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On the Polynomial Solution of Divided-Difference Equations of the Hypergeometric Type on Nonuniform Lattices

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Abstract: In this paper, we provide a formal proof of the existence of a polynomial solution of fixed degree for a second-order divided-difference equation of the hypergeometric type on non-uniform lattices, generalizing therefore previous work proving existence of the polynomial solution for second-order differential, difference or q-difference equation of hypergeometric type. This is achieved by studying the properties of the mean operator and the divided-difference operator as well as by defining explicitly, the right and the "left" inverse for the second operator. The method constructed to provide this formal proof is likely to play an important role in the characterization of orthogonal polynomials on non-uniform lattices and might also be used to provide hypergeometric representation (when it does exist) of the second solution—non polynomial solution—of a second-order divided-difference equation of hypergeometric type.

Keywords: second-order differential/difference/q-difference equation of hypergeometric type; non-uniform lattices; divided-difference equations; polynomial solution

1. Introduction

Classical orthogonal polynomials of a continuous variable (P_n) are known to satisfy a second-order differential equation of hypergeometric type

$$\sigma(x)\,y''(x)+\tau(x)\,y'(x)+\lambda\,y(x)=0, \tag{1}$$

where σ is a polynomial of degree at most 2, τ is a first degree polynomial and λ is a constant with respect to x.

In [1,2], it is shown that Equation (1) has a polynomial solution of exactly n degree for a specific given constant $\lambda=\lambda_n$. This is achieved mainly by showing that :

- the nth derivative $y^{(n)}$ of any solution y of (1) satisfies an equation of the same type (hypergeometric aspect), that is, an equation of the form

$$\sigma(x)\,y''(x)+\tau_n(x)\,y'(x)+\lambda_n\,y(x)=0, \tag{2}$$

where τ_n is a first degree polynomial and λ_n is a constant given by

$$\tau_n(x) = \tau + n\,\sigma'(x),\ \lambda_n = \lambda + n\,\tau' + \frac{n(n-1)}{2}\,\sigma''. \tag{3}$$

- Any solution of (2) can be written as the nth derivative of a solution of (1), provided that $\lambda_j \neq 0,\ j = 1...n-1$.

The fact that the constant solution of (2) when $\lambda_n = 0$ is the nth derivative of a solution of (1) leads to the existence of a polynomial solution of (1), of exactly n degree, when

$$\lambda = \lambda_n = -n\,\tau' - \frac{n(n-1)}{2}\,\sigma''.$$

This result proves not only the existence of a polynomial solution for Equation (1) but also allows for establishing the Rodrigues formula expressing the polynomial solution in the term of the nth derivative:

$$P_n(x) = \frac{B_n}{\rho(x)}\,[\sigma^n(x)\,\rho(x)]^{(n)}\,,$$

where B_n is a constant and ρ is the weight function satisfying the Pearson equation

$$(\sigma(x)\,\rho(x))' = \tau(x)\,\rho(x).$$

It is worth mentioning that Hermite, Laguerre, Jacobi and Bessel polynomials are the polynomial eigenfunctions of the second-order linear differential operation given in (1).

Using the same approach, similar results have been established in [3] (See also [4]) for the classical orthogonal polynomials of a discrete variable satisfying instead a second-order difference equation of hypergeometric type

$$\sigma(x)\,\Delta\nabla y(x) + \tau(x)\,\Delta y(x) + \lambda_n\,y(x) = 0, \tag{4}$$

where Δ and ∇ are the forward and the backward operators defined by

$$\Delta f(s) = f(s+1) - f(s),\ \nabla f(s) = f(s) - f(s-1).$$

Furthermore, it should be noticed that Charlier, Krawtchuk, Meixner and Hahn polynomials are the polynomial eigenfunctions of the second-order linear difference operation given in (4).

