axioms

MDPI

Article

Relation Theoretic Common Fixed Point Results for Generalized Weak Nonlinear Contractions with an Application

Atiya Perveen [1,*], Idrees A. Khan [2] and Mohammad Imdad [1]

1 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India; mhimdad@yahoo.co.in
2 Department of Mathematics, Integral University, Lucknow 226026, India; idrees_maths@yahoo.com
* Correspondence: atiya.rs@amu.ac.in

Received: 11 March 2019; Accepted: 18 April 2019; Published: 24 April 2019

Abstract: In this paper, by introducing the concept of generalized Ćirić-type weak $(\phi_g, \mathcal{R})$-contraction, we prove some common fixed point results in partial metric spaces endowed with binary relation $\mathcal{R}$. We also deduce some useful consequences showing the usability of our results. Finally, we present an application to establish the solution of a system of integral equations.

Keywords: common fixed point; binary relation; preserving mapping; $(\phi_g, \mathcal{R})$-contraction; partial ordering

MSC: 54H25; 47H10

1. Introduction

With a view to enhance the domain of applicability, Matthews [1] initiated the idea of a partial metric space by weakening the metric conditions and also proved an analogue of Banach contraction principle in such spaces. Thereafter, many well-known results of metric fixed point theory were extended to partial metric spaces (see [2–16] and references therein).

On the other hand, Turinici [17] initiated the idea of order theoretic metric fixed point results, which was put in more natural and systematic forms by Ran and Reurings [18], Nieto and Rodríguez-López [19,20], and some others. Very recently, Alam and Imdad [21] extended the Banach contraction principle to complete metric space endowed with an arbitrary binary relation. This idea has inspired intense activity in this theme, and by now, there exists considerable literature around this result (e.g., [6,21–25]).

Proving new results in metric fixed point theory by replacing contraction conditions with a generalized one continues to be the natural approach. In recent years, several well-known contraction conditions such as Kannan type, Chatterjee type, Ciric type, phi-contractions, and some others were introduced in this direction.

In this paper, we introduce some useful notions, namely, $\mathcal{R}$-precompleteness, $\mathcal{R}$-g-continuity and $\mathcal{R}$-compatibility, and utilize the same to establish common fixed point results for generalized weak ϕ-contraction mappings in partial metric spaces endowed with an arbitrary binary relation $\mathcal{R}$. We also derive several useful corollaries which are either new results in their own right or sharpened versions of some known results. Finally, an application is provided to validate the utility of our result.

2. Preliminaries

Matthews [1] defined partial metric space as follows:

Definition 1. *[1] Let M be a non-empty set. A mapping $\rho : M \times M \to [0,\infty)$ is said to be a partial metric if (for all $z_1, z_2, z_3 \in M$):*

(a) $z_1 = z_2 \iff \rho(z_1, z_1) = \rho(z_1, z_2) = \rho(z_2, z_2)$;
(b) $\rho(z_1, z_1) \le \rho(z_1, z_2)$;
(c) $\rho(z_1, z_2) = \rho(z_2, z_1)$;
(d) $\rho(z_1, z_2) \le \rho(z_1, z_3) + \rho(z_3, z_2) - \rho(z_3, z_3)$.

The pair (M, ρ) is called a partial metric space.

Notice that in partial metric, the self-distance of any point need not be zero. A metric on a non-empty set M is a partial metric with the condition that for all $z \in M$, $\rho(z,z) = 0$.

A partial metric ρ generates a T_0-topology, say τ_ρ on M, with base the family of open balls $\mathcal{B}_\rho(z,\epsilon)$ ($z \in M$ and $\epsilon > 0$) defined as:

$$\mathcal{B}_\rho(z,\epsilon) = \{w \in M : \rho(z,w) \le \rho(z,z) + \epsilon\}.$$

If ρ is a partial metric on M, then the function $d_\rho : M \times M \to [0,\infty)$ defined by:

$$d_\rho(z_1, z_2) = 2\rho(z_1, z_2) - \rho(z_1, z_1) - \rho(z_2, z_2),$$

is a metric on M.

Definition 2. *[1] Let (M, ρ) be a partial metric space. Then:*

(a) *A sequence $\{z_n\}$ is said to be convergent to a point $z \in M$ if $\lim_{n\to\infty} \rho(z_n, z) = \rho(z,z)$.*
(b) *A sequence $\{z_n\}$ is said to be Cauchy if $\lim_{m,n\to\infty} \rho(z_n, z_m)$ exists and is finite.*
(c) *(M,ρ) is said to be complete if every Cauchy sequence $\{z_n\}$ in M converges (with respect to τ_ρ) to a point a $z \in M$ and $\rho(z,z) = \lim_{n\to\infty} \rho(z_n, z_m)$.*

Remark 1. *In a complete partial metric space, every closed subset is complete.*

The following lemmas are needed in the sequel.

Lemma 1. *[1] Let (M, ρ) be a partial metric space. Then:*

(a) *A sequence $\{z_n\}$ is Cauchy in (M,ρ) if and only if it is Cauchy in (M, d_ρ).*
(b) *(M,ρ) is complete if and only if the metric space (M, d_ρ) is complete. In addition:*

$$\lim_{n\to\infty} d_\rho(z_n, z) = 0 \iff \rho(z,z) = \lim_{n\to\infty} \rho(z_n, z) = \lim_{m,n\to\infty} \rho(z_n, z_m).$$

Lemma 2. *[2] Let (M,ρ) be a partial metric space and $\{z_n\}$ a sequence in M such that $\{z_n\} \to w$, for some $w \in M$ with $\rho(w,w) = 0$. Then, for any $z \in M$, we have $\lim_{n\to\infty} \rho(z_n, z) = \rho(w,z)$.*

Definition 3. *Let S and g be two self-mappings on a non-empty set M.*

(a) *An element $z \in M$ is said to be a coincidence point of S and g if $Sz = gz$.*
(b) *An element $z^* \in M$ is said to be a point of coincidence if $z^* = Sz = gz$, for some $z \in M$.*
(c) *If $z \in M$ is a point of coincidence of S and g such that $z = Sz = gz$, then z is called a common fixed point.*

3. Relation Theoretic Notions and Auxiliary Results

Let M be a non-empty set. A binary relation $\mathcal{R}$ on M is a subset of $M \times M$. For $z_1, z_2 \in M$, we write $(z_1, z_2) \in \mathcal{R}$ if z_1 is related to z_2 under $\mathcal{R}$. Sometimes, we denote it as $z_1 \mathcal{R} z_2$ instead of $(z_1, z_2) \in \mathcal{R}$.

Further, if $(z_1, z_2) \in \mathcal{R}$ such that z_1 and z_2 are distinct, then we write $(z_1, z_2) \in \mathcal{R}^{\neq}$ (sometimes as $z_1 \mathcal{R}^{\neq} z_2$). It is observed that $\mathcal{R}^{\neq} \subseteq \mathcal{R}$ is also a binary relation on M. $M \times M$ and $\emptyset$ are trivial binary relations on M, specifically called a universal relation and empty relation, respectively. The inverse, transpose or dual relation of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined as $\mathcal{R}^{-1} = \{(z_1, z_2) \in M \times M : (z_2, z_1) \in \mathcal{R}\}$. We denote by $\mathcal{R}^s$ the symmetric closure of $\mathcal{R}$, which is defined as $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.

Throughout this manuscript, M is a non-empty set, $\mathcal{R}$ stands for a binary relation on M and I_M denotes an identity mapping, and S and g are self-mappings on M.

Definition 4. *[26] For a binary relation $\mathcal{R}$:*

(a) *Two elements $z_1, z_2 \in M$ are said to be $\mathcal{R}$-comparative if $(z_1, z_2) \in \mathcal{R}$ or $(z_2, z_1) \in \mathcal{R}$. We denote it by $[z_1, z_2] \in \mathcal{R}$.*

(b) *$\mathcal{R}$ is said to be complete if $[z_1, z_2] \in \mathcal{R}$, for all $z_1, z_2 \in M$.*

Proposition 1. *[21] For a binary relation $\mathcal{R}$ on M, we have (for all $z_1, z_2 \in M$):*

$$(z_1, z_2) \in \mathcal{R}^s \iff [z_1, z_2] \in \mathcal{R}.$$

Definition 5. *[21] A sequence $\{z_n\} \subseteq M$ is said to be $\mathcal{R}$-preserving if $(z_n, z_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$.*

Here, we follow the notion (of $\mathcal{R}$-preserving) as used by Alam and Imdad [21]. Notice that Roldán and Shahzad [27] and Shahzad et al. [28] used the term "$\mathcal{R}$-nondecreasing" instead of "$\mathcal{R}$-preserving".

