative.

Consider the initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive fractional differential equations with state dependent delay (NIFrDDE) with $q \in (0, 1)$:

$$ \begin{aligned} &{}_{t_i}^{C}D_t^q x(t) = f(t, x(t), x_{\rho(t,x_t)}) \text{ for } t \in (t_i, s_{i+1}], i = 0, 1, 2, \ldots, \\ &x(t) = \phi_i(t, x(s_i)), \ t \in (s_i, t_i], \ i = 1, 2, \ldots, \\ &x(t + t_0) = \varphi(t) \text{ for } t \in [-r, 0], \end{aligned} \tag{1} $$

where $x \in \mathbb{R}^n$, ${}_{t_i}^{C}D_t^q x(t)$ denotes the Caputo fractional derivative with lower limit t_i for the state $x(t)$, the functions $f : [0, s_1] \bigcup_{i=1}^\infty [t_i, s_{i+1}] \times R^n \times PC_0 \to \mathbb{R}^n$; $\rho : [0, s_1] \bigcup_{i=1}^\infty [t_i, s_{i+1}] \times PC_0 \to \mathbb{R}$, $\varphi \in PC_0$; $\phi_i : [s_i, t_i] \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots$. Here, $x_t(s) = x(t + s), s \in [-r, 0]$, i.e., represents the history of the state from time $t - r$ up to the present time t. Note that for any $t \geq 0$ we let $x_{\rho(t,x_t)} = x(\rho(t, x(t + s))), s \in [-r, 0]$, i.e., the function ρ determines the state-dependent delay. Note, the integer order differential equations with non-instantaneous impulses and state dependent delay are studied in [15].

Let $\mathcal{PC}[t_0, \infty)$ be the space of all functions $y : [t_0 - r, \infty) \to \mathbb{R}^n$ which are piecewise continuous on $[t_0 - r, \infty)$ with points of discontinuity $s_i,\ i = 1, 2, \ldots$, the limits $y(s_i - 0) = \lim_{t\to s_i,\, t<s_i} y(t) = y(s_i)$

and $y(s_i+) = \lim_{t\to s_i,\, t>s_i} y(t)$ exist, for any $t \in (t_i, s_i]$ the Caputo fractional derivative ${}_{t_i}^{C}D_t^q y(t), i = 0, 1, \dots,$ exists and it is endowed with the norm $||y||_{\mathcal{PC}} = \sup_{t\in[t_0-r,\infty)}\{||y(t)|| : y \in \mathcal{PC}[t_0,\infty)\}$ where $||.||$ is a norm in $\mathbb{R}^n$.

Define the set $PC^q[t_0,\infty) = \{y \in C(\bigcup_{i=0}^{\infty}(t_i,s_i],\mathbb{R}^n)$ such that for any $t \in (t_i,s_i]$: $\int_{t_i}^{t}(t-s)^{q-1}y(s)ds < \infty,\ i = 1,2,\dots\}$.

We introduce the assumptions:

A1. The function $f \in C([0,s_1]\bigcup_{i=1}^{\infty}[t_i,s_{i+1}] \times \mathbb{R}^n \times PC_0, \mathbb{R}^n)$ is such that for any $y \in \mathbb{R}^n, u \in PC_0$ the inclusion $f(.,y,u) \in PC^q[0,\infty)$ holds.

A2. The function $\rho \in C([0,s_1]\bigcup_{i=1}^{\infty}[t_i,s_{i+1}] \times PC_0, [-r,\infty))$ and for any $(t,y) \in \bigcup_{i=0}^{k}[t_i,s_{i+1}] \times PC_0$ the inequalities $t - r \le \rho(t,y) \le t$ holds.

A3. The functions $\phi_i \in C([s_i,t_i] \times \mathbb{R}^n, \mathbb{R}^n),\ i = 1,2,\dots.$

A4. The function $\varphi \in PC_0$.

A5. The function $f(t,0) = 0$ for $t \in [0,s_1]\bigcup_{i=1}^{\infty}[t_i,s_{i+1}]$ and $\phi_i(t,0) = 0$ for $t \in [s_i,t_i],\ i = 1,2,\dots.$

Remark 1. *Assumption A5 guarantees the existence of the zero solution of IVP for NIFrDDE (Equation (1)) with the zero initial function* $\varphi \equiv 0$.

Remark 2. *Assumption A2 guarantees the delay of the argument in Equation (1).*

Definition 1. *Let the conditions A1–A4 be satisfied. The function* $x \in \mathcal{PC}[t_0,\infty)$ *is a solution of the IVP in Equation (1) iff it satisfies the following integral-algebraic equation*

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ \varphi(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s),x_{\rho(s,x_s)})ds, & t \in (0,s_1], \\ \phi_i(t,x(s_i)), & t \in (s_i,t_i],\ i = 1,2,\dots, \\ \phi_i(t_i,x(s_i)) + \frac{1}{\Gamma(q)}\int_{t_i}^t (t-s)^{q-1} f(s,x(s),x_{\rho(s,x_s)})ds, & t \in (t_i,s_{i+1}],\ i = 1,2,\dots. \end{cases} \tag{2}$$

Definition 2. *The functions* f, ρ *are defined only on the intervals without impulses on which the differential equation is given.*

We generalize Lipschitz stability ([1]) for ordinary differential equations to systems of Caputo fractional non-instantaneous impulsive differential equations with state dependent delay.

Definition 3. *The zero solution of NIFrDDE (Equation (1)) is said to be:*

- *Uniformly Lipschitz stable if there exists* $M \ge 1$ *and* $\delta > 0$ *such that, for any for any initial time* $t_0 \in [0,s_1]\bigcup_{k=1}^{\infty}[t_k,s_k]$ *and any initial function* $\varphi \in PC_0$*, the inequality* $||\varphi||_{PC_0} < \delta$ *implies* $||x(t;t_0,\varphi)|| \le M||\varphi||_{PC_0}$ *for* $t \ge t_0$ *where* $x(t;t_0,\varphi)$ *is a solution of Equation (1).*
- *Globally uniformly Lipschitz stable if there exists* $M \ge 1$ *such that, for any initial time* $t_0 \in [0,s_1]\bigcup_{k=1}^{\infty}[t_k,s_k]$ *and any initial function* $\varphi \in PC_0$*, the inequality* $||\varphi||_{PC_0} < \infty$ *implies* $||x(t;t_0,\varphi)|| \le M||\varphi||_{PC_0}$ *for* $t \ge t_0$.

Let $J \subset \mathbb{R}_+,\ 0 \in J, \rho > 0$. Consider the following sets:

$$\begin{aligned} M(J) &= \{a \in C[J,\mathbb{R}^+] : a(0) = 0,\ \text{a(r) is strictly increasing in } J, \text{ and} \\ &\quad a^{-1}(\alpha r) \le r q_a(\alpha) \ \text{ for some function } \ q_a :\ q_a(\alpha) \ge 1, \text{ if } \alpha \ge 1\}, \\ K(J) &= \{a \in C[J,\mathbb{R}^+] : a(0) = 0,\ \text{a(r) is strictly increasing in } J, \text{ and} \\ &\quad a(r) \le K_a r \ \text{ for some constant } \ K_a > 0\}, \\ S_\rho &= \{x \in \mathbb{R}^n :\ ||x|| \le \rho\}. \end{aligned}$$

Remark 3. *The function* $a(u) = K_1 u,\ K_1 > 0$ *is from the class* $K(\mathbb{R}_+)$ *with* $K_a = K_1$. *The function* $a(u) = K_2 u^2,\ K_2 \in (0,1]$ *is from the class* $M([1,\infty))$ *with* $q(u) = \sqrt{\frac{u}{K_2}} \geq 1$ *for* $u \geq 1$.