The same result can be established in the same way to for the classical orthogonal polynomials of a q-discrete variable satisfying a second-order q-difference equation of hypergeometric type [5] (See also [6,7])

$$\sigma(x)\,D_q^2 y(x) + \tau(x)\,D_q y(x) + \lambda_n\,y(x) = 0, \tag{5}$$

where D_q is the Hahn operator [8] defined by

$$D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)\,x},\ x \neq 0,\ D_q f(0) := f'(0),$$

provided that $f'(0)$ exists. Orthogonal polynomials which are eigenfunctions of the second-order q-difference operator given defined by (5) are [5]: Big q-Jacobi, Big q-Laguerre, Little q-Jacobi, Little q-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam–Carlitz I, Al-Salam–Carlitz II, Stieltjes–Wigert, Discrete q–Hermite I, Discrete q–Hermite II, q-Hahn, q-Meixner, Quantum q-Krawtchouk, q-Krawtchouk, Affine q-Krawtchouk, the q-Charlier and the q-Charlier II polynomials.

Classical orthogonal polynomials on non-uniform lattices (including but not limited to Askey–Wilson polynomials, Racah and q-Racah polynomials), are known to satisfy a second-order divided-difference equation of the form [9,10] (see also [11])

$$\phi(x(s))\,\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\frac{\nabla y(x(s))}{\nabla x(s)}\right]+\frac{\psi(x(s))}{2}\left[\frac{\Delta y(x(s))}{\Delta x(s)}+\frac{\nabla y(x(s))}{\nabla x(s)}\right]+\lambda_n\,y(x(s))=0, \tag{6}$$

where ψ and ϕ are polynomials of degree 1 and at most 2, respectively; λ_n is a constant depending on n and on the leading coefficients of ϕ and ψ. The lattice $x(s)$ is defined by [9,10]

$$x(s)=\begin{cases} c_1\,q^{-s}+c_2\,q^s+c_3 & \text{if} \quad q\neq 1,\\ c_4\,s^2+c_5\,s+c_6 & \text{if} \quad q=1,\end{cases} \tag{7}$$

is known as non-uniform lattice and fulfills various important properties.

Equation (6) can be transformed into equation [12]

$$\phi(x(s))\mathbb{D}_x^2 y(x(s))+\psi(x(s))\mathbb{S}_x\mathbb{D}_x y(x(s))+\lambda_n y(x(s))=0, \tag{8}$$

called divided-difference equation of the hypergeometric type by means of the two companion operators $\mathbb{D}_x$ (called divided-difference operator) and $\mathbb{S}_x$ (called mean operator) defined as [9,10,12,13]

$$\mathbb{D}_x\,f(x(s))=\frac{f(x(s+\frac{1}{2}))-f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2})-x(s-\frac{1}{2})},\quad \mathbb{S}_x\,f(x(s))=\frac{f(x(s+\frac{1}{2}))+f(x(s-\frac{1}{2}))}{2}. \tag{9}$$

Using appropriate bases, computer algebra software has been used to solve divided-difference Equation (8) for specific families of classical orthogonal polynomials on a non-uniform lattice. For some special values of the parameter for the specific case of Askey–Wilson polynomials, non-polynomial solution has been recovered together with the polynomial one [14] (see page 15, Equations (62) and (63)). In addition, the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ have played a decisive role not only for establishing the functional approach of the characterization theorem of classical orthogonal polynomials on non-uniform lattices, but also for providing algorithmic solution to linear homogeneous divided-difference equations with polynomial coefficients, allowing to solve explicitly [13] the first-order divided-difference equations satisfied by the basic exponential function

$$\mathbb{D}_x y(x(s))=\frac{2wq^{\frac{1}{4}}}{1-q}\,y(x(s)),$$

and the second-order divided-difference equation satisfied by the basic trigonometric functions

$$\mathbb{D}_x^2 y(x(s))=-\left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^2\,y(x(s)),$$

where w is a given constant.

The aim of this work is:

1. redto define the right and the “left” inverses of the operator $\mathbb{D}_x$;
2. to provide a formal proof of the existence of a polynomial solution of a preassigned degree of the divided-difference equation of hypergeometric type (8), extending and generalising therefore—by means of specialisation and limiting situations on the lattice $x(s)$—similar results obtained for second-order differential, difference or q-difference equation of hypergeometric type.

2. Preliminary Results: Known and New Properties

Since the main result of this paper is based on the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ which are defined by using the lattice $x(s)$, we will provide in this section some known and basic properties of $x(s)$, $\mathbb{D}_x$ and $\mathbb{S}_x$. We will also derive new properties such as the right and the "left" inverses of the operator $\mathbb{D}_x$, required in the next section.