Definition 6. *[29] Let $N \subseteq M$. If for each $z_1, z_2 \in N$, there exists a point $z_3 \in M$ such that $(z_1, z_3) \in \mathcal{R}$ and $(z_2, z_3) \in \mathcal{R}$, then N is said to be $\mathcal{R}$-directed.*

Definition 7. *[30] For $z_1, z_2 \in M$, a path of length $l \in \mathbb{N}$ in $\mathcal{R}$ from z_1 to z_2 is a finite sequence $\{p_0, p_1, ..., p_l\} \subseteq M$ such that $p_0 = z_1$, $p_l = z_2$ and $(p_i, p_{i+1}) \in \mathcal{R}$, for each $0 \leq i \leq l-1$.*

Definition 8. *[31] Let $N \subseteq M$. If for each $z_1, z_2 \in N$, there exists a path in $\mathcal{R}$ from z_1 to z_2, then N is said to be $\mathcal{R}$-connected.*

Definition 9. *[21] $\mathcal{R}$ is said to be S-closed if $(z_1, z_2) \in \mathcal{R}$ implies that $(Sz_1, Sz_2) \in \mathcal{R}$, for all $z_1, z_2 \in M$.*

Definition 10. *[31] $\mathcal{R}$ is said to be (S, g)-closed if $(gz_1, gz_2) \in \mathcal{R}$ implies that $(Sz_1, Sz_2) \in \mathcal{R}$, for all $z_1, z_2 \in M$.*

Observe that on setting $g = I_M$, Definition 10 reduces to Definition 9.

Proposition 2. *[31] If $\mathcal{R}$ is (S, g)-closed, then $\mathcal{R}^s$ is also (S, g)-closed.*

Definition 11. *[23] $\mathcal{R}$ is said to be locally S-transitive if for each $\mathcal{R}$-preserving sequence $\{z_n\} \subseteq S(M)$ with range $E = \{z_n : n \in \mathbb{N}_0\}$, the binary relation $\mathcal{R}|_E$ is transitive.*

Motivated by Alam and Imdad [31], we introduce the notion of $\mathcal{R}$-continuity and $\mathcal{R}$-g-continuity in the context of partial metric space as follows:

Definition 12. *Let (M, ρ) be a partial metric space endowed with a binary relation $\mathcal{R}$. A self-mapping S on M is said to be $\mathcal{R}$-continuous at a point $z \in M$ if for any $\mathcal{R}$-preserving sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$, we have $\{Sz_n\} \to Sz$. S is $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of M.*

Definition 13. *Let (M, ρ) be a partial metric space endowed with a binary relation $\mathcal{R}$. A self mapping S is said to be $(g, \mathcal{R})$-continuous at a point $z \in M$ if for any sequence $\{z_n\} \subseteq M$ with $\{gz_n\}$ $\mathcal{R}$-preserving and $\{gz_n\} \to gz$, we have $\{Sz_n\} \to Sz$. S is $\mathcal{R}$-g-continuous if it is $\mathcal{R}$-g-continuous at each point of M.*

Remark 2. *Notice that for $g = I_M$, Definition 13 reduces to Definition 12.*

In the next definition, we introduce $\mathcal{R}$-compatibility.

Definition 14. *Let (M, ρ) be a partial metric space endowed with binary relation $\mathcal{R}$ and $S, g : M \to M$. S and g are said to be $\mathcal{R}$-compatible if for any sequence $\{z_n\}$ such that $\{Sz_n\}$ and $\{gz_n\}$ are $\mathcal{R}$-preserving and $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} gz_n$, we have:*

$$\lim_{n\to\infty} d_\rho(g(Sz_n), S(gz_n)) = 0.$$

Inspired by Imdad et al. [24], we introduce the following notions in the setting of partial metric spaces in the similar way.

Definition 15. *Let (M, ρ) be a partial metric space endowed with a binary relation $\mathcal{R}$. A subset $N \subseteq M$ is said to be $\mathcal{R}$-precomplete if each $\mathcal{R}$-preserving Cauchy sequence $\{z_n\} \subseteq N$ converges to some $z \in M$.*

Remark 3. *Every $\mathcal{R}$-complete subset of M is $\mathcal{R}$-precomplete.*

Proposition 3. *Every $\mathcal{R}$-closed subspace of an $\mathcal{R}$-complete partial metric space is $\mathcal{R}$-complete.*

Proposition 4. *An $\mathcal{R}$-complete subspace of a partial metric space is $\mathcal{R}$-closed.*

Next, we introduce the notion of ρ-self closedness in the setting of partial metric spaces.

Definition 16. *Let (M, ρ) be a partial metric space endowed with binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be ρ-self closed if for each $\mathcal{R}$-preserving sequence $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $[z_{n_k}, z] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$.*

We now state the following lemma needed in our subsequent discussion.

Lemma 3. *Let M be a non-empty set and $g : M \to M$. Then there exists a subset $N \subseteq M$ with $g(N) = g(M)$ and $g : N \to M$ is one–one.*

We use the following notations in our subsequent discussions:
$Coin(S, g)$: Set of all coincidence points of S and g;
$M(g, S, \mathcal{R})$: The collection of all points $z \in M$ such that $[gz, Sz] \in \mathcal{R}$.

4. Main Results

Let Φ denote the set of all mappings $\phi : [0, \infty) \to [0, \infty)$ satisfying the following:

(Φ1) ϕ is non-decreasing;
(Φ2) $\phi(\delta) = 0$ iff $\delta = 0$ and $\liminf_{n\to\infty} \phi(\delta_n) > 0$ if $\lim_{n\to\infty} \delta_n > 0$.

Notice that Reference [32] used the condition that ϕ is continuous. Inspired by Reference [33], we replace their condition by a more weaker condition (Φ2). In fact, this condition is also weaker than that ϕ is lower semi-continuous. Indeed, if ϕ is a lower semi-continuous function, then for a sequence $\{\delta_n\}$ with $\lim_{n\to\infty} \delta_n = \delta > 0$, we have $\liminf_{n\to\infty} \phi(\delta_n) \geq \phi(\delta) > 0$.

Before presenting our main result, we define the following.

Definition 17. *Let M be a non-empty set endowed with an arbitrary binary relation $\mathcal{R}$ and $N \subseteq M$. Then, N is said to be $(S, g, \mathcal{R})$-directed if for each $z_1, z_2 \in N$, there exists a point $z_3 \in M$ such that $(gz_i, gz_3) \in \mathcal{R}$, for $i = 1, 2$ and $(gz_3, Sz_3) \in \mathcal{R}$.*

Definition 18. *Let M be a non-empty set endowed with an arbitrary binary relation $\mathcal{R}$ and $N \subseteq M$. Then, N is said to be $(S, g, \mathcal{R})$-connected if for each $z_1, z_2 \in N$, there exists a path $\{gp_0, gp_1, ..., gp_l\} \subseteq g(M)$ between z_1 and z_2 such that $(gp_i, Sp_i) \in \mathcal{R}$, for $1 \le i \le l - 1$.*

Remark 4. *For $g = I_M$, Definitions 17 and 18 reduce to $(S, \mathcal{R})$-directed and $(S, \mathcal{R})$-connected.*

Now, we state and prove our first main result, which runs as follows:

Theorem 1. *Let (M, ρ) be a partial metric space equipped with a binary relation $\mathcal{R}$, $N \subseteq M$, an $\mathcal{R}^{\neq}$-precomplete subspace in M and $S, g : M \to M$. Assume that the following conditions are satisfied:*

- (a) $M(g, S, \mathcal{R}) \neq \emptyset$;
- (b) *$\mathcal{R}$ is (S, g)-closed;*
- (c) $S(M) \subseteq g(M) \cap N$;
- (d) *$\mathcal{R}$ is locally S-transitive;*
- (e) *S satisfies generalized Ćirić-type weak $(\phi_g, \mathcal{R})$-contraction, i.e.,*

$$\rho(Sz, Sw) \le \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)), \tag{1}$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$, where:

$$\mathcal{M}_{\rho,g}(z, w) = \max\left\{\rho(gz, gw), \rho(gz, Sz), \rho(gw, Sw), \frac{\rho(gz, Sw) + \rho(gw, Sz)}{2}\right\};$$

- (f) *$(f1)$ S and g are $\mathcal{R}^{\neq}$-compatible;*
  $(f2)$ S and g are $\mathcal{R}^{\neq}$-continuous;
  or alternatively:
- (f^*) *(f^*1) $N \subseteq g(M)$;*
  (f^*2) either S is $(g, \mathcal{R}^{\neq})$-continuous or S and g are continuous or $\mathcal{R}^{\neq}|_N$ is ρ-self closed.

Then, S and g have a coincidence point.