4. Lyapunov Functions and Their Derivatives among Nonlinear Non-Instantaneous Caputo Delay Fractional Differential Equations

One approach to study Lipschitz stability of solutions of Equation (1) is based on using Lyapunov-like functions. The first step is to define a Lyapunov function. The second step is to define its derivative among the fractional equation.

We use the class Λ of Lyapunov-like functions, defined and used for impulsive differential equations in [16].

Definition 4. *Let* $J \in \mathbb{R}_+$ *be a given interval, and* $\Delta \subset \mathbb{R}^n$ *be a given set. We say that the function* $V(t,x) : J \times \Delta \to \mathbb{R}_+$, *belongs to the class* $\Lambda(J,\Delta)$ *if*

- *The function* $V(t,x)$ *is continuous on* $J/\{s_k \in J\} \times \Delta$ *and it is locally Lipschitz with respect to its second argument.*
- *For each* $s_k \in J$ *and* $x \in \Delta$, *there exist finite limits*

$$V(s_k,x) = V(s_k - 0,x) = \lim_{t\uparrow s_k} V(t,x) \text{ and } V(s_k + 0,x) = \lim_{t\downarrow s_k} V(t,x).$$

In connection with the Caputo fractional derivative, it is necessary to define in an appropriate way the derivative of Lyapunov functions among the studied equation. We give a brief overview of the derivatives of Lyapunov functions among solutions of fractional differential equations known and used in the literature. There are mainly three types of derivatives of Lyapunov functions from the class $\Lambda(J,\Delta)$ used in the literature to study stability properties of solutions of Caputo fractional differential in Equation (1):

- *First type*: **the Caputo fractional derivative** of the function $V(t,x(t)) \in \Lambda([a,b),\Delta)$ defined by

$${}_{t_k}^{c}D^q V(t,x(t)) = \frac{1}{\Gamma(1-q)} \int_{t_k}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s,x(s))\Big)ds, \qquad t \in [t_k, s_{k+1}) \tag{3}$$

  where $x(t)$ is a solution of Equation (1).
- *Second type*: **Dini fractional derivative** of the Lyapunov function $V \in \Lambda([t_0,\infty),\mathbb{R}^n)$ among Equation (1): Let $\phi \in PC_0$ and $t \in (t_k, s_{k+1})$ for a non-negative integer k. Then,

$$\begin{aligned} &D^+_{(1)} V(t,\phi(0),t_k,\phi) = \\ &\limsup_{h\to 0} \frac{1}{h^q}\Big[V(t,\phi(0)) - \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r+1}\, {}_qC_r V(t-rh, \phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t)))\Big] \end{aligned} \tag{4}$$

  where $\phi_0(s) = \psi(s)$ and $\phi(\rho(t,\phi_0)-t) = \phi(\rho(t,\phi(s))-t)$ for any $s \in [-r,0]$. We note that, because of Assumption A2, the inequality $t - r < \rho(t,\phi(s)) < t$ holds ($-r < \rho(t,\phi(s)) - t < 0$), i.e $\phi(\rho(t,\phi(s))-t)$ is well defined.

  The derivative of Equation (4) keeps the concept of fractional derivatives because it has a memory.

- *Third type*: **Caputo fractional Dini derivative** of a Lyapunov function $V \in \Lambda([t_0,\infty),\mathbb{R}^n)$ among Equation (1): Let the initial function $\varphi \in PC_0$ be given and the function $\phi \in PC_0$ and $t \in (t_k, s_{k+1})$ for a non-negative integer k. Then,

$$\begin{aligned}&{}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0)) = \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,\phi(0)) - V(t_k,\varphi(0)) \\ &- \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r+1}\, {}_qC_r \Big(V(t-rh,\phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t))) - V(t_k,\varphi(0))\Big)\Big\},\end{aligned} \tag{5}$$

or its equivalence

$$\begin{aligned}&{}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0)) = \\ &\limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,\phi(0)) + \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r}\, {}_qC_r V(t-rh,\phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t)))\Big\} \\ &- \frac{V(t_k,\varphi(0))}{(t-t_k)^q\Gamma(1-q)}.\end{aligned} \tag{6}$$

The derivative ${}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0))$ given by Equation (6) depends significantly on both the fractional order q and the initial data (t_k,φ) of IVP for FrDDE (Equation (1)) and it makes this type of derivative close to the idea of the Caputo fractional derivative of a function.

Remark 4. *For any initial data $(t_k,\varphi) \in \mathbb{R}_+ \times PC_0$ of the IVP for NIFrDDE (Equation (1)) and any function $\phi \in PC_0$ and any point $t \in (t_k, s_{i+1})$ for a non-negative integer k the relations*

$${}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0)) = D^+_{(1)} V(t,\phi(0),t_k,\phi) - {}^{RL}_{t_k}D^q\Big(V(t_k,\varphi(0))\Big),$$

$${}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0)) = D^+_{(1)} V(t,\phi(0),t_k\phi), \quad \text{if } V(t_k,\varphi(0)) = 0 \tag{7}$$

$${}^{c}_{(1)}D^q_+ V(t,\phi;t_k,\varphi(0)) < D^+_{(1)} V(t,\phi(0),t_k,\phi), \quad \text{if } V(t_k,\varphi(0)) > 0. \tag{8}$$

are satisfied.

Remark 5. *A derivative of $V(t,x) \in \Lambda(J,\Delta)$ among a system of Caputo fractional differential equations without delays was introduced by V. Lakshmikantham et al. [17] in 2009. Later, it was generalized for fractional equations with delays ([18–20]):*

$$D^+_{(1)} V(t,\phi(0),\phi) = \limsup_{h\to 0} \frac{1}{h^q}\Big[V(t,\phi(0)) - V(t-h,\phi(0) - h^q f(t,\phi))\Big], \quad t \geq t_0 \tag{9}$$

where $\phi \in C([-\tau,0],\Delta)$.

This definition is a direct generalization of the well known Dini derivative among differential equations with ordinary derivatives. However, for equations with fractional derivatives, it seems strange. It does not depend on the order q of the fractional derivative nor on the initial time t_0. The operator defined by Equation (9) has no memory, which is typical for the fractional derivative.

The derivative $D^+_{(1)} V(t,\phi(0),\phi)$ defined by Equation (9) is applied in [18] to study stability of fractional delay differential equations where in the proof of the main comparison result (Theorem 4.3 [18]) the derivative $D^+_{(1)} V(t,\phi(0),\phi)$ is incorrectly substituted by the Caputo fractional derivative (see Equations (20) and (30) in [18]). A similar situation occurs with the application of the derivative of Equation (9) in [20] for studying stability of impulsive fractional differential equations.

In the next example to simplify the calculations and to emphasize the derivatives and their properties, we consider the scalar case, i.e. $n = 1$.

Example 1. *(Lyapunov function depending directly on the time variable). Let* $V(t,x) = m(t)\ x^2$ *where* $m \in C^1(\mathbb{R}_+, \mathbb{R}_+)$.

Case 1. Caputo fractional derivative. Let x be a solution of NIFrDDE (Equation (1)). Then, the fractional derivative

$$ {}_{t_0}^{c}D^q V(t,x(t)) = {}_{t_0}^{c}D^q \Big(m(t)\ x^2(t) \Big) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{m'(s)x^2(s) + 2m(s)x(s)x'(s)}{(t-s)^q} ds $$

is difficult to obtain in the general case for any solution of Equation (1). In addition, the solution $x(t)$ might not be differentiable on the intervals of impulses.