2.1. Known Properties of the Lattice $x(s)$

Taking into account the notation

$$x_\mu(s) = x(s + \frac{\mu}{2}),$$

the non-uniform lattice $x(s)$ defined by Equation (7) satisfies

$$\begin{aligned} x(s+k) - x(s) &= \gamma_k \nabla x_{k+1}(s), && (10)\\ \frac{x(s+k) + x(s)}{2} &= \alpha_k x_k(s) + \beta_k, && (11) \end{aligned}$$

for $k = 0, 1, \ldots$, with

$$\alpha_0 = 1, \alpha_1 = \alpha, \beta_0 = 0, \beta_1 = \beta, \gamma_0 = 0, \gamma_1 = 1, \tag{12}$$

where the sequences (α_k), (β_k), (γ_k) satisfy the following relations:

$$\begin{aligned} \alpha_{k+1} - 2\alpha\,\alpha_k + \alpha_{k-1} &= 0,\\ \beta_{k+1} - 2\beta_k + \beta_{k-1} &= 2\beta\,\alpha_k,\\ \gamma_{k+1} - \gamma_{k-1} &= 2\alpha_k, \end{aligned} \tag{13}$$

and are given explicitly by [9,10]

$$\alpha_n = 1,\ \beta_n = \beta\, n^2,\ \gamma_n = n,\ \text{for } \alpha = 1, \tag{14}$$

and

$$\alpha_n = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2},\ \beta_n = \frac{\beta(1-\alpha_n)}{1-\alpha},\ \gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},\ \text{for } \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \tag{15}$$

2.2. Known Properties of the Operators $\mathbb{D}_x$ and $\mathbb{S}_x$

The operators $\mathbb{S}_x$ and $\mathbb{D}_x$ fulfil the so-called Product rules I [13,14]:

$$\begin{aligned} \mathbb{D}_x\left(f(x(s))g(x(s))\right) &= \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), && (16)\\ \mathbb{S}_x\left(f(x(s))g(x(s))\right) &= U_2(x(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), && (17) \end{aligned}$$

where U_2 is a polynomial of degree 2

$$U_2(x(s)) = (\alpha^2 - 1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \eta_x, \tag{18}$$

and η_x is a constant given by [14]

$$\eta_x = \frac{x^2(0) + x^2(1)}{4\alpha^2} - \frac{(2\alpha^2 - 1)}{2\alpha^2}x(0)\,x(1) - \frac{\beta\,(\alpha+1)}{\alpha^2}(x(0) + x(1)) + \frac{\beta^2\,(\alpha+1)^2}{\alpha^2}. \tag{19}$$

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the so-called Product Rules II [13,14]:

$$\mathbb{D}_x\,\mathbb{S}_x = \alpha\,\mathbb{S}_x\,\mathbb{D}_x + U_1(x(s))\,\mathbb{D}_x^2;\ \ \mathbb{S}_x^2 = U_1(x(s))\,\mathbb{S}_x\,\mathbb{D}_x + \alpha\,U_2(x(s))\,\mathbb{D}_x^2 + \mathbb{I}, \tag{20}$$

where $\mathbb{I}$ is the identity operator $\mathbb{I}f(x) = f(x)$, and

$$U_1(x(s)) = (\alpha^2 - 1)\, x(s) + \beta\, (\alpha + 1).$$

2.3. An Appropriate Basis for the Operators $\mathbb{D}_x$ and $\mathbb{S}_x$

Searching for a polynomial basis on which the action of the companion operators will give a linear combination of at most two elements of the basis, Foupouagnigni et al. proved in [14] that the polynomial F_n defined by

$$F_n(x(s)) = F_n(x(s), x(\varepsilon)), \text{ with } F_n(x(s), x(\varepsilon)) = \prod_{j=1}^{n} \left[x(s) - x_j(\varepsilon)\right], \tag{21}$$

where ε is the unique solution (provided that the lattice $x(s)$ is quadratic or q-quadratic: i.e., the constants c_j in (7) satisfy $c_1\, c_2 \neq 0$ or $c_4 \neq 0$) in the variable t of the equation $x_1(t) = x(t)$, is the right basis for the operators $\mathbb{S}_x$ and $\mathbb{D}_x$ because it satisfies the following properties

$$\mathbb{D}_x F_n(x(s)) = \gamma_n\, F_{n-1}(x(s)), \tag{22}$$

$$\mathbb{S}_x F_n(x(s)) = \alpha_n\, F_n(x(s)) + \frac{\gamma_n}{2}\, \nabla x_{n+1}(z_x) F_{n-1}(x(s)), \tag{23}$$

where the constants α_n and γ_n are given in (14) and (15).

