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Fixed Point Theorems through Modified ω-Distance and Application to Nontrivial Equations

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Abstract: In this manuscript, we utilize the concept of modified ω-distance mapping, which was introduced by Alegre and Marin [Alegre, C.; Marin, J. Modified ω-distance on quasi metric spaces and fixed point theorems on complete quasi metric spaces. *Topol. Appl.* **2016**, *203*, 120–129] in 2016 to introduce the notions of (ω, φ)-Suzuki contraction and generalized (ω, φ)-Suzuki contraction. We employ these notions to prove some fixed point results. Moreover, we introduce an example to show the novelty of our results. Furthermore, we introduce some applications for our results.

Keywords: quasi metric space; Suzuki contractions; fixed point theorems; modified ω-distance; almost perfect functions

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1. Introduction and Preliminaries

Constructing new contractions and formulating new fixed point theorems are very important subjects in mathematics since active researchers employ the existence and uniqueness of the fixed point to solve some integral equations, differential equations, etc.

Banach was the first pioneer mathematician who constructed and formulated the first fixed point theorem, which was called after him as the Banach contraction principle [1].

Suzuki [2] introduced a new contraction and generalized the Banach contraction principle.

In the rest of this paper, the letter d refers to a metric on a set B and f_1 refers to self-mappings on B.

One of the important contractions is the Kannan contraction [3]:

$$d(f_1 l_1, f_1 l_2) \leq \alpha[d(l_1, f_1 l_1) + d(l_2, f_1 l_2)] \text{ for all } l_1, l_2 \in B,$$

where $\alpha \in [0, \frac{1}{2})$.

Moreover, Kannan proved that if f_1 satisfies Kannan contraction, then f_1 has a unique fixed point.

In 1931, Wilson [4] generalized the notion of metric spaces to a new notion called quasi metric spaces.

Definition 1. *We call $q : B \times B \to [0, \infty)$ a quasi metric if q satisfies:*

(i) $q(l_1, l_2) = 0 \Longleftrightarrow l_1 = l_2$
and:

(ii) $q(l_1, l_2) \leq q(l_1, l_3) + q(l_3, l_2)$ *for all* $l_1, l_2, l_3 \in B$.

(B, q) is called a quasi metric space.

From now on, by (B, q), we mean a quasi metric space.

Defining $q_m : B \times B \to [0, +\infty)$ via

$$q_m(l_1, l_2) = \max\{q(l_1, l_2), q(l_2, l_1)\},$$

we generate a metric on B.

Recall the following definitions.

Definition 2. *[5,6] The sequence (l_t) converges to $l \in B$ if $\lim_{t\to\infty} q(l_t, l) = \lim_{n\to\infty} q(l, l_t) = 0$.*

Definition 3. *[6] Let (l_t) be a sequence in (B, q). Then, we say that:*

(i) (l_t) *is left-Cauchy if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(l_t, l_m) < \varepsilon \; \forall \; t \geq m > n_0$.*

(ii) (l_t) *is right-Cauchy if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(l_t, l_m) < \varepsilon \; \forall \; m \geq t > n_0$.*

Definition 4. *[5,6] We say that (l_t) is Cauchy if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $q(l_t, l_m) \leq \epsilon \; \forall \; t, m > n_0$.*

We note that (l_t) in (B, q) is Cauchy if and only if (l_t) is right and left Cauchy.

Definition 5. *[5,6] We say that (B, q) is complete if every Cauchy sequence in B is convergent.*

For some theorems in quasi-metric space, see [5–9].

Alegre and Marin [10] introduced the concept of modified ω-distance mappings on (B, d).

Definition 6. *[10] A modified ω-distance (shortened as $m\omega$-distance) on (B, q) is a function $p : B \times B \to [0, \infty)$, which satisfies:*

(W1) $p(l_1, l_2) \leq p(l_1, l_3) + p(l_3, l_2)$ *for all* $l_1, l_2, l_3 \in B$;

(W2) $p(l, .) : B \to [0, \infty)$ *is lower semi-continuous for all* $l \in B$; *and*

(mW3) for each $\varepsilon > 0$, there exist $\nu > 0$ such that if $p(l_1, l_2) \leq \nu$ and $p(l_2, l_3) \leq \nu$, then $q(l_1, l_3) \leq \epsilon$ for all $l_1, l_2, l_3 \in B$.