Proof. Choose $z_0 \in M$ as in (a) and construct a sequence $\{gz_n\}$ in M as follows:

$$gz_n = Sz_{n-1} = S^n z_0,\ \forall n \in \mathbb{N}_0.$$

If there is some $m_0 \in \mathbb{N}_0$ such that $gz_{m_0} = gz_{m_0+1}$, then z_{m_0} is the coincidence point of the pair (S, g) and we are done. Henceforth, assume that $gz_n \neq gz_{n+1}$, for all $n \in \mathbb{N}_0$. In view of condition (b), we have $(gz_n, gz_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$. Employing condition (e), we have:

$$\rho(Sz_{n-1}, Sz_n) \le \mathcal{M}_{\rho,g}(z_{n-1}, z_n) - \phi(\rho(Sz_{n-1}, Sz_n)), \tag{2}$$

which implies:

$$\rho(gz_n, gz_{n+1}) = \rho(Sz_{n-1}, Sz_n) \le \mathcal{M}_{\rho,g}(z_{n-1}, z_n), \tag{3}$$

where:

$$\begin{aligned}\mathcal{M}_{\rho,g}(z_{n-1},z_n) &= \max\Big\{\rho(gz_{n-1},gz_n),\rho(gz_{n-1},Sz_{n-1}),\rho(gz_n,Sz_n),\\ &\qquad \frac{\rho(gz_{n-1},Sz_n)+\rho(gz_n,Sz_{n-1})}{2}\Big\}\\ &= \max\Big\{\rho(gz_{n-1},gz_n),\rho(gz_{n-1},gz_n),\rho(gz_n,gz_{n+1}),\\ &\qquad \frac{\rho(gz_{n-1},gz_{n+1})+\rho(gz_n,gz_n)}{2}\Big\}\\ &\le \max\Big\{\rho(gz_{n-1},gz_n),\rho(gz_n,gz_{n+1}),\frac{\rho(gz_{n-1},gz_n)+\rho(gz_n,gz_{n+1})}{2}\Big\}\\ &= \max\{\rho(gz_{n-1},gz_n),\rho(gz_n,gz_{n+1})\}.\end{aligned}$$

Now, if $\mathcal{M}_{\rho,g}(z_{n-1},z_n)=\rho(gz_n,gz_{n+1})$, then Equation (2) becomes:

$$\rho(gz_n,gz_{n+1})\le\rho(gz_n,gz_{n+1})-\phi(\rho(gz_n,gz_{n+1})),$$

a contradiction. Hence, we have $\mathcal{M}_{\rho,g}(z_{n-1},z_n)=\rho(gz_{n-1},gz_n)$ and Equation (3) implies that $\{\rho(gz_n,gz_{n+1})\}$ is non-decreasing (also bounded below by 0). Thus, there exists $r\ge 0$ such that $\lim_{n\to\infty}\rho(gz_n,gz_{n+1})=r$. Next, we show that $r=0$. Suppose, by contrast, that it is not so, i.e., $r>0$. Passing the limit $n\to\infty$ in Equation (2), we get:

$$r\le r-\liminf_{n\to\infty}\phi(\rho(gz_n,gz_{n+1}))$$

which is a contradiction. Hence:

$$\lim_{n\to\infty}\rho(gz_n,gz_{n+1})=0. \tag{4}$$

We also have:

$$\begin{aligned}d_\rho(gz_n,gz_{n+1}) &= 2\rho(gz_n,gz_{n+1})-\rho(gz_n,gz_n)-\rho(gz_{n+1},gz_{n+1})\\ &\le 2\rho(gz_n,gz_{n+1}),\end{aligned}$$

which, on letting $n\to\infty$ and applying Equation (4), yields that:

$$\lim_{n\to\infty}d_\rho(gz_n,gz_{n+1})=0.$$

Now, our claim is that $\{gz_n\}$ is a Cauchy sequence in (N,d_ρ). Otherwise, there exist two subsequences $\{gz_{m_k}\}$ and $\{gz_{n_k}\}$ of $\{gz_n\}$ such that n_k is the smallest integer for which:

$$n_k>m_k>k \text{ and } d_\rho(gz_{m_k},gz_{n_k})\ge\epsilon. \tag{5}$$

Since $d_\rho(z,w)\le 2\rho(z,w)$, for all $z,w\in M$, Equation (5) gives:

$$n_k>m_k>k,\ \rho(gz_{m_k},gz_{n_k})\ge\frac{\epsilon}{2} \text{ and } \rho(gz_{m_k},gz_{n_k})<\frac{\epsilon}{2}.$$

Now, using triangular inequality, we have:

$$\begin{aligned}\frac{\epsilon}{2}\le\rho(gz_{m_k},gz_{n_k}) &\le \rho(gz_{m_k},gz_{n_k-1})+\rho(gz_{n_k-1},gz_{n_k})\\ &< \frac{\epsilon}{2}+\rho(gz_{n_k-1},gz_{n_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain:

$$\lim_{k\to\infty} \rho(gz_{m_k}, gz_{n_k}) = \frac{\epsilon}{2}. \tag{6}$$

Again, the triangle inequality yields the following:

$$\rho(gz_{m_k}, gz_{n_k-1}) \le \rho(gz_{m_k}, gz_{n_k}) + \rho(gz_{n_k}, gz_{n_k-1})$$

and:

$$\rho(gz_{m_k}, gz_{n_k}) \le \rho(gz_{m_k}, gz_{n_k-1}) + \rho(gz_{n_k-1}, gz_{n_k})$$

which together give rise to:

$$|\rho(gz_{m_k}, gz_{n_k-1}) - \rho(gz_{m_k}, gz_{n_k})| \le \rho(gz_{n_k-1}, gz_{n_k}).$$

Now, on taking $k \to \infty$, the above inequality gives:

$$\lim_{k\to\infty} \rho(gz_{m_k}, gz_{n_k-1}) = \frac{\epsilon}{2}.$$

In a similar manner, one can show that:

$$\lim_{k\to\infty} \rho(gz_{m_k-1}, gz_{n_k-1}) = \lim_{k\to\infty} \rho(gz_{m_k-1}, gz_{n_k}) = \frac{\epsilon}{2}.$$

Thus, we get:

$$\lim_{k\to\infty} \mathcal{M}_{\rho,g}(z_{m_k-1}, z_{n_k-1}) = \frac{\epsilon}{2}. \tag{7}$$

Using (d), we have $(gz_{m_k-1}, gz_{n_k-1}) \in \mathcal{R}$ and hence, Equation (1) implies:

$$\rho(gz_{m_k}, gz_{n_k}) \le \mathcal{M}_{\rho,g}(z_{m_k-1}, z_{n_k-1})) - \phi(\rho(gz_{m_k}, gz_{n_k})).$$

Using Equations (6) and (7) and letting $k \to \infty$ in the above inequality, we get:

$$\frac{\epsilon}{2} \le \frac{\epsilon}{2} - \liminf_{k\to\infty} \phi(\rho(gz_{m_k}, gz_{n_k})),$$

a contradiction. Hence, $\{gz_n\}$ is Cauchy in (N, d_ρ) (as $\{gz_n\} \subseteq S(M) \subseteq N$) which is also $\mathcal{R}^{\neq}$-preserving. Lemma 1 ensures that it is also Cauchy in (N, ρ). Thus, the $\mathcal{R}^{\neq}$-precompleteness of N in M ensures the existence of a point $\bar{z} \in M$ such that:

$$\lim_{n\to\infty} gz_n = \bar{z}. \tag{8}$$

Thus, we also have:

$$\lim_{n\to\infty} d_\rho(gz_n, \bar{z}) = 0. \tag{9}$$

Now, by Equation (9) and Lemma 1, we get:

$$\rho(\bar{z}, \bar{z}) = \lim_{m,n\to\infty} \rho(gz_n, \bar{z}) = \lim_{m,n\to\infty} \rho(gz_m, gz_n) = 0. \tag{10}$$