After reviewing some properties of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$, we now state and prove the following proposition providing the left and right inverse of the operator $\mathbb{D}_x$, to be used in the next section to complete the proof of the main theorem of this paper.

Proposition 1.
Let $\mathbb{F}_x$ be a linear operator defined on the basis $(F_n)_n$ by

$$\mathbb{F}_x\, F_n = \frac{F_{n+1}}{\gamma_{n+1}}, \; n \geq 0, \; \mathbb{F}_x 0 := 0. \tag{24}$$

Then, $\mathbb{F}_x$ satisfies the following relations:

$$\mathbb{D}_x\, \mathbb{F}_x = \mathbb{I}, \; \mathbb{F}_x\, \mathbb{D}_x = \mathbb{I} - \delta_{x(\varepsilon)}, \tag{25}$$

where $\mathbb{I}$ is the identity operator and $\delta_{x(\varepsilon)}$ is the Dirac delta distribution defined by

$$\langle \delta_{x(\varepsilon)}, P \rangle = P(x(\varepsilon)), \; \forall P,$$

with ε is defined in (21).

Proof. For all positive integer n, F_n defined by (21) is a polynomial of degree exactly n. (F_n) is therefore a basis of $\mathbb{C}[x]$. Letting $f \in \mathbb{C}_n[x]$, there exist $f_0, \ldots, f_n \in \mathbb{C}[x]$ such that

$$f(x(s)) = \sum_{j=0}^{n} f_j F_j(x(s)).$$

We have

$$\langle \delta_{x(\varepsilon)}, f \rangle = f(x(\varepsilon)) = \sum_{j=1}^{n} f_j F_j(x(\varepsilon)) + f_0 = f_0. \tag{26}$$

$$\begin{aligned} \mathbb{D}_x \mathbb{F}_x f(x(s)) &= \sum_{j=0}^{n} f_j \mathbb{D}_x \mathbb{F}_x F_j(x(s)) \\ &= \sum_{j=0}^{n} \frac{f_j \mathbb{D}_x F_{j+1}(x(s))}{\gamma_{j+1}} \\ &= \sum_{j=0}^{n} f_j F_j(x(s)) = f(x(s)). \end{aligned}$$

Hence, the first part of Relation (24) holds. Using (26), we have

$$\begin{aligned} \mathbb{F}_x \mathbb{D}_x f(x(s)) + \langle \delta_\varepsilon, f \rangle &= \mathbb{F}_x \mathbb{D}_x f(x(s)) + f_0 \\ &= \sum_{j=0}^{n} f_j \mathbb{F}_x \mathbb{D}_x F_j(x(s)) + f_0 \\ &= \sum_{j=1}^{n} f_j \gamma_j \mathbb{F}_x F_{j-1}(x(s)) + f_0 \\ &= \sum_{j=1}^{n} f_j F_j(x(s)) + f_0 \\ &= f(x(s)). \end{aligned}$$

The second part of (25) is therefore satisfied. □

3. Existence of the Polynomial Solution of the Divided-Difference Equation of the Hypergeometric Type

Having stated and proved required properties of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$, we will now state and prove the main theorem of this paper.

Theorem 1. *Let n be a nonnegative integer, ψ and ϕ be two polynomials of degree 1 and at most 2, respectively, such that*

$$\forall\, k \in \mathbb{N},\ \eta_k := \phi_2\, \gamma_k + \psi_1\, \alpha_k \neq 0. \tag{27}$$

Then, the divided-difference equation

$$\phi(x(s))\, \mathbb{D}_x^2 y(x(s)) + \psi(x(s))\, \mathbb{S}_x \mathbb{D}_x y(x(s)) + \lambda_{n,0}\, y(x(s)) = 0, \tag{28}$$

with

$$\lambda_{n,0} = -\gamma_n\, (\phi_2\, \gamma_{n-1} + \psi_1\, \alpha_{n-1}) = -\gamma_n \eta_{n-1}, \tag{29}$$

where ψ_1 and ϕ_2 are leading coefficients of polynomials ψ and ϕ respectively, has a polynomial solution of exactly n degree.