Case 2. Dini fractional derivative. Let $\phi \in PC_0$ and $t \in (t_k, s_{k+1})$ for a non-negative integer k. Then, applying Equation (4), we obtain

$$ \begin{aligned} & D^{+}_{(1)} V(t,\phi(0),t_k,\phi) \\ & = \limsup_{h\to 0} \frac{1}{h^q} \Big[m(t)\ (\phi(0))^2 - \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r+1}\ {}_qC_r m(t-rh)(\phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t))))^2 \Big] \\ & = \limsup_{h\to 0} \frac{1}{h^q} \Big[m(t) \Big((\phi(0))^2 - (\phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t))))^2 \Big) \\ & \quad + (\phi(0) - h^q f(t,\phi(0),\phi_0))^2 \sum_{r=0}^{[\frac{t-t_k}{h}]} (-1)^r\ {}_qC_r m(t-rh) \Big] \\ & = \phi(0)\ m(t) f(t,\phi(0),\phi(\rho(t,\phi_0)-t))) + (\phi(0))^2\ {}_{t_k}^{RL}D^q \Big(m(t) \Big). \end{aligned} $$

Case 2. Caputo fractional Dini derivative. Let $\varphi,\ \phi \in PC_0$ and $t \in (t_k, s_{k+1})$ for a non-negative integer k. Then, we use Equation (6) and obtain

$$ \begin{aligned} & {}_{(1)}^{c}D^q_{+} V(t,\phi;t_k,\varphi(0)) \\ & = \limsup_{h\to 0^+} \frac{1}{h^q} \Big\{ \phi(0)^2 m(t) - \sum_{r=1}^{[\frac{t-t_k}{h}]} (-1)^{r+1}\ {}_qC_r m(t-rh)\ (\phi(0) - h^q f(t,\phi(0),\phi(\rho(t,\phi_0)-t))))^2 \Big\} \\ & \quad - (\varphi(0))^2 m(t_k) \frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\ & = 2\phi(0) m(t) f(t,\phi(0),\phi(\rho(t,\phi_0)-t))) + (\phi(0))^2\ {}_{t_k}^{RL}D^q \Big(m(t) \Big) - (\varphi(0))^2 m(t_k) \frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\ & = D^{+}_{(1)} V(t,\phi(0),t_k,\phi) - V(t_k,\varphi(0)) \frac{(t-t_k)^{-q}}{\Gamma(1-q)}. \end{aligned} $$

□

5. Comparison Results

Lemma 1. *[17]. Let* $v \in C([a,b],\mathbb{R})$ *be such that* $(t-a)^{1-q}v \in C([a,b],\mathbb{R})$ *and there exists a point* $\tau \in (a,b]$: $v(\tau) = 0$ *and* $v(t) \le 0$ *for* $t \in [a,\tau]$. *Then,* ${}_{a}^{C}D^q_{\tau} v(\tau) \ge 0$.

We use the following comparison scalar fractional differential equation with non-instantaneous impulses:

$$\begin{aligned} &{}^{C}_{t_i}D^q_t u(t) = g(t, u(t)) \ \text{ for } t \in (t_i, s_{i+1}], i = 0, 1, 2, \dots, \\ &u(t) = \psi_i(t, u(s_i - 0)), \ \ t \in (s_i, t_i], \ i = 1, 2, \dots, \\ &u(t_0) = u_0, \end{aligned} \tag{10}$$

where $u, u_0 \in \mathbb{R}, g : [0, s_1] \cup_{k=1}^{\infty} [t_k, s_k] \times \mathbb{R} \to \mathbb{R}, \psi_k : [s_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$ $(k = 1, 2, 3, \dots)$.

We obtain some comparison results. Note some comparison results for fractional time delay differential equations are obtained in [18] by applying the derivative defined by Equation (9) and substituting it incorrectly as a Caputo fractional derivative (see Remark 5).

We introduce the following conditions:

A6. The function $g(t, u) \in C([0, s_1] \bigcup_{k=1}^{\infty} [t_k, s_{k+1}] \times \mathbb{R}_+, \mathbb{R})$ is strictly decreasing with respect to its second argument, and for any $k = 1, 2, \dots$ the functions $\psi_k : [s_k, t_k] \times \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing with respect to their second argument.

A7. The function $g(t, 0) = 0$ for $t \in [0, s_1] \bigcup_{k=1}^{\infty} [t_k, s_{k+1}]$ and for any $k = 1, 2, \dots$ the function $\psi_k(t, 0) = 0$ for $t \in [s_k, t_k]$.

A8. For all $k = 1, 2, \dots,$ the functions ψ_k satisfies $\psi_k(t, u) \le u, \ t \in [s_k, t_k], \ u \in \mathbb{R}$.

In our main results, we use the Lipschitz stability of the zero solution of the scalar comparison non-instantaneous impulsive fractional differential in Equation (10).

Example 2. *Let* $t_k = 2k, k = 0, 1, \dots$ *and* $s_k = 2k - 1, \ k = 1, 2, \dots$*. Consider the scalar non-instantaneous impulsive fractional differential equation*

$$\begin{aligned} &{}^{C}_{t_i}D^{0.25}_t u(t) = u(t) \ \textit{ for } t \in (t_i, s_{i+1}], i = 0, 1, 2, \dots, \\ &u(t) = \psi_k(t, u(s_k - 0)), \ \ t \in (s_i, t_i], \ i = 1, 2, \dots, \\ &u(0) = u_0, \end{aligned} \tag{11}$$

where $u, u_0 \in \mathbb{R}$.

Case 1. Suppose for all natural numbers $k = 1, 2, \dots$ the equality $\psi_k(t, u) = \frac{u}{2t}, \ u \in \mathbb{R}, t \in [s_k, t_k]$ holds. Then, the solution of Equation (11) is given by

$$u(t) = \begin{cases} u_0 E_{0.25}(t^{0.25}), & t \in (0, 1], \\ \frac{u_0 (E_{0.25}(1))^k}{2t \prod_{i=1}^{k-1}(4i)}, & t \in (2k - 1, 2k], \ k = 1, 2, \dots, \\ \frac{u_0 (E_{0.25}(1))^k}{\prod_{i=1}^{k}(4i)} E_{0.25}((t - 2k)^{0.25}), & t \in (2k, 2k + 1], \ k = 1, 2, \dots. \end{cases} \tag{12}$$

The solution of Equation (11) is uniformly Lipschitz stable with $M_1 = 30$ (see Figure 1 for the graph of the solutions with various initial values).

Case 2. Suppose for all natural numbers $k = 1, 2, \dots$ the equality $\psi_k(t, u) = tu, \ u \in \mathbb{R}, t \in [s_k, t_k]$ holds. Then, the solution of Equation (11) is given by

$$u(t) = \begin{cases} u_0 E_{0.25}(t^{0.25}), & t \in (0, 1], \\ u_0 (E_{0.25}(1))^k \prod_{i=1}^{k-1}(2i) \ t, & t \in (2k - 1, 2k], \ k = 1, 2, \dots, \\ u_0 (E_{0.25}(1))^k \prod_{i=1}^{k}(2i) E_{0.25}((t - 2k)^{0.25}), & t \in (2k, 2k + 1], \ k = 1, 2, \dots. \end{cases} \tag{13}$$

The solution of Equation (11) is unbounded (see Figure 2 for the graph of the solution).