Further, by the definition of $\{gz_n\}$ and Equation (8), we have:

$$\lim_{n\to\infty} Sz_n = \bar{z}. \tag{11}$$

Finally, to prove the existence of coincidence point of S and g, we make use of conditions (f) and (f^*). Firstly, assume that (f) holds. Now, as $(gz_n, gz_{n+1}) \in \mathcal{R}^{\neq}$, so using assumption $(f2)$ and Equation (8), we obtain:

$$\lim_{n\to\infty} g(gz_n) = g(\lim_{n\to\infty} gz_n) = g\bar{z}. \tag{12}$$

By the definition of $\{gz_n\}$, we have $\{Sz_n\}$ is also $\mathcal{R}^{\neq}$-preserving (i.e., $(Sz_n, Sz_{n+1}) \in \mathcal{R}^{\neq}$, for all n), so using assumption $(f2)$ and Equation (11), we get:

$$\lim_{n\to\infty} g(Sz_n) = g(\lim_{n\to\infty} Sz_n) = g\bar{z}. \tag{13}$$

By using Equation (8) and $\mathcal{R}^{\neq}$-continuity of S, we obtain:

$$\lim_{n\to\infty} S(gz_n) = S(\lim_{n\to\infty} gz_n) = S\bar{z}. \tag{14}$$

As $\{Sz_n\}$ and $\{gz_n\}$ are $\mathcal{R}^{\neq}$-preserving and $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} gz_n = \bar{z}$, by the condition $(f1)$, we have:

$$\lim_{n\to\infty} d_\rho(g(Sz_n), S(gz_n)) = 0. \tag{15}$$

Now, from Equations (13)–(15) and continuity of d_ρ, it follows that:

$$\begin{aligned} d_\rho(g\bar{z}, S\bar{z}) &= d_\rho(\lim_{n\to\infty} g(Sz_n), \lim_{n\to\infty} S(gz_n)) \\ &= \lim_{n\to\infty} d_\rho(g(Sz_n), S(gz_n)) \\ &= 0, \end{aligned}$$

i.e., $g\bar{z} = S\bar{z}$ and we are done. Secondly, suppose that (f^*) is satisfied. Then, by (f^*1), there exists some $z \in M$ such that $\bar{z} = gz$. Hence, Equations (8) and (11) respectively reduce to:

$$\lim_{n\to\infty} gz_n = gz, \tag{16}$$

and:

$$\lim_{n\to\infty} Sz_n = gz. \tag{17}$$

Next, to accomplish that z is a coincidence point of S and g, we utilize (f^*2). Thus, suppose that S is $\mathcal{R}^{\neq}$-g-continuous, then using Equation (16), we obtain:

$$\lim_{n\to\infty} Sz_n = Sz. \tag{18}$$

Now, by virtue of uniqueness of limit, Equations (17) and (18) give $Sz = gz$.

Next, assume that S and g are continuous. Then owing to Lemma 3, there exists $D \subseteq M$ such that $g(D) = g(M)$ and $g : D \to M$ is injective. Now, define a mapping $\bar{S} : g(D) \to g(M)$ by:

$$\bar{S}(gt) = St, \ \forall gt \in g(D). \tag{19}$$

As $g : D \to M$ is injective and $S(M) \subseteq g(M)$, $\bar{S}$ is well-defined. Further, due to the continuity of S and g, $\bar{S}$ is continuous. The fact that $g(D) = g(M)$, assumptions (c) and (f^*1) imply that:

$$S(M) \subseteq g(D) \cap N \text{ and } N \subseteq g(D).$$

Thus, without loss of generality, we can construct $\{z_n\} \subseteq D$, satisfying Equation (16) with $z \in D$. On using Equations (16), (17), and (19) with continuity of $\bar{S}$, we obtain:

$$Sz = \bar{S}(gz) = \bar{S}(\lim_{n\to\infty} gz_n) = \lim_{n\to\infty} \bar{S}(gz_n) = \lim_{n\to\infty} Sz_n = gz,$$

and we are done. Alternatively, if $\mathcal{R}^{\neq}|_N$ is ρ-self closed, then for any $\mathcal{R}^{\neq}$-preserving sequence $\{gz_n\}$ in N with $\{gz_n\} \to gz$, there exists a subsequence $\{gz_{n_k}\}$ of $\{gz_n\}$ such that $[gz_{n_k}, gz] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. Suppose $\rho(gz, Sz) > 0$, then we have:

$$\mathcal{M}_{\rho,g}(z_{n_k}, z) = \max\left\{\rho(gz_{n_k}, gz), \rho(gz_{n_k}, Sz_{n_k}), \rho(gz, Sz), \frac{\rho(gz_{n_k}, Sz) + \rho(gz, Sz_{n_k})}{2}\right\}.$$

Letting $k \to \infty$ and using Equation (8), we get:

$$\lim_{k\to\infty} \mathcal{M}_{\rho,g}(z_{n_k}, z) = \rho(gz, Sz). \tag{20}$$

Now, applying $z = z_{n_k}$ and $w = z$, condition (e) gives:

$$\rho(Sz_{n_k}, Sz) \leq \mathcal{M}_{\rho,g}(z_{n_k}, z) - \phi(\rho(Sz_{n_k}, Sz)),$$

which, on letting $n \to \infty$ and using Equations (8) and (20) and Lemma 2, yields that:

$$\rho(gz, Sz) \leq \rho(gz, Sz) - \liminf_{k\to\infty} \phi(\rho(gz_{n_k+1}, Sz)),$$

a contradiction. Hence $\rho(gz, Sz) = 0$, i.e., $gz = Sz$. This completes the proof. □

Now, we present a corresponding uniqueness result.

Theorem 2. *In addition to the assumptions of Theorem 1, if we assume that the following condition is satisfied:*

(g) *$S(M)$ is $(S, g, \mathcal{R}^s)$-connected,*

then S and g have a unique point of coincidence. Moreover, if:

(h) *S and g are weakly compatible,*

then S and g have a unique common fixed point.

Proof. Firstly, Theorem 1 ensures that $Coin(S, g) \neq \emptyset$. Let $\bar{z}, \bar{w} \in Coin(S, g)$. Then, there exists $z, w \in M$ such that $\bar{z} = Sz = gz$ and $\bar{w} = Sw = gw$. Our claim is that $\bar{z} = \bar{w}$. Now, owing to hypothesis (g), there exists a path, say $\{gp_0, gp_1, gp_2, ..., gp_l\} \subseteq M$ of some finite length l in $\mathcal{R}|^s_{g(M)}$ from Sz to Sw with:

$$gp_0 = Sz,\ gp_l = Sw \text{ and } [gp_i, gp_{i+1}] \in \mathcal{R},\ \text{for each } 0 \leq i \leq l-1 \tag{21}$$

and:

$$[gp_i, Sp_i] \in \mathcal{R},\ \text{for each } 1 \leq i \leq l-1. \tag{22}$$

Define constant sequences $\{p^0_n = z\}$ and $\{p^l_n = w\}$, then we have $gp^0_{n+1} = Sp^0_n = Sz = \bar{z}$ and $gp^l_{n+1} = Sp^l_n = Sw = \bar{w}$, for all $n \in \mathbb{N}_0$. Further, set $p^i_0 = p_i$, for each $0 \leq i \leq l$ and define sequences $\{p^1_n\}$, $\{p^2_n\}, ..., \{p^{k-1}_n\}$ by:

$$gp^i_{n+1} = Sp^i_n,\ \forall n \in \mathbb{N}_0 \text{ and for each } 1 \leq i \leq l-1.$$

Hence:

$$gp^i_{n+1} = Sp^i_n,\ \forall n \in \mathbb{N}_0 \text{ and for each } 0 \leq i \leq l.$$

By mathematical induction, we will prove that:

$$[gp_n^i, gp_n^{i+1}] \in \mathcal{R},\ \forall n \in \mathbb{N}_0 \text{ and for each } 0 \le i \le l-1.$$

In view of Equation (21), the result holds for $n = 0$. Now, suppose it holds for $n = k > 0$, i.e.:

$$[gp_k^i, gp_k^{i+1}] \in \mathcal{R},\ \text{for each } 0 \le i \le l-1.$$

By (S, g)-closedness of $\mathcal{R}$ and Proposition 2, we have:

$$[Sp_k^i, Sp_k^{i+1}] = [gp_{k+1}^i, gp_{k+1}^{i+1}] \in \mathcal{R},\ \text{for each } 0 \le i \le l-1,$$

i.e., the result holds for $n = k+1$ and hence, it holds for all $n \in \mathbb{N}_0$. Also from Equation (22), we have $[gp_0^i, gp_1^i] \in \mathcal{R}$ and $\mathcal{R}$ is (S, g)-closed, so by Proposition 2 and Equation (4), we have:

$$\lim_{n\to\infty} \rho(gp_n^i, gp_{n+1}^i) = 0. \tag{23}$$

Now, for all $n \in \mathbb{N}_0$ and for each $0 \le i \le l-1$, define $f_n^i = \rho(gp_n^i, gp_n^{i+1})$. Our claim is that:

$$\lim_{n\to\infty} f_n^i = 0.$$

Suppose, by contrast, that $\lim_{n\to\infty} f_n^i = f > 0$. Since $[gp_n^i, gp_n^{i+1}] \in \mathcal{R}$, $(gp_n^i, gp_n^{i+1}) \in \mathcal{R}$ or $(gp_n^{i+1}, gp_n^i) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$ and for each $0 \le i \le l-1$. Making use of Equation (1), we have:

$$\rho(Sp_n^i, Sp_n^{i+1}) \le \mathcal{M}_{\rho,g}(p_n^i, p_n^{i+1}) - \phi(\rho(Sp_n^i, Sp_n^{i+1}))$$

or:

$$\rho(gp_{n+1}^i, Sp_{n+1}^{i+1}) \le \mathcal{M}_{\rho,g}(p_n^i, p_n^{i+1}) - \phi(\rho(gp_{n+1}^i, gp_{n+1}^{i+1})), \tag{24}$$

where:

$$\begin{aligned}
\mathcal{M}_{\rho,g}(p_n^i, p_n^{i+1}) &= \max\Big\{\rho(gp_n^i, gp_n^{i+1}), \rho(gp_n^i, Sp_n^i), \rho(gp_n^{i+1}, Sp_n^{i+1}),\\
&\qquad \frac{\rho(gp_n^i, Sp_n^{i+1}) + \rho(gp_n^{i+1}, Sp_n^i)}{2}\Big\}\\
&= \max\Big\{\rho(gp_n^i, gp_n^{i+1}), \rho(gp_n^i, gp_{n+1}^i), \rho(gp_n^{i+1}, gp_{n+1}^{i+1}),\\
&\qquad \frac{\rho(gp_n^i, gp_{n+1}^{i+1}) + \rho(gp_n^{i+1}, gp_{n+1}^i)}{2}\Big\}\\
&\le \max\Big\{\rho(gp_n^i, gp_n^{i+1}), \rho(gp_n^i, gp_{n+1}^i), \rho(gp_n^{i+1}, gp_{n+1}^{i+1}),\\
&\qquad \frac{\rho(gp_n^i, gp_{n+1}^i) + \rho(gp_{n+1}^i, gp_{n+1}^{i+1}) + \rho(gp_n^{i+1}, gp_n^i) + \rho(gp_n^i, gp_{n+1}^i)}{2}\Big\}.
\end{aligned}$$

Now, letting $n \to \infty$ and using Equation (23), we obtain:

$$\lim_{n\to\infty} \mathcal{M}_{\rho,g}(p_n^i, p_n^{i+1}) = f,$$

which, on applying Equation (24) after taking limit, yields that:

$$f \le f - \liminf_{n\to\infty} \phi(\rho(p_{n+1}^i, p_{n+1}^{i+1})),$$

a contradiction. Therefore, $\lim_{n\to\infty} f_n^i = 0$.

Next, we have:

$$\begin{aligned}\rho(\bar{z},\bar{w}) = \rho(gp_n^0, gp_n^l) &\leq \sum_{i=0}^{k-1} \rho(gp_n^i, gp_n^{i+1}) - \sum_{i=1}^{k-1} \rho(gp_n^i, gp_n^{i+1}) \\ &\leq \sum_{i=0}^{k-1} \rho(gp_n^i, gp_n^{i+1}) \\ &= \sum_{i=0}^{k-1} f_n^i \to 0 \text{ (as } n\to\infty).\end{aligned}$$

Hence, $\bar{z} = \bar{w}$, i.e., $Sz = Sw$. Thus, S and g have a unique point of coincidence.

Secondly, to justify the existence of a unique common fixed point, we consider $z \in Coin(S,g)$, i.e., $Sz = gz = \bar{z}$, for some $\bar{z} \in M$. By the condition (h), S and g commute at their coincidence points, i.e.,

$$S(gz) = g(Sz) = g(gz), \tag{25}$$

thereby yielding $S\bar{z} = g\bar{z}$, i.e., $\bar{z} \in Coin(S,g)$. Thus, by the uniqueness of point of the coincidence point, we have:

$$\bar{z} = g\bar{z} = S\bar{z}.$$

The uniqueness of the common fixed point is a direct consequence of the uniqueness of the coincidence point. This finishes the proof. □

We present the following example to support our result.

Example 1. *Let $M = [0,\infty)$ with partial metric $\rho : M \times M \to [0,\infty)$ defined by:*

$$\rho(z_1,z_2) = \max\{z_1,z_2\}.$$

Define a binary relation $\mathcal{R} = \{(z_1,z_2) \in M \times M : z_1 \geq z_2\}$. Clearly, (M,ρ) is a complete partial metric space. Define $S, g : M \to M$ by:

$$Sz = \frac{z}{3} \text{ and } gz = \frac{z}{2},\ \forall z \in M.$$

It is clear that $\mathcal{R}$ is (S,g)-closed and S and g are continuous. Next, define $\phi : [0,\infty) \to [0,\infty)$ by:

$$\phi(t) = \frac{t}{6},\ \forall t \in [0,\infty).$$

Clearly, $\phi \in \Phi$. Observe that all the conditions of Theorems 1 and 2 are fulfilled (with $N = M$). Hence, S and g have a unique common fixed point (namely 0).

Next, we present the following corollaries.

Corollary 1. *The conclusion of Theorem 2 remains valid if we replace the condition (g) by any one of the following:*

- $(g1)$ *$\mathcal{R}|_{g(M)}$ is complete;*
- $(g2)$ *$S(M)$ is $(S,g,\mathcal{R}^s)$-directed.*

Proof. If $(g1)$ holds true, then for any $z_1,z_2 \in S(M)$, we have $z_1 = gw_1$ and $z_2 = gw_2$, for some $w_1,w_2 \in M$ (as $S(M) \subseteq g(M)$). In view of $(g1)$, we have $[gw_1,gw_2] \in \mathcal{R}|_{g(M)}$, i.e., $\{gw_1,gw_2\}$ is a path of length 1 in $\mathcal{R}|^s_{g(M)}$ from z_1 to z_2. Hence, condition (g) of Theorem 2 is fulfilled and the result is concluded by Theorem 2.

On the other hand, if condition $(g2)$ holds, then for each $z_1, z_2 \in S(M)$ (such that $z_1 = gw_1$ and $z_2 = gw_2$, for $w_1, w_2 \in M$), there exists $w_3 \in M$ such that $[gw_1, gw_3]$, $[gw_2, gw_3] \in \mathcal{R}|_{g(M)}$, i.e., $\{gw_1, gw_3, gw_2\}$ is a path of length 2 in $\mathcal{R}|^s_{g(M)}$ from z_1 to z_2 and $[gw_3, Sw_3] \in \mathcal{R}|_{g(M)}$. Hence, condition (g) of Theorem 2 is fulfilled and again by Theorem 2, the conclusion follows. □

Corollary 2. *The conclusions of Theorems 1 and 2 remain true if we replace assumption (e) by the following one:*

$(e1)$ *S satisfies*

$$\rho(Sz, Sw) \leq \rho(gz, gw) - \phi(\rho(Sz, Sw)), \tag{26}$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$.

Proof. As $\rho(gz, gw) \leq \mathcal{M}_{\rho,g}(z, w)$, we have:

$$\rho(Sz, Sw) \leq \rho(gz, gw) - \phi(\rho(Sz, Sw)) \implies \rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)),$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$. Thus, all the assumptions of Theorems 1 and 2 are satisfied and the conclusions hold. □

Following Reference [32], it can be easily seen that in a partial metric space (M, ρ), for all $(gz, gw) \in \mathcal{R}^{\neq}$, the conditions:

$$\rho(Sz, Sw) \leq \rho(gz, gw) - \phi(\rho(Sz, Sw)), \tag{27}$$

and:

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)), \tag{28}$$

are more weaker than:

$$\rho(Sz, Sw) \leq \rho(gz, gw) - \phi(\rho(gz, gw)), \tag{29}$$

and:

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\mathcal{M}_{\rho,g}(z, w)), \tag{30}$$

respectively. However, the converse need not be true in general (even the above assertion is true for any $z, w \in M$). This leads us to our next corollary.

Corollary 3. *The conclusions of Theorems 1 and 2 remain true if we replace assumption (e) by the following one:*

$(e2)$ *S satisfies:*

$$\rho(Sz, Sw) \leq \rho(gz, gw) - \phi(\rho(gz, gw)), \tag{31}$$

or:

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\mathcal{M}_{\rho,g}(z, w)), \tag{32}$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$.

By setting $\phi(t) = (1 - k)t$, with $k \in [0, 1)$ and $t \in [0, \infty)$ in Corollary 3, we deduce the following corollaries:

Corollary 4. *The conclusions of Theorems 1 and 2 remain true if we replace assumption (e) with the following one:*

$(e3)$ *there exists $k \in [0, 1)$ such that:*

$$\rho(Sz, Sw) \leq k\rho(gz, gw),$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$.