The proof of Theorem 1 will be organized as follows: we split the proof in five lemmas which we first state, prove, and then put these lemmas together in combination with Proposition 1 to deduce the proof of this theorem.

Lemma 1.
If the function y_0 is a solution of (28), then the function $y_1 = \mathbb{D}_x\, y_0$ satisfies

$$\phi^{[1]}(x(s))\, \mathbb{D}_x^2 y(x(s)) + \psi^{[1]}(x(s))\, \mathbb{S}_x \mathbb{D}_x y(x(s)) + \lambda_{n,1}\, y(x(s)) = 0, \tag{30}$$

where

$$\begin{cases} \phi^{[1]}(x(s)) = \mathbb{S}_x\phi(x(s)) + \alpha\, U_2(x(s))\, \mathbb{D}_x\psi + U_1(x(s))\, \mathbb{S}_x\psi(x(s)), \\ \psi^{[1]}(x(s)) = \mathbb{D}_x\phi(x(s)) + U_1(x(s))\, \mathbb{D}_x\psi + \alpha\, \mathbb{S}_x\psi(x(s)), \\ \lambda_{n,1} = \lambda_{n,0} + \mathbb{D}_x\psi. \end{cases} \tag{31}$$

Proof. Assume that y_0 satisfies (28). Applying the operator $\mathbb{D}_x$ to (28) in which y is replaced by y_0 and using the product rule I in (16) and (17), we obtain

$$\mathbb{D}_x\phi(x(s))\, \mathbb{S}_x\mathbb{D}_x^2 y_0(x(s)) + \mathbb{S}_x\phi(x(s))\, \mathbb{D}_x^3 y_0(x(s)) + \mathbb{D}_x\psi(x(s))\, \mathbb{S}_x^2\mathbb{D}_x y_0(x(s))$$
$$+\mathbb{S}_x\psi(x(s))\, \mathbb{D}_x\mathbb{S}_x\mathbb{D}_x y_0(x(s)) + \lambda_{n,0}\, \mathbb{D}_x y_0(x(s)) = 0.$$

Using the product rules II in (20) to replace $\mathbb{S}_x^2$ and $\mathbb{D}_x\mathbb{S}_x$ in the previous equation, we have

$$\phi^{[1]}(x(s))\, \mathbb{D}_x^3 y_0(x(s)) + \psi^{[1]}(x(s))\, \mathbb{S}_x\mathbb{D}_x^2 y_0(x(s)) + \lambda_{n,1}\, \mathbb{D}_x y_0(x(s)) = 0,$$

where $\phi^{[1]}(x(s))$, $\psi^{[1]}(x(s))$ and $\lambda_{n,1}$ are defined by (31). Therefore, $y_1 = \mathbb{D}_x\, y_0$ is a solution of Equation (30). □

Lemma 2.
If the function y_0 is a solution of (28), then the function $y_k = \mathbb{D}_x^k\, y_0$ is a solution of the equation

$$\phi^{[k]}(x(s))\, \mathbb{D}_x^2 y(x(s)) + \psi^{[k]}(x(s))\, \mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda_{n,k}\, y(x(s)) = 0, \tag{32}$$

where the polynomials $\phi^{[k]}$, $\psi^{[k]}$ and the constant $\lambda_{n,k}$ satisfy

$$\begin{cases} \phi^{[k+1]}(x(s)) = \mathbb{S}_x\phi^{[k]}(x(s)) + \alpha\, U_2(x(s))\, \mathbb{D}_x\psi^{[k]} + U_1(x(s))\, \mathbb{S}_x\psi^{[k]}(x(s)), \\ \psi^{[k+1]}(x(s)) = \mathbb{D}_x\phi^{[k]}(x(s)) + U_1(x(s))\, \mathbb{D}_x\psi^{[k]} + \alpha\, \mathbb{S}_x\psi^{[k]}(x(s)), \\ \lambda_{n,k+1} = \lambda_{n,k} + \mathbb{D}_x\psi^{[k]}, \end{cases} \tag{33}$$

with the following initial values: $\phi^{[0]} := \phi$, $\psi^{[0]} := \psi$.