Therefore, for $\psi_k(t, u) = \frac{u}{2t} \le u$ the solution is Lipschitz stable but for $\psi_k(t, u) = \frac{u}{2t} \ge u$ it is not (compare with condition (A8)). □

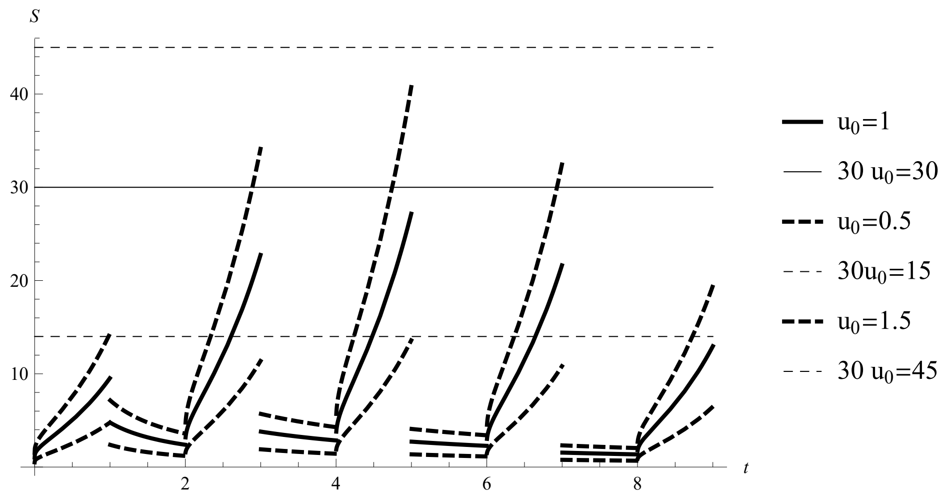

Figure 1. Example 2. Graph of the solution of Equation (11) with $\psi_k(t,u) = \frac{u}{2t}$ for various initial values.

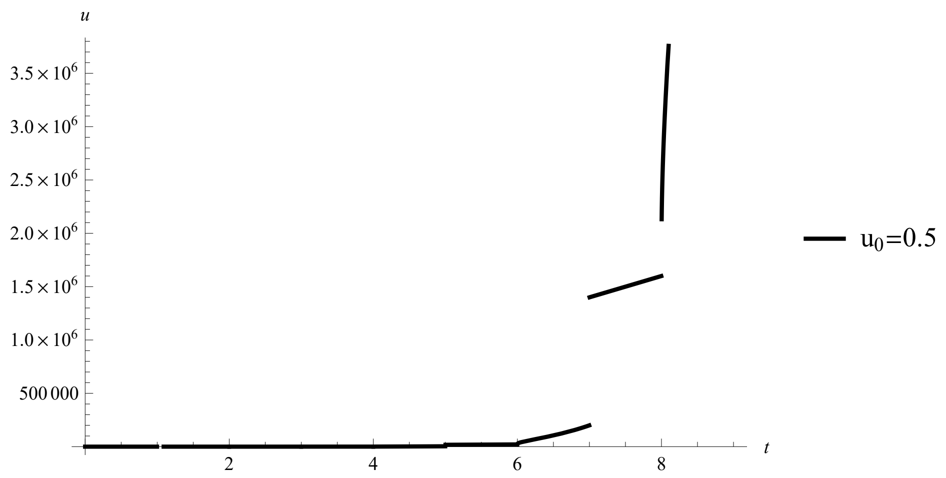

Figure 2. Example 2. Graph of the solution of Equation (11) with $\psi_k(t,u) = tu$.

In our study, we use some comparison results. When the Caputo fractional derivative is used, then the comparison result is:

Lemma 2. *(Caputo fractional derivative). Assume the following conditions are satisfied:*

1. *Assumptions A1–A4 and A6 are satisfied.*
2. *The function $x^*(t) = x(t;t_0,\varphi) : [t_0,T) \to \Delta, x^* \in PC^q([t_0,T))$ is a solution of Equation (1) where $\Delta \subset \mathbb{R}^n, 0 \in \Delta, T \leq \infty$.*
3. *The function $V \in \Lambda([t_0,T),\Delta)$ is such that*

   (i) *For any $i = 0,1,2,\cdots:\ (t_i,s_{i+1}) \cap [t_0,T) \neq \emptyset$ and for $t \in (t_i,s_{i+1}) \cap [t_0,T)$, the inequality*

$$ {}_{t_i}^{C}D_t^q V(t,x^*(t)) \leq g(t,V(t,x^*(t))) $$

   holds.

   (ii) *For all $i = 1,2,3,\cdots:\ (s_i,t_i) \cap [t_0,T) \neq \emptyset$ the inequality*

$$ V(t,\phi_i(t,x^*(s_i-0))) \leq \psi_i(t,V(s_i-0,x^*(s_i-0))) \quad for \ \ t \in (s_i,t_i] \cap [t_0,T) $$

   holds.

If $sup_{s\in[-r,0]}V(t_0,\varphi(s)) \leq u_0$, *then the inequality* $V(t,x^*(t)) \leq r(t)$ *for* $t \in [t_0,T)$ *holds, where* $r(t) = r(t;t_0,u_0)$ *is the maximal solution on* $[t_0,T)$ *of Equation (10) with* $u_0 \geq 0$.

Proof. We use induction with respect to the intervals to prove Lemma 2. Let $m(t) = V(t,x^*(t)), t \geq t_0$. We prove

$$m(t) \leq u(t), \quad t \geq t_0. \tag{14}$$

Let $t \in [t_0,s_1]$. Let $\varepsilon > 0$ be an arbitrary number. We prove

$$m(t) < u(t) + \varepsilon, \quad t \geq [t_0,s_1]. \tag{15}$$

Note $m(t_0) = V(t_0,\varphi(0)) \leq \sup_{s\in[-r,0]} V(t_0,\varphi(s)) \leq u_0$, i.e. the inequality in Equation (15) holds for $t = t_0$. If the inequality in Equation (15) is not true, then there exists a point $t^* \in (t_0,s_1]$ such that $m(t^*) = u(t^*) + \varepsilon, \ m(t) < u(t) + \varepsilon, t \in [t_0,t^*)$.

From Lemma 1 with $a = t_0$, $b = s_1$, $\tau = t^*$ and $v(t) = m(t) - u(t) - \varepsilon$ the inequality ${}_{t_0}^{C}D_t^q m(t^*) \geq {}_{t_0}^{C} D_t^q u(t^*) = g(t^*,u(t^*))$ holds.

From Assumption A6 and Condition 3(i), the inequality ${}_{t_0}^{C}D_t^q m(t^*) \leq g(t^*,m(t^*)) = g(t^*,u(t^*) + \varepsilon) < g(t^*,u(t^*))$ holds. The contradiction proves the validity of Equation (15). Since ε is an arbitrary positive number, we obtain the inequality in Equation (14) for $t \in [t_0,s_1]$.

Let $t \in (s_1,t_1]$. Then, from the impulsive equality in Equation (1), Condition 3(ii), Assumption A6 and the inequality in Equation (14) for $t = s_1 - 0$, we obtain $m(t) = V(t,x^*(t)) = V(t,\phi_1(t,x^*(s_1 - 0))) \leq \psi_1(t,V(s_1 - 0,x^*(s_1 - 0))) = \psi_1(t,m(s_1 - 0)) \leq \psi_1(t,u(s_1 - 0)) = u(t)$, i.e. Equation (14) holds on $(s_1,t_1]$.

Let $t \in (t_1,s_2]$. Let $\varepsilon > 0$ be an arbitrary number. We prove Equation (15) for $t \in [t_1,s_2]$. Note that Equation (15) is true for $t = t_1$. If the inequality in Equation (15) is not true, then there exists a point $t^* \in (t_1,s_2]$ such that $m(t^*) = u(t^*) + \varepsilon, \ m(t) < u(t) + \varepsilon, t \in [t_1,t^*)$.

From Lemma 1 with $a = t_1$, $b = s_2$, $\tau = t^*$ and $v(t) = m(t) - u(t) - \varepsilon$, the inequality ${}_{t_1}^{C}D_t^q m(t^*) \geq {}_{t_1}^{C} D_t^q u(t^*) = g(t^*,u(t^*))$ holds.