We see that the above corollary is a relatively new and somewhat refined version of Alam and Imdad [31] type result in partial metric space with some refinement, e.g.:

- We use $\mathcal{R}^{\neq}$-precompleteness of subspace $N \subseteq M$ in place of $\mathcal{R}$-completeness.
- We use $\mathcal{R}^{\neq}$-analogous of compatibility, continuity, closedness and ρ-self closedness instead of their $\mathcal{R}$-analogous.

Corollary 5. *The conclusions of Theorems 1 and 2 remain true if we replace assumption (e) with the following one:*

$(e4)$ *S satisfies:*
$$\rho(Sz, Sw) \leq k\mathcal{M}_{\rho,g}(z, w), \tag{33}$$
for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$.

By considering $g = I_M$, the following fixed point result can be deduced easily from Theorems 1 and 2.

Corollary 6. *Let (M, ρ) be a partial metric space equipped with a binary relation $\mathcal{R}$, $N \subseteq M$ an $\mathcal{R}^{\neq}$-precomplete subspace in M and $S : M \to M$. Assume that the following assumptions are satisfied:*

(a) *There exists $z_0 \in M$ such that $(z_0, Sz_0) \in \mathcal{R}$;*
(b) *$\mathcal{R}$ is S-closed;*
(c) *$S(M) \subseteq N$;*
(d) *$\mathcal{R}$ is locally S-transitive;*
(e) *S satisfies generalized Ćirić-type weak $(\phi, \mathcal{R})$-contraction, i.e.:*

$$\rho(Sz, Sw) \leq M(z, w) - \phi(\rho(Sz, Sw)),$$

for all $z, w \in M$ with $(z, w) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$, where:

$$M(z, w) = \max\left\{\rho(z, w), \rho(z, Sz), \rho(w, Sw), \frac{\rho(z, Sw) + \rho(w, Sz)}{2}\right\};$$

(f) *either S is $\mathcal{R}^{\neq}$-continuous or $\mathcal{R}^{\neq}|_N$ is ρ-self closed.*

Then, S has a fixed point. In addition, if:

(g) *N is $(S, \mathcal{R}^s)$-connected,*

then the fixed point is unique.

In place of $\mathcal{R}^{\neq}$-precomplete of N, if we use the $\mathcal{R}^{\neq}$-completeness of the whole space M, then we find a particular version of Theorem 1.

Corollary 7. *Let $(M, \rho, \mathcal{R})$ be an $\mathcal{R}^{\neq}$-complete partial metric space and $S, g : M \to M$ satisfy the following assumptions:*

(a) *$M(g, S, \mathcal{R}) \neq \emptyset$;*
(b) *$\mathcal{R}$ is (S, g)-closed;*
(c) *$S(M) \subseteq g(M)$;*
(d) *$\mathcal{R}$ is locally S-transitive;*
(e) *S satisfies generalized Ćirić-type weak $(\phi_g, \mathcal{R})$-contraction, i.e.,:*

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)), \tag{34}$$

for all $z, w \in M$ with $(gz, gw) \in \mathcal{R}^{\neq}$ and $\phi \in \Phi$, where:

$$\mathcal{M}_{\rho,g}(z,w) = \max\left\{\rho(gz,gw), \rho(gz,Sz), \rho(gw,Sw), \frac{\rho(gz,Sw)+\rho(gw,Sz)}{2}\right\};$$

(f) ($f1$) *S and g are $\mathcal{R}^{\neq}$-compatible;*
($f2$) *S and g are $\mathcal{R}^{\neq}$-continuous;*
or alternatively:

(f^*) (f^*1) *there exists an $\mathcal{R}^{\neq}$-closed subspace N of M such that $S(M) \subseteq N \subseteq g(M)$;*
(f^*2) *either S is $\mathcal{R}^{\neq}$-g-continuous or S and g are continuous or $\mathcal{R}^{\neq}|_N$ is ρ-self closed.*

Then, S and g have a coincidence point.

Proof. The result follows by Proposition 3 and Remark 3. □

Moreover, in Corollary 7, if we assume g to be surjective, then assumption (c) as well as assumption (f^*1) can be removed trivially since $N = g(M) = M$.

5. Consequences

5.1. Results in Abstract Spaces

By considering $\mathcal{R}$ to be the universal relation, i.e., $\mathcal{R} = M \times M$, we deduce the following results from Theorems 1 and 2.

Corollary 8. *Let (M, ρ) be a partial metric space and $S, g : M \to M$. Assume that the following conditions are satisfied:*

(a) *$S(M) \subseteq g(M) \cap N$;*

(b) *S satisfies:*

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z,w) - \phi(\rho(Sz,Sw)),$$

for all $z, w \in M$ with $gz \neq gw$ and $\phi \in \Phi$;

(c) (c_1) *S and g are compatible;*
(c_2) *S and g are continuous;*
or alternatively:

(c^*) *$N \subseteq g(M)$.*

Then, S and g have a coincidence point.

Corollary 9. *Moreover, if S and g are weakly compatible, then S and g have a unique common fixed point.*

In view of Corollary 4 under $\mathcal{R} = M \times M$, it can be easily seen that Corollary 8 is a more generalized and sharpened version of Goebel and Jungck type results in partial metric spaces.

5.2. Results in Ordered Partial Metric Spaces via Increasing Mappings

The idea under consideration was initiated by Turinici [17], which was later generalized by several authors, e.g., Ran and Reurings [18], Nieto and Rodríguez-López [19], and some others, e.g., the authors of [34–37]. In this section, from now on, $\preceq$ denotes a partial order on a non-empty set M, $(M, \preceq)$ denotes a partially ordered set, and $(M, \rho, \preceq)$ stands for a partial metric space with partial order $\preceq$, which we call ordered partial metric space.

Now, we recall the following notions which are needed in the sequel.

Definition 19. *[38] A mapping $S : M \to M$ is said to be g-increasing if $Sz_1 \preceq Sz_2$, for any $z_1, z_2 \in M$ with $gz_1 \preceq gz_2$.*

Remark 5. *Notice that S is g-increasing and the notion $\preceq$ is (S,g)-closed in our sense coincide with each other.*

Definition 20. *[39] Let $\{z_n\}$ be a sequence in an ordered set $(M,\preceq)$. Then:*

(a) *$\{z_n\}$ is said to be increasing if for all $m,n \in \mathbb{N}_0$:*

$$m \leq n \implies z_m \preceq z_n.$$

(b) *$\{z_n\}$ is said to be decreasing if for all $m,n \in \mathbb{N}_0$:*

$$m \leq n \implies z_n \preceq z_m.$$

(c) *$\{z_n\}$ is said to be monotone if it is either increasing or decreasing.*

Now, we introduce the notion of increasing-convergence-upper bound (ICU) property in the setting of ordered partial metric spaces.

Definition 21. *Let $(M,\rho,\preceq)$ be an ordered partial metric space. We say that $(M,\rho,\preceq)$ has ICU (increasing-convergence-upper bound) property if every increasing sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$ is bounded above by limit, i.e., $z_n \preceq z$, for all $n \in \mathbb{N}$.*

Remark 6. *It is observed that $(M,\rho,\preceq)$ has ICU property is equivalent to $\preceq$ is ρ-self closed.*

Notice that Alam et al. [40] defined ICU property in the setting of ordered metric spaces.

Definition 22. *In an ordered partial metric space $(M,\rho,\preceq)$, we define the following:*

(a) *$(M,\rho,\preceq)$ is said to be $\overline{O}$-complete (resp. $\underline{O}$-complete, O-complete) if every increasing (resp. decreasing, monotone) Cauchy sequence in M converges in M.*

(b) *a self-mapping S on M is said to be $(g,\overline{O})$-continuous (resp. $(g,\underline{O})$-continuous, (g,O)-continuous) at $z \in M$, if for any increasing (resp. decreasing, monotone) sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$, we have $\{Sz_n\} \to Sz$.*
S is $(g,\overline{O})$-continuous (resp. $(g,\underline{O})$-continuous, (g,O)-continuous) on M if it is $(g,\overline{O})$-continuous (resp. $(g,\underline{O})$-continuous, (g,O)-continuous) at every $z \in M$.

(c) *two self-mappings S and g are said to be $\overline{O}$-compatible (resp. $\underline{O}$-compatible, O-compatible) if for any sequence $\{z_n\}$ and $z \in M$ such that $\{Sz_n\}$ and $\{gz_n\}$ are increasing (resp. decreasing and monotone) and $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} gz_n = z$, we have:*

$$\lim_{n\to\infty} \rho(S(gz_n), g(Sz_n)) = 0.$$

Remark 7. *Notice that for $g = I$, $(g,\overline{O})$-continuity reduces to $\overline{O}$-continuity, and the same happens to the others.*

The above notions were defined by Kutbi et al. [41] in the setting of ordered metric spaces. Now, we introduce the following notion.