Proof. Lemma 1 assures the validity of the result for $k = 1$.

Let k be a positive integer. Assume that y_k is solution of Equation (32). Applying the operator $\mathbb{D}_x$ to (32) in which y is replaced by y_k and using the Product Rules I, we obtain

$$\mathbb{D}_x\phi^{[k]}(x(s))\, \mathbb{S}_x\mathbb{D}_x^2 y_k(x(s)) + \mathbb{S}_x\phi^{[k]}(x(s))\, \mathbb{D}_x^3 y_k(x(s)) + \mathbb{D}_x\psi^{[k]}(x(s))\, \mathbb{S}_x^2\mathbb{D}_x y_k(x(s))$$
$$+\mathbb{S}_x\psi^{[k]}(x(s))\, \mathbb{D}_x\mathbb{S}_x\mathbb{D}_x y_k(x(s)) + \lambda_{n,k}\, \mathbb{D}_x y_k(x(s)) = 0.$$

Using the products rule II to replace $\mathbb{S}_x^2$ and $\mathbb{D}_x\mathbb{S}_x$ in the previous equation, we have

$$\phi^{[k+1]}(x(s))\, \mathbb{D}_x^3 y_k(x(s)) + \psi^{[k+1]}(x(s))\, \mathbb{S}_x\mathbb{D}_x^2 y_k(x(s)) + \lambda_{n,k+1}\, \mathbb{D}_x y_k(x(s)) = 0,$$

where $\phi^{[k+1]}(x(s))$, $\psi^{[k+1]}(x(s))$ and $\lambda_{n,k+1}$ are defined by (33). Thus, $y_{k+1} = \mathbb{D}_x\, y_k$ satisfies

$$\phi^{[k+1]}(x(s))\, \mathbb{D}_x^2 y_{k+1}(x(s)) + \psi^{[k+1]}(x(s))\, \mathbb{S}_x\mathbb{D}_x y_{k+1}(x(s)) + \lambda_{n,k+1}\, y_{k+1}(x(s)) = 0.$$

□

Lemma 3.
If a given function y_1 satisfies (30) with $\lambda_{n,0} \neq 0$, then there exists a function y_0 satisfying (28) such that

$$y_1 = \mathbb{D}_x y_0. \tag{34}$$

Proof. Let y_1 be a solution of (30) with $\lambda_{n,0} \neq 0$. If there would exist a solution v_0 of (28) such that $y_1 = \mathbb{D}_x\, v_0$, then from (28) we can express v_0 as:

$$v_0(x(s)) = -\frac{1}{\lambda_{n,0}}\, \left[\phi(x(s))\, \mathbb{D}_x y_1(x(s)) + \psi(x(s))\, \mathbb{S}_x y_1(x(s))\right]. \tag{35}$$

Now, it remains to verify that the function v_0 defined in terms of y_1 by (35) satisfies Equation (28) with

$$\mathbb{D}_x v_0 = y_1. \tag{36}$$

By applying $\mathbb{D}_x$ to (35) and using product rules I, II and the fact that y_1 is solution of (30), we get

$$\begin{aligned} -\lambda_{n,0}\, \mathbb{D}_x v_0(x(s)) &= \mathbb{D}_x \left[\phi(x(s))\, \mathbb{D}_x y_1(x(s)) + \psi(x(s))\, \mathbb{S}_x y_1(x(s))\right] \\ &= \phi^{[1]}(x(s))\, \mathbb{D}_x^2 y_1(x(s)) + \psi^{[1]}(x(s))\, \mathbb{S}_x \mathbb{D}_x y_1(x(s)) + (\lambda_{n,1} - \lambda_{n,0})\, y_1(x(s)) \\ &= -\lambda_{n,0}\, y_1(x(s)). \end{aligned}$$

Therefore, $\mathbb{D}_x v_0 = y_1$ since $\lambda_{n,0} \neq 0$.