From Assumption A6 and Condition 3(i), the inequality ${}_{t_1}^{C}D_t^q m(t^*) \leq g(t^*,m(t^*)) = g(t^*,u(t^*) + \varepsilon) < g(t^*,u(t^*))$ holds. The contradiction proves the validity of Equation (15) and the inequality in Equation (14) for $t \in (t_1,s_2]$. Continuing this process and an induction argument prove Equation (14) and Lemma 2. □

Lemma 3. *[10] Let* $m \in C([t_0,T],\mathbb{R})$ *and there exists* $\tau \in (t_0,T]$, *such that* $m(\tau) = 0$ *and* $m(t) < 0$ *for* $t \in [t_0,\tau)$. *Then, the inequality* ${}_{t_0}^{GL}D_+^q m(\tau) > 0$ *holds.*

When the Dini fractional derivative defined by Equation (4) or Caputo fractional Dini derivative defined by Equation (5) is used then the comparison result is:

Lemma 4. *(Dini fractional derivative/Caputo fractional Dini derivative). Assume:*

1. *Assumptions A1–A4 and A6 are satisfied.*
2. *The function* $x^*(t) = x(t;t_0,\varphi) : [t_0,T) \to \Delta, x^* \in PC^q([t_0,T))$ *is a solution of Equation (1) where* $\Delta \subset \mathbb{R}^n, 0 \in \Delta, T \leq \infty$.
3. *The function* $V \in \Lambda([t_0,T),\Delta)$ *is such that*

   (i) *For any* $i = 0,1,2,\cdots: \ (t_i,s_{i+1}) \cap [t_0,T) \neq \emptyset$ *and for* $t \in (t_i,s_{i+1}) \cap [t_0,T)$, *the inequality*

$$\mathcal{D}_{(1)}V(t,\phi,t_i) \leq g(t,V(t,\phi(0))$$

holds where $\phi(\Theta) = x^*(t+\Theta)$, $\Theta \in [-r,0]$, and $\mathcal{D}_{(1)}V(t,\phi,t_i)$ is one of the following two derivatives: the Dini fractional derivative $D^+_{(1)}V(t,\phi(0),t_i,\phi)$ defined by Equation (4) or the Caputo fractional Dini derivative ${}^{c}_{(1)}D^q_+V(t,\phi;t_i,\varphi(0))$ defined by Equation (5).

(ii) For all $i = 1,2,3,\cdots:\ (s_i,t_i)\cap[t_0,T) \neq \emptyset$, the inequality

$$V(t,\phi_i(t,x^*(s_i-0))) \leq \psi_i(t,V(s_i-0,x^*(s_i-0))) \quad for \ \ t \in (s_i,t_i]\cap[t_0,T)$$

holds.

If $sup_{s\in[-r,0]}V(t_0,\varphi(s)) \leq u_0$, then the inequality $V(t,x^*(t)) \leq r(t)$ for $t \in [t_0,T)$ holds, where $r(t) = r(t;t_0,u_0)$ is the maximal solution on $[t_0,T)$ of Equation (10) with $u_0 \geq 0$.

Proof. The proof is similar to the one in Lemma 2 where instead of the Caputo fractional derivative of the Lyapunov function, we use the Dini fractional derivative or the Caputo fractional Dini derivative which are less restrictive with respect to the properties of Lyapunov functions (for example, differentiability is not required). We sketch the proof emphasizing the differences with Lemma 2.

Case 1. Let $\mathcal{D}_{(1)}V(t,\phi,t_i) = {}^{c}_{(1)}D^q_+V(t,\phi;t_i,\varphi(0)), i = 1,2,\ldots$ in Condition 3(i) of Lemma 4.

We use induction with respect to the intervals to prove Lemma 4. We prove the inequality in Equation (14).

Case 1.1. Let $t \in [t_0,s_1]$. We prove Equation (15) with $\varepsilon > 0$ an arbitrary number. Note that Equation (15) holds for $t = t_0$. If the inequality in Equation (15) is not true, then there exists a point $t^* \in (t_0,s_1]$ such that $p(t^*) = 0$ and $p(t^*) < 0$ for $t \in [t_0,t^*)$ where $p(t) = m(t) - u(t) - \varepsilon$. From Lemma 3 with $\tau = t^*$ we get the inequality

$${}^{GL}_{t_0}D^q_+m(t^*) >^{GL}_{t_0} D^q_+u(t^*) +^{GL}_{t_0} D^q_+\varepsilon.$$

Thus

$$\begin{aligned} {}^{C}_{t_0}D^q_t m(t^*) &=^{GL}_{t_0} D^q_+(m(t^*) - m(t_0)) =^{GL}_{t_0} D^q_+m(t^*) - {}^{GL}_{t_0}D^q_+m(t_0) \\ &> {}^{GL}_{t_0}D^q_+(u(t^*) - u_0) =^{C}_{t_0} D^q_t u(t^*). \end{aligned} \tag{16}$$

Following the proof of Lemma 3 [3] from the choice of the point t^*, the definition of the function $m(t)$, the definition of the derivative ${}^{c}_{(1)}D^q_+V(\tau,\phi(0);t_0,\varphi(0))$, Assumption A2 and $x(t+s) = \phi(s)$, $x_{\rho(t,x_t)} = x_{\rho(t,\phi_0)} = x(\rho(t,\phi_0)) = x(t+(\rho(t,\phi_0)-t)) = \phi(\rho(t,\phi_0)-t)$, Assumption A6 and Condition 3(i) of Lemma 4, we obtain the inequality

$$\begin{aligned} {}^{C}_{t_0}D^q_t m(\tau) = {}^{GL}_{t_0}D^q_+(m(t^*) - m(t_0)) &\leq {}^{c}_{(1)}D^q_+V(t^*,\phi(0);t_0,\varphi(0)) \\ &\leq g(t^*,V(t^*,\phi(0)) = g(t^*,m(t^*)) = g(t^*,u(t^*)+\varepsilon) \leq g(t^*,u(t^*)) \\ &=^{C}_{t_0} D^q_t u(t^*) \end{aligned} \tag{17}$$

with $\phi(\Theta) = x(\tau+\Theta)$, $\Theta \in [-\tau,0]$.

The inequality in Equation (17) contradicts the inequality in Equation (16). The contradiction proves the validity of Equation (15) and, therefore, the validity of Equation (14) on $[t_0,s_1]$.

Case 1.2. Let $t \in (s_1,t_1]$. From the impulsive equality in Equation (1), Condition 3(ii) of Lemma 4, Assumption A6 and the inequality in Equation (14) for $t = s_1 - 0$, we obtain for $t \in (s_1,t_1]$ the inequalities $m(t) = V(t,x^*(t)) = V(t,\phi_1(t,x^*(s_1-0))) \leq \psi_1(t,V(s_1-0,x^*(s_1-0))) = \psi_1(t,m(s_1-0)) \leq \psi_1(t,u(s_1-0)) = u(t)$, i.e. Equation (14) holds on (s_1,t_1).

Case 1.3. Let $t \in (t_1,s_2]$. The proof of the inequality in Equation (15) for $t \geq (t_1,s_2]$ is similar to the one in Case 1.1 by replacing t_0 with t_1.