Definition 23. *A subset N of an ordered partial metric space $(M,\rho,\preceq)$ is said to be $\overline{O}$-precomplete (resp. $\underline{O}$-precomplete, O-precomplete) if every increasing (resp. decreasing, monotone) Cauchy sequence in N converges to a point of M.*

Under consideration of Remarks 5 and 6 and $\mathcal{R} = \preceq$, we obtained the below result from Theorem 1, which is new for the existing literature.

Corollary 10. *Let $(M, \rho, \preceq)$ be an ordered partial metric space, $N \subseteq M$ an $\bar{O}$-precomplete subspace in M and $S, g : M \to M$. Assume that the following assumptions are satisfied:*

- (a) *There exists $z_0 \in M$ such that $gz_0 \preceq Sz_0$;*
- (b) *S is g-increasing;*
- (c) *$S(M) \subseteq g(M) \cap N$;*
- (d) *S satisfies generalized Ćirić-type weak $(\phi_g, \preceq)$-contraction, i.e.,*

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)), \tag{35}$$

for all $z, w \in M$ with $gz \preceq gw$ and $\phi \in \Phi$, where:

$$\mathcal{M}_{\rho,g}(z, w) = \max\left\{\rho(gz, gw), \rho(gz, Sx), \rho(gw, Sw), \frac{\rho(gz, Sw) + \rho(gw, Sz)}{2}\right\};$$

- (e) $(e1)$ *S and g are $\overline{O}$-compatible;*
  $(e2)$ *S and g are $\overline{O}$-continuous;*
  or alternatively:
- (e^*) (e^*1) *$N \subseteq g(M)$;*
  (e^*2) *either S is $(g, \overline{O})$-continuous or S and g are continuous or $(N, \rho, \preceq)$ has ICU property.*

Then, S and g have a coincidence point.

5.3. Results in Ordered Partial Metric Spaces via Comparable Mappings

Definition 24. *[42] For $S, g : M \to M$, S is said to be g-comparable if for all $z_1, z_2 \in M$ such that $gz_1 \prec\succ gz_2$, we have $Sz_1 \prec\succ Sz_2$.*

Remark 8. *Observe that the notion S is g-comparable is equivalent to saying that $\prec\succ$ is (S, g)-closed.*

Definition 25. *[43] Let $(M, \preceq)$ be an ordered set and $\{z_n\}$ a sequence in M. Then:*

- *(a)* *$\{z_n\}$ is said to be termwise bounded if there is an element $z \in M$ such that each term of $\{z_n\}$ is comparable with z, i.e., $z_n \prec\succ z$, for all $n \in \mathbb{N}_0$ and z is a c-bound of $\{z_n\}$.*
- *(b)* *$\{z_n\}$ is said to be termwise monotone if consecutive terms of $\{z_n\}$ are comparable, i.e., $z_n \prec\succ z_{n+1}$, for all $n \in \mathbb{N}_0$.*

Now, we introduce TCC property in the setting of ordered partial metric spaces.

Definition 26. *We say that an ordered partial metric space $(M, \rho, \preceq)$ has TCC property if every termwise monotone convergent sequence $\{z_n\}$ in M has a subsequence, which is termwise bounded by the limit (of the sequence) as a c-bound, i.e.:*

$$z_n \updownarrow z \implies \text{there exists a subsequence } \{z_{n_k}\} \text{ of } \{z_n\} \text{with } z_{n_k} \prec\succ z,\ \forall k \in \mathbb{N}_0.$$

Remark 9. *It is observed that $(M, \rho, \preceq)$ has TCC property which is equivalent to $\prec\succ$, which is ρ-self closed.*

In view of Remarks 8 and 9 and using $\mathcal{R} = \prec\succ$ in Theorem 1, we again obtained a new result for the existing literature.

Corollary 11. *Let $(M, \rho, \preceq)$ be an ordered partial metric space, $N \subseteq M$, an O-precomplete subspace in M and $S, g : M \to M$. Assume that the following assumptions are satisfied:*

- (a) *There exists $z_0 \in M$ such that $gz_0 \prec\succ Sz_0$;*

(b) *S is g-increasing;*

(c) $S(M) \subseteq g(M) \cap N$;

(d) *S satisfies generalized Ćirić-type weak $(\phi_g, \mathcal{R})$-contraction, i.e.:*

$$\rho(Sz, Sw) \leq \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)), \tag{36}$$

for all $z, w \in M$ with $gz \prec\succ gy$ and $\phi \in \Phi$, where;

$$\mathcal{M}_{\rho,g}(z, w) = \max\left\{\rho(gz, gw), \rho(gz, Sz), \rho(gw, Sw), \frac{\rho(gz, Sw) + \rho(gw, Sz)}{2}\right\};$$

(e) $(e1)$ *S and g are O-compatible;*
$(e2)$ *S and g are O-continuous;*
or alternatively:

(e^*) (e^*1) $N \subseteq g(M)$;
(e^*2) *either S is (g, O)-continuous or S and g are continuous or $(N, \rho, \preceq)$ has TCC property.*

Then, S and g have a coincidence point.

6. Application

Let us consider the following system of equations:

$$\begin{cases} z(t) &= \int_0^T K_1(t, \tau, z(\tau))d\tau + a(t); \\ z(t) &= \int_0^T K_2(t, \tau, z(\tau))d\tau + a(t), \end{cases} \tag{37}$$

for all $t \in \Omega = [0, T]$, $T > 0$, where $K_1, K_2 : \Omega \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $a : \Omega \to \mathbb{R}^n$.

Our aim is to provide an existence theorem in order to find the solution of the above system of integral equations using Theorem 1.

Let $\mathcal{R}$ be an arbitrary transitive binary relation on $\mathbb{R}^n$ and $M = \mathcal{C}(\Omega, \mathbb{R}^n)$, set of all continuous mappings from $\Omega \to \mathbb{R}^n$, with sup norm $\|z\|_M = \max_{t \in \Omega} \|z(t)\|$, $z \in M$. Consider a binary relation $\mathcal{R}_M$ on M as:

$$(z_1, z_2) \in \mathcal{R}_M \iff (z_1(t), z_2(t)) \in \mathcal{R},\ \forall t \in \Omega.$$

For any $\mathcal{R}_M$-preserving sequence $\{z_n\}$ in M converging to $z \in M$, we have $(z_n(t), z(t)) \in \mathcal{R}$, for all $t \in \Omega$. Further, define $S, g : M \to M$ by:

$$Sz(t) = \int_0^T K_1(t, \tau, z(\tau))d\tau + a(t) \text{ and } gz(t) = \int_0^T K_2(t, \tau, z(\tau))d\tau + a(t),$$

for all $t \in \Omega$, where g is surjective.

Theorem 3. *Suppose the following conditions are satisfied:*

(A) $K_1, K_2 : \Omega \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $a : \Omega \to \mathbb{R}^n$ *are continuous;*

(B) *There exists some $z_0 \in M$ such that:*

$$\left(\int_0^T K_2(t, \tau, z_0(\tau))d\tau + a(t), \int_0^T K_1(t, \tau, z_0(\tau))d\tau + a(t)\right) \in \mathcal{R},\ \forall t \in \Omega;$$

(C) $(gz(t), gw(t)) \in \mathcal{R} \implies (Sz(t), Sw(t)) \in \mathcal{R}, \forall t \in \Omega$;

(D) *For each $z, w \in M$ such that $(z, w) \in \mathcal{R}^{\neq}$ and $t, \tau \in \Omega$, there exists a number $\lambda \in [0, \frac{1}{T}]$ such that:*

$$\|K_1(t, \tau, z(\tau)) - K_1(t, \tau, w(\tau))\| \leq \lambda \|gz(t) - gw(t)\|.$$

Then, Equation (37) has a solution in M.