We prove that v_0 is solution of (28) by replacing y_1 in the Equation (35) by $\mathbb{D}_x v_0$. □

Lemma 4.
For any positive integer n, the coefficients $\lambda_{n,k}$ defined by relation (33) satisfy

$$\lambda_{n,k} = \lambda_{n,0} - \lambda_{k,0},\ 0 \leq k \leq n, \tag{37}$$

$$\lambda_{n,k} \neq 0, \qquad \text{for } 0 \leq k \leq n-1, \text{ and } (n,k) \neq (0,0), \tag{38}$$

where

$$\lambda_{n,0} = -\gamma_n(\phi_2\gamma_{n-1} + \psi_1\alpha_{n-1}).$$

Proof. If we denote by $\phi^{[k]}(x(s)) = \phi_2^{[k]}\, F_2(x(s)) + \phi_1^{[k]}\, F_1(x(s)) + \phi_0^{[k]}$ and $\psi^{[k]}(x(s)) = \psi_1^{[k]}\, F_1(x(s)) + \psi_0^{[k]}$, then from (33), we have the following system of recurrence equation

$$\left\{ \begin{aligned} \phi_2^{[k+1]} &= \alpha_2\, \phi_2^{[k]} + \alpha\, \gamma_1 (\alpha^2 - 1)\, \psi_1^{[k]} + \alpha_1\, (\alpha^2 - 1)\, \psi_1^{[k]}, \\ \psi_1^{[k+1]} &= \gamma_2\, \phi_2^{[k]} + (\alpha^2 - 1)\, \gamma_1\, \psi_1^{[k]} + \alpha\, \alpha_1\, \psi_1^{[k]}, \\ \lambda_{n,k+1} &= \lambda_{n,k} + \psi_1^{[k]}. \end{aligned} \right.$$

Using relations

$$\alpha_2 = 2\,\alpha^2 - 1,\ \gamma_2 = 2\,\alpha,$$

derived from Equations (12) and (13), the previous system of equations becomes

$$\left\{ \begin{aligned} \phi_2^{[k+1]} &= (2\,\alpha^2 - 1)\, \phi_2^{[k]} + 2\,\alpha\, (\alpha^2 - 1)\, \psi_1^{[k]}, \\ \psi_1^{[k+1]} &= 2\,\alpha\, \phi_2^{[k]} + (2\,\alpha^2 - 1)\, \psi_1^{[k]}, \\ \lambda_{n,k+1} &= \lambda_{n,k} + \psi_1^{[k]}. \end{aligned} \right.$$

Solving this system of recurrence equations with the initial values $\phi_2^{[0]} = \phi_2$, $\psi_1^{[0]} = \psi_1$, we obtain for the q-quadratic lattice

$$\lambda_{n,k} = \frac{(q^k - q^n)}{(q-1)^2 q^k q^n} \left[\sqrt{q}(q^k q^n - q)\phi_2 + \frac{1}{2}(q-1)(q + q^k q^n)\psi_1\right]. \tag{39}$$

Using the definition of $\lambda_{n,0}$ which of course coincides with the one of $\lambda_{n,k}$ for $k = 0$, we derive (37) from (39).

Solving the following the equation

$$\lambda_{n,k} = 0,$$

in terms of the unknown k keeping in mind (27), gives a unique solution $k = n$. Thus, relation (38) is satisfied. It can easily be proved in the same way that relation (38) is satisfied for the quadratic lattice. □

Lemma 5.
Let n be a fixed positive integer and let k be an integer such that $0 \leq k \leq n$. Then, if y_k is a solution of Equation (32), then there exists y_0 solution of Equation (28) such that

$$y_k = \mathbb{D}_x^k\, y_0.$$

Proof. Let k be a nonnegative integer with $k \leq n$. Assume that y_k satisfies (32). Then, we obtain that there exists a function y_{k-1} solution of the equation obtained by replacing k in (32) by $k-1$, namely,

$$\phi^{[k-1]}(x(s))\,\mathbb{D}_x^2 y(x(s)) + \psi^{[k-1]}(x(s))\,\mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda_{n,k-1}\,y(x(s)) = 0, \tag{40}$$

such that

$$y_k = \mathbb{D}_x\, y_{k-1}.$$

This is achieved using the fact that $\lambda_{n,k-1} \neq 0$ thanks to Lemma 4, and also using Lemma 3 but with the functions y_0 and y_1 replaced, respectively, by the functions y_{k-1} and y_k while Equation (28) is replaced by Equation (40). In addition, Equation (30) is replaced by the Equation (32). □

The proof is completed by repeating the same process for $y_{k-1}, y_{k-2}, \ldots, y_1$ and using Lemmas 3 and 4.