Case 2. Let $\mathcal{D}_{(1)}V(t,\phi,t_i)$ in Condition 3(i) of Lemma 4 be the Dini fractional derivative $D^{+}_{(1)}V(t,\phi(0),t_i,\varphi)$ defined by Equation (4). Then, based on the proof in Case 1 and Remark 4, we establish Lemma 4. □

6. Main Results

Theorem 1. (Caputo fractional derivative) *Let the following conditions be satisfied:*

1. *Assumptions A1–A8 are fulfilled.*
2. *There exist a function $V \in \Lambda(\mathbb{R}_+,\mathbb{R}^n)$ and*
   - *(i) The inequalities*
   $$b(||x||) \leq V(t,x) \leq a(||x||), \ \ x \in \mathbb{R}^n, t \in \mathbb{R}_+$$
   holds, where $a \in K([0,\rho])$, $b \in M([0,\rho])$, $\rho > 0$;
   - *(ii) For any initial data and any solution $x(t)$ of Equation (1) defined on $[t_0,\infty)$ such that for any $\tau \in (t_k,s_{k+1})$, k is a non-negative integer, such that $x(t) \in S_\rho$, $t \in [t_0,\tau]$ and $V(\tau,x(\tau)) \geq V(s,x(s))$ for $s \in [t_0,\tau]$ the inequality*
   $${}_{t_i}^{C}D_t^q V(t,x(t)) \leq g(t,V(t,x(t))), \ \ t \in (t_i,s_{i+1}] \cap [t_0,\tau], i = 0,1,2,\ldots,k$$
   holds.
   - *(iii) For any $k = 0,1,2,\ldots$ and $t \in (s_k,t_{k+1}]$, $y \in S_\rho$ the inequality*
   $$V(t,\phi_k(t,y)) \leq \psi_k(t,V(s_k - 0,y))$$
   holds.
3. *The zero solution of Equation (10) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).*

Then, the zero solution of Equation (1) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).

Proof. Let the zero solution of Equation (10) be uniformly Lipschitz stable. Let $t_0 \geq 0$ be an arbitrary. Without loss of generality, we assume $t_0 \in [0,s_1)$. From Condition 3, there exist $M \geq 1$, $\delta_1 > 0$ such that for any $u_0 \in \mathbb{R}: \ |u_0| < \delta_1$ the inequality

$$|u(t;t_0,u_0)| \ \leq M\,|u_0| \text{ for } t \geq t_0 \tag{18}$$

holds, where $u(t;t_0,u_0)$ is a solution of Equation (10) with the initial data (t_0,u_0).

From the inclusions $a \in K([0,\rho])$ and $b \in M([0,\rho])$, there exist a function $q_b(u)$ and a positive constant K_a. Without loss of generality, we can assume $K_a \geq 1$. Choose the constant M_1 such that $M_1 > \max\{1,q_b(K_a),q_b(M)K_a\}$ and $\delta_2 \leq \frac{\rho}{2M_1}$. Therefore, $2M_1\delta_2 \leq \rho$.

Let $\delta = \min\left\{\delta_1,\delta_2,\frac{\delta_1}{K_a}\right\}$. Choose the initial function $\varphi \in PC_0([-r,0])$ such that $||\varphi||_{PC_0} < \delta$. Therefore, $||\varphi||_{PC_0} < \delta \leq \delta_2 \leq \rho$, i.e. $\varphi(s) \in S_\rho$ for $s \in [-r,0]$. Consider the solution $y(t) = y(t;t_0,\varphi)$ of the system in Equation (1) for the chosen initial data (t_0,φ).

Let $u_0^* = \sup_{s\in[-r,0]} V(t_0,\varphi(s))$. From the choice of φ and the properties of the function $a(u)$ applying condition 2(i) we get $u_0^* = V(t_0,\varphi(\xi)) \leq a(||\varphi(\xi)||) \leq a(||\varphi||_{PC_0}) \leq K_a||\varphi||_{PC_0} < K_a\delta \leq \delta_1$. Therefore, the function $u^*(t)$ satisfies Equation (18) for $t \geq t_0$ with $u_0 = u_0^*$, where $u^*(t) = u(t;t_0,u_0^*)$ is a solution of Equation (10) with initial data (t_0,u_0^*).

Let $\varepsilon \in (0,M_1\delta]$ be an arbitrary number. We prove

$$V(t,y(t)) < b(M_1||\varphi||_{PC_0} + \varepsilon), \quad t \geq t_0. \tag{19}$$

For $t = t_0$, we get $V(t_0,y(t_0)) = V(t_0,\varphi(0)) \leq a(||\varphi(0)||) \leq a(||\varphi||_{PC_0} \leq K_a||\varphi||_{PC_0} \leq b(q_b(K_a)||\varphi||_{PC_0}) \leq b(M_1||\varphi||_{PC_0}) < b(M_1||\varphi||_{PC_0} + \varepsilon)$, i.e. the inequality in Equation (19) holds.

Assume Equation (19) is not true.

Case 1. There exists a point $T > t_0$, $T \in \bigcup_{k=0}^{\infty}(t_k, s_{k+1}]$ such that $V(t, y(t)) < b(M_1||\varphi||_{PC_0} + \varepsilon)$ for $t \in [t_0, T)$, $V(T, y(T)) = b(M_1||\varphi||_{PC_0} + \varepsilon)$, i.e. $V(s, y(s)) \le V(T, y(T))$ for $s \in [t_0, T]$. Then, from Condition 2(i), we obtain the inequalities $||y(t)|| \le b^{-1}(V(t, y(t)) \le M_1||\varphi||_{PC_0} + \varepsilon < 2M_1\delta \le 2M_1\delta_2 \le \rho$ for $t \in [t_0, T]$, i.e., $y(t) \in S_\rho$ for $t \in [t_0, T]$ and, according to Condition 2(ii) of Theorem 1 with $\tau = T$, it follows that Condition 3(i) of Lemma 2 is satisfied for the solution $y(t)$ on the interval $[t_0, T]$ and $\Delta = S_\rho$.

According to Lemma 2, we get

$$V(t,\ y(t)) \le u^*(t) \text{ for } t \in [t_0, T]. \tag{20}$$

From the inequality in Equation (20) and Condition 2(i), we obtain

$$\begin{aligned} M_1||\varphi||_{PC_0} &= b^{-1}(V(T, y(T))) \le b^{-1}(u^*(T)) \\ &\le b^{-1}(M\,|u_0^*|) = b^{-1}(MV(t_0, \varphi(\xi))) \le q_b(M)V(t_0, \varphi(\xi)) \\ &\le q_b(M)a(||\varphi(\xi)||) \le q_b(M)a(||\varphi||_0) \le q_b(M)K_a||\varphi)||_{PC_0} < M_1||\varphi||_{PC_0}. \end{aligned} \tag{21}$$

The contradiction proves the validity of Equation (19). From the inequality in Equation (19) and Condition 2(i), we have Theorem 1.

Case 2. There exists a point $T > t_0$, $T \in \bigcup_{k=1}^{\infty}(s_k, t_k)$ such that $V(t, y(t)) < b(M_1||\varphi||_{PC_0} + \varepsilon)$ for $t \in [t_0, T)$, $V(T, y(T)) = b(M_1||\varphi||_{PC_0} + \varepsilon)$. Then, as in Case 1 we get $y(t) \in S_\rho$ for $t \in [t_0, T]$. Let $T \in (s_j, t_{j+1})$ for a natural number j. According to Condition 2(iii) of Theorem 1, we obtain $b(M_1||\varphi||_{PC_0} + \varepsilon) = V(T, y(T)) = V(T, \phi_j(T, y(s_j - 0))) \le \psi_j(T, V(s_j - 0, y(s_j - 0))) \le \psi_j(T, b(M_1||\varphi||_{PC_0})) < \psi_j(T, b(M_1||\varphi||_{PC_0}) + \varepsilon)$. The contradiction proves this case is not possible.