Proof. Define $\rho : M \times M \to [0, \infty)$ as:

$$\rho(z, w) = \|z - w\|_M, \ \forall z, w \in M.$$

Now, for $(z, w) \in \mathcal{R}^{\neq}$, we have:

$$\begin{aligned}
\rho(Sz, Sw) &= \max_{t\in\Omega} \left\| \int_0^T (K_1(t, \tau, z(\tau)) - K_1(t, \tau, w(\tau)))d\tau \right\| \\
&\le \max_{t\in\Omega} \int_0^T \|K_1(t, \tau, z(\tau)) - K_1(t, \tau, w(\tau))\| d\tau \\
&\le \lambda \max_{t\in\Omega} \|gz(t) - gw(t)\| \int_0^T d\tau \\
&= \lambda T \|gz - gw\|_M \\
&= \lambda_1 \rho(gz, gw),
\end{aligned}$$

where $\lambda_1 = \lambda T$. Now, define $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = (1 - \lambda_1)t$, $\lambda_1 \in [0, 1)$. It can be easily seen that $\phi \in \Phi$. Applying it in the above inequality, we obtain:

$$\begin{aligned}
\rho(Sz, Sw) &\le \rho(gz, gw) - \phi(\rho(gz, gw)) \\
&\le \rho(gz, gw) - \phi(\rho(Sz, Sw)) \\
&\le \mathcal{M}_{\rho,g}(z, w) - \phi(\rho(Sz, Sw)),
\end{aligned}$$

where $\mathcal{M}_{\rho,g}$ is as defined in Theorem 1. By choosing $N = M$, it is also clear that $S(M) \subseteq M = g(M)$. Hence, by fulfilling all the necessary requirements of Theorem 1, S and g have a coincidence point. Hence, the system (Equation (37)) has a solution. This completes the proof. □

7. Conclusions

Essentially, inspired by Alam and Imdad [21] and Zhiqun Xue [32], we introduced a new contraction condition and used the same to prove some new fixed point results in the setting of partial metric space. To establish our claim, we deduced some corollaries which are still new and refined versions of earlier known results in literature. Finally, by presenting an application, we exhibited the usability of our main result.

Author Contributions: All the authors have contributed equally in the preparation of this manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
2. Abdeljawad, T.; Karapınar, E.; Taş, K. Existence and uniqueness of a common fixed point on partial metric spaces. *Appl. Math. Lett.* **2011**, *24*, 1900–1904. [CrossRef]
3. Altun, I.; Erduran, A. Fixed point theorems for monotone mappings on partial metric spaces. *Fixed Point Theory Appl.* **2011**, *2011*, 508730. [CrossRef]
4. Amini-Harandi, A. Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.* **2012**, *2012*, 204. [CrossRef]
5. Aydi, H. Some coupled fixed point results on partial metric spaces. *Int. J. Math. Math. Sci.* **2011**, *2011*, 647091. [CrossRef]

6. Bejenaru, A.; Pitea, A. Fixed point and best proximity point theorems on partial metric spaces. *J. Math. Anal.* **2016**, *7*, 25–44.
7. Chi, K.P.; Karapınar, E.; Thanh, T.D. Generalized contraction principle in partial metric spaces. *Math. Comput. Model.* **2012**, *55*, 1673–1681. [CrossRef]
8. Chi, K.P.; Karapınar, E.; Thanh, T.D. On the fixed point theorems for generalized weakly contractive mappings on partial metric spaces. *Bull. Iran. Math. Soc.* **2013**, *39*, 269–381.
9. Karapınar, E. Generalizations of Caristi Kirk's theorem on partial metric spaces. *Fixed Point Theory Appl.* **2011**. [CrossRef]
10. Karapınar, E. A note on common fixed point theorems in partial metric spaces. *Miscolc Math. Notes* **2011**, 12, 185–191. [CrossRef]
11. Karapınar, E.; Sadarangani, K. Fixed point theory for cyclic (ϕ-ψ)-contractions. *Fixed Point Theory Appl.* **2011**. [CrossRef]
12. Karapınar, E.; Erhan, I.M. Cyclic contractions and fixed point theorems. *FILOMAT* **2012**, *26*, 777–782. [CrossRef]
13. Karapınar, E.; Shatanawi, W.; Tas, K. Fixed point theorem on partial metric spaces involving rational expressions. *Miskolc Math. Notes* **2013**, *14*, 135–142. [CrossRef]
14. Shatanawi, W.; Postolache, M. Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 54. [CrossRef]
15. Valero, O. On Banach fixed point theorems for partial metric spaces. *Appl. Gen. Topol.* **2005**, *2*, 229–240. [CrossRef]
16. Karapınar, E.; Romaguera, S. Nonunique fixed point theorems in partial metric spaces. *FILOMAT* **2013**, *27*, 1305–1314. [CrossRef]
17. Turinici, M. Fixed points for monotone iteratively local contractions. *Demonstr. Math.* **1986**, *19*, 171–180.
18. Ran, A.C.M.; Reurings, M.C.B. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
19. Nieto, J.J.; Rodríguez-López, R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **2005**, *22*, 223–239. [CrossRef]
20. Nieto, J.J.; Rodríguez-López, R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Math. Sin. Engl. Ser.* **2007**, *23*, 2205–2212. [CrossRef]
21. Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [CrossRef]
22. Ahmadullah, M.; Ali, J.; Imdad, M. Unified relation-theoretic metrical fixed point theorems under an implicit contractive condition with an application. *Fixed Point Theory Appl.* **2016**, *2016*, 42. [CrossRef]
23. Alam, A.; Imdad, M. Nonlinear contractions in metric spaces under locally T-transitive binary relations. *Fixed Point Theory* **2018**, *19*, 13–24. [CrossRef]
24. Imdad, M.; Khan, Q.H.; Alfaqih, W.M.; Gubran, R. A relation theoretic $(F, \mathcal{R})$-contraction principle with applications to matrix equations. *Bull. Math. Anal. Appl.* **2018**, *10*, 1–12.
25. Sawangsup, K.; Sintunavarat, W. Fixed point and multidimensional fixed point theorems with applications to nonlinear matrix equations in terms of weak altering distance functions. *Open Math.* **2017**, *15*, 111–125. [CrossRef]
26. Maddux, R.D. *Relation Algebras*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006; Volume 150.
27. Roldán, A.; Shahzad, N. Fixed point theorems by combining Jleli and Samet's and Branciari's inequalities. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3822–3849.
28. Shahzad, N.; Roldán-López-de-Hierro, A.F.; Khojasteh, F. Some new fixed point theorems under $(\mathcal{A}, \mathcal{S})$-contractivity conditions. *RACSAM* **2017**, *111*, 307–324. [CrossRef]
29. Samet, B.; Turinici, M. Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications. *Commun. Math. Anal.* **2012**, *13*, 82–97.
30. Kolman, B.; Busby, R.C.; Ross, S.C. *Discrete Mathematical Structures*; Prentice-Hall, Inc.: Upper Saddle River, NJ, USA, 2003.
31. Alam, A.; Imdad, M. Relation-theoretic metrical coincidence theorems. *FILOMAT* **2017**, *31*, 4421–4439. [CrossRef]
32. Xue, Z. The convergence of fixed point for a kind of weak contraction. *Nonlinear Funct. Anal. Appl.* **2016**, *21*, 497–500.

33. Popescu, O. Fixed points for (ψ, ϕ)-weak contractions. *Appl. Math. Lett.* **2011**, *24*, 1–4. [CrossRef]
34. Aydi, H. Some fixed point results in ordered partial metric spaces. *arXiv* **2011**, arXiv:1103.3680.
35. Aydi, H. Common fixed point results for mappings satisfying (ψ, ϕ)-weak contractions in ordered partial metric spaces. *Int. J. Math. Stat.* **2012**, *12*, 53–64.
36. Gülyaz, S.; Karapınar, E. A coupled fixed point result in partially ordered partial metric spaces through implicit function. *Hacet. J. Math. Stat.* **2013**, *42*, 347–357.
37. Roldán, A.; Martínez-Moreno, J.; Roldán, C.; Karapınar, E. Multidimensional fixed-point theorems in partially ordered complete partial metric spaces under (ψ, φ)-contractivity conditions. *Abstr. Appl. Anal.* **2013**, *2013*, 634371. [CrossRef]
38. Ćirić, L.B.; Cakić, N.; Rajović, M.; Ume, J.S. Monotone generalized nonlinear contractions in partially ordered metric spaces. *Fixed Point Theory Appl.* **2008**, *2008*, 131294. [CrossRef]
39. Turinici, M. Abstract comparison principles and multivariable Gronwall-Bellman inequalities. *J. Math. Anal. Appl.* **1986**, *117*, 100–127. [CrossRef]
40. Alam, A.; Khan, A.R.; Imdad, M. Some coincidence theorems for generalized nonlinear contractions in ordered metric spaces with applications. *Fixed Point Theory Appl.* **2014**, *2014*, 1–30. [CrossRef]
41. Kutbi, M.A.; Alam, A.; Imdad, M. Sharpening some core theorems of Nieto and Rodríguez-López with application to boundary value problem. *Fixed Point Theory Appl.* **2015**, *2015*, 198. [CrossRef]
42. Alam, A.; Imdad, M. Comparable linear contractions in ordered metric spaces. *Fixed Point Theory* **2017**, *18*, 415–432. [CrossRef]
43. Alam, A.; Imdad, M. Monotone generalized contractions in ordered metric spaces. *Bull. Korean Math. Soc.* **2016**, *53*, 61–81. [CrossRef]