Proof of Theorem 1. Since, for $k = n$, $\lambda_{n,k} = \lambda_{n,n} = 0$ thanks to (37), Equation (32) admits a constant solution, namely $F_0(x(s)) = 1$. We therefore deduce from Lemma 5 that there exists a function v_0 solution of (28) such that

$$F_0(x(s)) = \mathbb{D}_x^n\, v_0(x(s)). \tag{41}$$

Next, we apply the operator $\mathbb{F}_x$ on both members of the previous equation and deduce by applying the second relation of Equation (25) of Proposition 1 that

$$\mathbb{F}_x\, F_0(x(s)) = \mathbb{F}_x\mathbb{D}_x\,\mathbb{D}_x^{n-1}v_0(x(s)) = \mathbb{D}_x^{n-1}v_0(x(s)) - \mathbb{D}_x^{n-1}v_0(x(s))\,|_{s=\varepsilon}\,.$$

Hence,

$$\mathbb{D}_x^{n-1}v_0(x(s)) = \mathbb{F}_x\,F_0(x(s)) + C_{n-1}\,F_0(x(s)),$$

where $C_{n-1} = \mathbb{D}_x^{n-1}v_0(x(s))\,|_{s=\varepsilon}\,.$

By applying again the operator $\mathbb{F}_x$ on both members of the previous equation and using the second relation of Equation (25), we get

$$\mathbb{D}_x^{n-2}v_0(x(s)) = \mathbb{F}_x^2\,F_0(x(s)) + C_{n-1}\,\mathbb{F}_x\,F_0(x(s)) + C_{n-2}\,F_0(x(s)),$$

where $C_{n-2} = \mathbb{D}_x^{n-2}v_0(x(s))\,|_{s=\varepsilon}\,.$ Repeating the same process, we express v_0 as

$$v_0(x(s)) = \mathbb{F}_x^n\,F_0(x(s)) + \sum_{j=0}^{n-1} C_j\,\mathbb{F}_x^j\,F_0(x(s)) = \frac{F_n(x(s))}{\prod_{l=1}^{n}\gamma_l} + \sum_{j=0}^{n-1} C_j\,\mathbb{F}_x^j\,F_0(x(s)),$$

where $C_j = \mathbb{D}_x^j v_0(x(s))\,|_{s=\varepsilon}\,.$ Therefore, $v_0(x(s))$ is a polynomial of degree exactly n in $x(s)$. □

4. Conclusions and Perspectives

In this work, we have derived the right and the "left" inverse of the operator $\mathbb{D}_x$ and used the properties of the inverse operators, as well as those of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$, to provide a formal proof that the divided-difference equation of hypergeometric type (28) has a polynomial solution of degree exactly n.

The novelty of our work is the formal proof of the existence of this polynomial solution, confirming therefore the fact that, in [14], by solving divided-difference (8) on a case by case basis and using most appropriate polynomial basis for each case, we have obtained for each family of classical orthogonal polynomials on non-uniform lattice, a hypergeometric or q-hypergeometric solution which happens to be a polynomial because of the form of one of the upper parameters obtained in the hypergeometric (or q-hypergeometric) representation of the obtained solution.

Finding hypergeometric representation of the non polynomial solution of (8) is not obvious and this was obtained unexpectedly for the Askey–Wilson polynomials when the parameters fulfill $b = a\,q^{\frac{1}{2}},\ d = a\,q^{\frac{1}{2}}$ [14] (see page 15, Equations (62) and (63)). The method developed here might help to understand when and why such a hypergeometric representation exists for non-polynomial solutions.

As an additional potential application of our paper, the right and the "left" inverse of the operator $\mathbb{D}_x$ are likely to play important role in the study of the properties of orthogonal polynomials on the non-uniform latices, and on the search of the solutions of divided-difference equations on non-uniform lattices, as well as on the hypergeometric representation (when they exist) of the second-solution—non polynomial solution—of Equation (28).

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