Case 3. There exists a natural number k such that $V(t, y(t)) < b(M_1||\varphi||_0 + \varepsilon)$ for $t \in [t_0, s_k]$ and $V(s_k + 0, y(s_k + 0)) > b(M_1||\varphi||_0 + \varepsilon)$. Therefore, $\psi_k(s_k, b(M_1||\varphi||_0)) \ge \psi_k(s_k, V(s_k, y(s_k))) \ge V(s_k + 0, \phi_k(s_k, y(s_k - 0))) = V(s_k + 0, y(s_k + 0)) > b(M_1||\varphi||_0)$. The contradiction proves this case is not possible.

The proof of globally uniformly Lipschitz stability is analogous so we omit it. □

Theorem 2. *Let the conditions of Theorem 1 be satisfied where Condition 2(i) is replaced by:*

$2^*(i)$ *the inequalities* $\lambda_1(t)||x||^2 \le V \le \lambda_2(t)||x||^2$, $x \in S_\rho, t \in \mathbb{R}^+$ *holds, where* $\lambda_1, \lambda_2 \in C(\mathbb{R}_+, (0, \infty))$ *and there exists positive constant* $A_1, A_2 :\ A_1 < A_2$ *such that* $\lambda_1(t) \ge A_1$, $\lambda_2(t) \le A_2$ *for* $t \ge 0$, *and* $\rho > 0$.

If the zero solution of Equation (10) is uniformly Lipschitz stable (uniformly globally Lipschitz stable), then the zero solution of Equation (1) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).

Proof. The proof is similar to the one in Theorem 1 where $M_1 = \sqrt{M\frac{A_2}{A_1}}$.

Theorem 3. (Dini fractional derivative/ Caputo fractional Dini derivative) *Let the following conditions be satisfied:*

1. *Assumptions A1–A8 are fulfilled.*
2. *There exist a function* $V(t, x) \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$, $\rho > 0$ *and*

   (i) *The inequalities*
   $$b(||x||) \le V(t, x) \le a(||x||),\ \ x \in \mathbb{R}^n, t \in \mathbb{R}_+$$
   holds, where $a \in K([0, \rho])$, $b \in M([0, \rho])$.

(ii) *For any function $\phi \in PC_0 : \ \phi(s) \in S_\rho$ for $s \in [-r, 0]$ such that for any $t : \ t \in (t_k, s_{k+1})$, k is a non-negative integer, such that $V(t+s, \phi(s)) \leq V(t, \phi(0)), \ s \in [-r, 0]$ the inequality*

$$\mathcal{D}_{(1)} V(t, \phi, t_k) \leq g(t, V(t, \phi(0)))$$

holds where $\mathcal{D}_{(1)} V(t, \phi, t_k)$ is one of the following two derivatives: the Dini fractional derivative $D^+_{(1)} V(t, \phi(0), t_k, \phi)$ defined by Equation (4) or the Caputo fractional Dini derivative ${}^c_{(1)}D^q_+ V(t, \phi; t_k, \varphi(0))$ defined by Equation (5) and $\rho > 0$.

(iii) *For any $k = 0, 1, 2, \ldots$ and $t \in (s_k, t_{k+1}], \ \ y \in S_\rho$ the inequality*

$$V(t, \phi_k(t, y)) \leq \psi_k(t, V(s_k - 0, y))$$

holds.

3. *The zero solution of Equation (10) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).*

Then, the zero solution of Equation (1) is uniformly Lipschitz stable (uniformly globally Lipschitz stable).

The proof of Theorem 3 is similar to the one in Theorem 1 where Lemma 4 is applied instead of Lemma 2.

Example 3. *Let $t_k = 2k, k = 0, 1, 2\ldots$ and $s_k = 2k - 1, \ k = 1, 2, \ldots.$ Consider the non-instantaneous impulsive fractional differential equations*

$$\begin{aligned}
&{}^C_{t_i}D^{0.25}_t x_1(t) = 0.25 x_1(t) - x_2(t) + 0.25 x_1(t) (x_{\rho(t,x_t)})_2^2, \\
&{}^C_{t_i}D^{0.25}_t x_2(t) = 0.25 x_2(t) + x_1(t) + 0.25 x_2(t) (x_{\rho(t,x_t)})_1^2 \\
&\qquad \text{for } t \in (t_i, s_{i+1}], i = 0, 1, 2, \ldots, \\
&x_1(t) = \frac{x_1(s_i - 0)}{\sqrt{2it}}, \ \ x_2(t) = \frac{x_2(s_i - 0)}{\sqrt{2it}}, \quad t \in (s_i, t_i], \ i = 1, 2, \ldots,
\end{aligned} \tag{22}$$

where $x = (x_1, x_2)$, $\rho(t, u) = t - \sin^2(u) : t - 1 \leq \rho(t, u) \leq t$, $x_{\rho(t,x_t)} = ((x_{\rho(t,x_t)})_1, (x_{\rho(t,x_t)})_2)$ and $(x_{\rho(t,x_t)})_i = x_i(t - \sin^2(x_i(t+s))), \ \ s \in [-1, 0], i = 1, 2.$

Let $V(t, x) = x_1^2 + x_2^2$, $x = (x_1, x_2)$.

Let $x(t)$ be a solution of Equation (22). Let the point $\tau \in (t_k, s_{k+1}]$, k is a non-negative integer, be such that $x(t) \in S_1, \ t \in [0, \tau]$ and $x_1(\tau)^2 + x_2(\tau)^2 \geq x_1(s)^2 + x_2(s)^2, \ s \in [0, \tau]$. Using the notation $x_{\rho(\tau,x_\tau)}$ and Assumption A2, it follows that $\rho(\tau, x_j(\tau + \Theta)) \in [\tau - r, \tau], j = 1, 2, \ \Theta \in [-r, 0]$ and therefore $(x_{\rho(\tau,x_\tau)})_1^2 + (x_{\rho(\tau,x_\tau)})_2^2 \leq x_1(\tau)^2 + x_2(\tau)^2$ or

$$x_1^2(t)(x_{\rho(t,x_t)})_2^2 \leq x_1^2(t)(x_{\rho(t,x_t)})_1^2 + x_1^2(t)(x_{\rho(t,x_t)})_2^2 \leq x_1^2(t)(x_1(\tau)^2 + x_2(\tau)^2) \leq x_1(t)^2$$

and

$$x_2^2(t)(x_{\rho(t,x_t)})_1^2 \leq x_2^2(t)(x_{\rho(t,x_t)})_1^2 + x_2^2(t)(x_{\rho(t,x_t)})_2^2 \leq x_2^2(t)(x_1(\tau)^2 + x_2(\tau)^2) \leq x_2(t)^2.$$

Then, for all $i = 0, 1, 2, \dots, k]$ and $t \in (t_i, s_{i+1}] \cap [0, \tau]$, we get the inequality

$$\begin{aligned} {}_{t_i}^{C}D_t^{0.25}V(t,x(t)) &= {}_{t_i}^{C}D_t^{0.25}x_1^2(t) + {}_{t_i}^{C}D_t^{0.25}x_2^2(t) \\ &\leq 2x_1(t)\,{}_{t_i}^{C}D_t^{0.25}x_1(t) + 2x_2(t)\,{}_{t_i}^{C}D_t^{0.25}x_2(t) \\ &= 2x_1(t)\Big(0.25x_1(t) - x_2(t) + 0.25x_1(t)(x_{\rho(t,x_t)})_2^2\Big) \\ &\quad + 2x_2(t)\Big(0.25x_2(t) + x_1(t) + 0.25x_2(t)(x_{\rho(t,x_t)})_1^2\Big) \\ &\leq V(t,x(t)). \end{aligned} \tag{23}$$

In addition, for any natural number i, $x \in S_1 \subset \mathbb{R}^2$ and $t \in [s_i, t_i] = [2i-1, 2i$, we get $V(t, \frac{x}{\sqrt{2it}}) = \left(\frac{x_1}{\sqrt{2it}}\right)^2 + \left(\frac{x_2}{\sqrt{2it}}\right)^2 = \frac{1}{2t}\left(\frac{x_1^2}{i} + \frac{x_2^2}{i}\right) \leq \frac{x_1^2+x_2^2}{2t} = \frac{V(s_i,x)}{2t} = \psi_i(t, V(s_i,x))$ with $\psi_i(t,u) = \frac{u}{2t}$.

According to Example 2, Case 1 and Theorem 1, the zero solution of Equation (22) is uniformly Lipschitz stable. □

Example 4. *Let $t_k = 2k, k = 0, 1, 2\dots$ and $s_k = 2k-1,\ k = 1, 2, \dots$. Consider the non-instantaneous impulsive fractional differential equations*

$$\begin{aligned} &{}_{t_i}^{C}D_t^{0.25}x_1(t) = 0.5x_1(t) - x_2(t) + 0.5x_1(t)(x_{\rho(t,x_t)})_2^2 - x_1(t)\frac{{}_{t_i}^{RL}D^q(\cos(0.5\pi(t+t_i+1)) + 1.1)}{\cos(0.5\pi(t+t_i+1)) + 1.1}, \\ &{}_{t_i}^{C}D_t^{0.25}x_2(t) = 0.5x_2(t) + x_1(t) + 0.5x_2(t)(x_{\rho(t,x_t)})_1^2 - x_2(t)\frac{{}_{t_i}^{RL}D^q(\cos(0.5\pi(t+t_i+1)) + 1.1)}{\cos(0.5\pi(t+t_i+1)) + 1.1} \\ &\qquad \text{for } t \in (t_i, s_{i+1}], i = 0, 1, 2, \dots, \\ &x_1(t) = \frac{x_1(s_i - 0)}{\sqrt{2it}},\ \ x_2(t) = \frac{x_2(s_i - 0)}{\sqrt{2it}},\quad t \in (s_i, t_i],\ i = 1, 2, \dots, \end{aligned} \tag{24}$$

where $x = (x_1, x_2)$, $\rho(t,u) = t - \sin^2(u)$, $t - 0.5 \leq \rho(t,u) \leq t$, $x_{\rho(t,x_t)} = ((x_{\rho(t,x_t)})_1, (x_{\rho(t,x_t)})_2)$ and $(x_{\rho(t,x_t)})_i = x_i(t - 0.5\sin^2(x_i(t+s)))$, $s \in [-0.5, 0], i = 1, 2$, $p(t) = \cos(0.5\pi(t + t_k + 1)) + 1.1$ for $t \in [t_k, s_{k+1}]$.

Note that, for any $t \in [t_k, s_{k+1}]$, the inequality $p(t) \leq p(t+s), s \in [-0.5, 0]$ holds.

In this case, the quadratic function and Theorem 1 does not work (as it did in Example 3) because ${}_{t_i}^{C}D_t^{0.25}V(t,x(t)) \leq 2V(t,x(t))(1 - \frac{{}_{t_k}^{RL}D^q p(t)}{p(t))}) \leq 2V(t,x(t))(1 - \frac{10}{11}\,{}_{t_k}^{RL}D^q(p(t))$ and the solution of the comparison Equation (10) with $g(t,u) = 2u(1 - \frac{10}{21}\,{}_{t_k}^{RL}D^q(\cos(0.5\pi(t+t_k+1)) + 1.1)$ is difficult to obtain.

Consider the Lyapunov function $V(t,x) = p(t)(x_1^2 + x_2^2)$, $x = (x_1, x_2)$.

Let the function $\phi \in PC_0$, $r = 0.5$ be such that $\phi(s) \in S_1$ for $s \in [-0.5, 0]$. Let $t: \ t \in (t_k, s_{k+1})$, k is a non-negative integer, be such that $p(t+s)(\phi_1(s)^2 + \phi_2(s)^2) \leq p(t)(\phi_1(0)^2 + \phi_2(0)^2)$, $s \in [-1, 0]$. From the definition of the function ρ, it follows that $\rho(t, \phi_j(s)) - t = -0.5\sin^2(\phi_j(s)) \in [-0.5, 0]$ for $s \in [-1, 0]$, $j = 1, 2$ and therefore $p(t+s)((\phi_1(\rho(t,\phi_1(s)) - t))^2 + (\phi_2(\rho(t,\phi_2(s)) - t))^2) \leq p(t)(\phi_1(0)^2 + \phi_2(0)^2)$, $s \in [-0.5, 0]$. Then

$$\begin{aligned} p(t)(\phi_2(\rho(t,\phi_0) - t))^2 &\leq p(t+s)(\phi_2(\rho(t,\phi_2(s)) - t))^2 \\ &\leq p(t+s)\Big((\phi_1(\rho(t,\phi_1(s)) - t))^2 + (\phi_2(\rho(t,\phi_2(s)) - t))^2\Big) \\ &\leq p(t)\Big(\phi_1(0)^2 + \phi_2(0)^2\Big), \quad s \in [-0.5, 0]. \end{aligned} \tag{25}$$

Similarly, we get $p(t)(\phi_1(\rho(t,\phi_0) - t))^2 \leq p(t)\Big(\phi_1(0)^2 + \phi_2(0)^2\Big)$.

Then, using Example 1, Case 2 and the notations $x_t = \phi_0$, $x_{\rho(t,x_t)} = \phi(\rho(t,\phi_0)-t)$, i.e., $f(t,x(t),x_{\rho(t,x_t)}) = f(t,\phi(0),\phi(\rho(t,\phi_0)-t))$, we get the inequality

$$\begin{aligned}
D^+_{(24)}V(t,\phi(0),t_k,\phi) &= \phi_1(0)p(t)f_1(t,\phi(0),\phi(\rho(t,\phi_0)-t)) + \phi_2(0)p(t)f_2(t,\phi(0),\phi(\rho(t,\phi_0)-t)) \\
&\quad + \Big(\phi_1^2(0)+\phi_2^2(0)\Big)\, {}^{RL}_{t_k}D^q p(t) \\
&= \phi_1(0)p(t)\Big(0.5\phi_1(0) - \phi_2(0) + 0.5\phi_1(0)(\phi_2(\rho(t,\phi_0)-t))^2 - \phi_1(0)\frac{{}^{RL}_{t_k}D^q p(t)}{p(t)}\Big) \\
&\quad + \phi_2(0)p(t)\Big(0.5\phi_2(0) + \phi_1(0) + 0.5\phi_2(0)(\phi_1(\rho(t,\phi_0)-t))^2 - \phi_2(0)\frac{{}^{RL}_{t_k}D^q p(t)}{p(t)}\Big) \\
&\quad + \Big(\phi_1^2(0)+\phi_2^2(0)\Big)\, {}^{RL}_{t_k}D^q p(t) \\
&\le V(t,\phi(0)).
\end{aligned}$$

In addition, for any natural number i, $x \in S_1 \subset \mathbb{R}^2$ and $t \in [s_i,t_i] = [2i-1, 2i$, we get $V(t, \frac{x}{\sqrt{2it}}) = \Big(\frac{x_1}{\sqrt{2it}}\Big)^2 + \Big(\frac{x_2}{\sqrt{2it}}\Big)^2 = \frac{1}{2t}\Big(\frac{x_1^2}{i} + \frac{x_2^2}{i}\Big) \le \frac{x_1^2+x_2^2}{2t} = \frac{V(s_i,x)}{2t} = \psi_i(t, V(s_i,x))$ with $\psi_i(t,u) = \frac{u}{2t}$.

According to Example 2, Case 1 and Theorem 3, the zero solution of Equation (24) is uniformly Lipschitz stable.

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