Definition 7. *[10] We call an $m\omega$-distance function a p strong $m\omega$-distance if p is lower semi-continuous on its second coordinate.*

Remark 1. *[10] If q is a quasi metric on B, then q is $m\omega$-distance.*

Lemma 1. *[11] Let (α_t), (β_t) be two sequences of nonnegative real numbers converging to zero. Assume that p is $m\omega$-distance. Then, we have the following:*

(i) If $p(l_t, l_m) \leq \alpha_t$ for any $t, m \in \mathbb{N}$ with $m \geq t$, then (l_t) is right Cauchy in (B, q).

(ii) If $p(l_t, l_m) \leq \beta_m$ for any $t, m \in \mathbb{N}$ with $t \geq m$, then (l_t) is left Cauchy in (B, q).

Remark 2. *[11] The above lemma implies that if $\lim_{m,t\to\infty} p(l_t, l_m) = 0$, then (l_t) is Cauchy in (B, q).*

For some works on ω-distance, we ask the readers to see [11–13].

Abodayeh et al. [14] generalized the definition of altering the distance function [15] to the concept of the almost perfect function.

Definition 8. *We call a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ almost perfect if φ satisfies:*

(i) $\varphi(l) = 0$ if and only if $l = 0$.

(ii) If (l_t) is a sequence in $[0, \infty)$ such that $\lim_{t\to\infty} \varphi(l_t) = 0$, then $\lim_{t\to\infty} l_t = 0$.

2. Main Results

We begin our work with the following definition:

Definition 9. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an almost perfect function and p be modified ω-distance on B. We say that p is bounded with respect to φ if there exists an integer $A > 0$ such that:*

$$\varphi p(l,e) \leq A \textit{ for all } l,e \in B.$$

Definition 10. *Equip (B,q) with an mω-distance mapping p. Then, we call that $f_1 : B \to B$ an (ω,φ)-Suzuki contraction if there are an almost perfect function φ and a constant $k \in [0,1)$ such that for all $l,e \in X$ and $t \in \mathbb{N}$, we have:*

$$(1-k)p\left(l, {f_1}^t l\right) \leq p(l,e) \Longrightarrow \varphi p(f_1 l, f_1 e) \leq k\varphi p(l,e),$$

and:

$$(1-k)p\left({f_1}^t l, l\right) \leq p(e,l) \Longrightarrow \varphi p(f_1 e, f_1 l) \leq k\varphi p(e,l).$$

Now, we introduce and prove our first result.

Theorem 1. *Equip (B,q) with an mω-distance mapping p. Let p be bounded with respect to the almost perfect function φ and f_1 be an (ω,φ)-Suzuki contraction mapping. Suppose that:*

(i) *f_1 is continuous,*
or
(ii) *if $u^* \in B$ and $u^* \neq f_1 u^*$, then:*

$$\inf\left\{p\left(e,u^*\right) + p\left(f_1 e, u^*\right) : e \in B\right\} > 0. \tag{1}$$

Then, f_1 has a unique fixed point in B.

Proof. By starting with $l_0 \in B$, we produce a sequence (l_t) in B inductively by putting $l_{t+1} = f_1 l_t$ for all $t \in \mathbb{N} \cup \{0\}$. Given $m,t \in \mathbb{N} \cup \{0\}$ with $m > t$, then $m = t+s$ for some $s \in \mathbb{N}$. From the definition, we have:

$$\begin{aligned}(1-k)p(l_{t-1}, l_{m-1}) &= (1-k)p(l_{t-1}, l_{t+s-1}) \\ &\leq p(l_{t-1}, l_{t+s-1}).\end{aligned}$$

Therefore, we get that:

$$\begin{aligned}\varphi p(l_t, l_m) &= \varphi p(f_1 l_{t-1}, {f_1}^s l_{t-1}) \\ &= \varphi p(f_1 l_{t-1}, f_1 l_{t+s-1}) \\ &\leq k\varphi p(l_{t-1}, l_{t+s-1}).\end{aligned} \tag{2}$$

Repeating (2) t-times, we get that:

$$\varphi p(l_t, l_m) \leq k^t \varphi p(l_0, l_s). \tag{3}$$

Since (B,p) is bounded with respect to φ, then we have:

$$\varphi p(l_t, l_m) \leq k^t A \text{ for some integer } A > 0. \tag{4}$$

By letting $t,m \to \infty$, we get that:

$$\lim_{t,m\to\infty} \varphi p(l_t, l_m) = 0. \tag{5}$$

By the definition of φ, we get that:

$$\lim_{t,m\to\infty} p(l_t, l_m) = 0. \tag{6}$$

Since $m > t$, Lemma 1 implies that (l_t) is right Cauchy. Now, suppose that $t, m \in \mathbb{N} \cup \{0\}$ with $t > m$. Then, $t = m + q$ for some $q \in \mathbb{N}$. We note that:

$$(1-k)p(l_{t-1}, l_{m-1}) \quad \leq p(l_{m+q-1}, l_{m-1}).$$

Therefore, we get that:

$$\begin{aligned} \varphi p(l_t, l_m) &= \varphi p(f_1 l_{t-1}, f_1 x_{m-1}) \\ &\leq \cdots \leq k^m \varphi p(l_q, l_0) \end{aligned} \tag{7}$$

$$\varphi p(l_t, l_m) \leq k^m \varphi p(l_q, l_0). \tag{8}$$

Since (B, p) is bounded with respect to φ, we get that:

$$\varphi p(l_n, l_m) \leq k^m A \text{ for some integer } A > 0. \tag{9}$$

By letting $t, m \to \infty$, we have:

$$\lim_{t,m\to\infty} \varphi p(l_n, l_m) = 0. \tag{10}$$

Therefore,

$$\lim_{t,m\to\infty} p(l_t, l_m) = 0. \tag{11}$$

Since $t > m$, Lemma 1 implies that (l_t) is left Cauchy. Therefore, we deduce that (l_t) is Cauchy. The completeness of (B, q) implies that there exists an element $l^* \in B$ such that $l_t \to l^*$. If f_1 is continuous, then $l_{t+1} = f_1 l_t$ converges to $f_1 l^*$. The uniqueness of the limit ensures that $f_1 l^* = l^*$. Let $\epsilon > 0$. Since $\lim_{t,m\to\infty} p(l_t, l_m) = 0$, we choose $k_0 \in \mathbb{N}$ such that $p(l_t, l_m) \leq \frac{\epsilon}{2}$ for all $l, m \geq k_0$. The lower semi continuity of p implies that:

$$p(l_t, l^*) \leq \lim_{j\to\infty} \inf p(l_t, l_j) \leq \frac{\epsilon}{2} \text{ for all } n \geq k_0.$$

Assume that $l^* \neq f_1 l^*$. Then, by (1), we have:

$$\begin{aligned} \inf\{p(e, l^*) + p(f_1 e, l^*) : e \in B\} &\leq \inf\{p(l_t, l^*) + p(f_1 l_t, l^*) : t \in \mathbb{N}\} \\ &= \inf\{p(l_t, l^*) + p(l_{t+1}, l^*) : t \in \mathbb{N}\} \leq \epsilon, \end{aligned}$$

a contradiction. Therefore, $l^* = f_1 l^*$. Now, assume that $z^* \in B$ is a fixed point of f_1. Therefore:

$$(1-k)p(z^*, {f_1}^t z^*) = (1-k)p(z^*, z^*) \leq p(z^*, z^*).$$

Thus,

$$\varphi p(z^*, z^*) = \varphi p(f_1 z^*, f_1 z^*) \leq k \varphi p(z^*, z^*).$$

Since $k < 1$ and φ is an almost perfect function, we conclude that $p(z^*, z^*) = 0$. Assume that there exists $v^* \in B$ such that $v^* = f_1 v^*$. Since $p(z^*, z^*) = 0$, we have:

$$(1-k)p(z^*, f_1^t z^*) = (1-k)p(z^*, z^*) \leq p(z^*, v^*).$$

Therefore,

$$\varphi p(z^*, v^*) = \varphi p(f_1 z^*, f_1 v^*) \leq k \varphi p(z^*, v^*).$$

Thus, we have $\varphi p(z^*, v^*) = 0$, and so, $p(z^*, v^*) = 0$. Hence, by (mW3), we have $q(z^*, v^*) = 0$. Thus, $v^* = z^*$. Therefore, the fixed point of f_1 is unique. $\square$

Corollary 1. *Equip (B, q) with an mω-distance mapping p. Assume p is bounded with respect to φ. Assume for all $e, l \in B$, we have:*

$$\varphi p(f_1 e, f_1 l) \leq k\varphi(p(e, l)), \text{ where } k \in [0, 1)\,. \tag{12}$$

Furthermore, assume that:

(i) *f_1 is continuous,*
or
(ii) *if $u^* \in B$ and $u^* \neq f_1 u^*$, then:*

$$\inf \{p\,(e, u^*) + p\,(f_1 e, u^*) : e \in B\} > 0.$$

Then, f_1 has a unique fixed point in B.

By taking the almost perfect function φ in Corollary 1 as follows:
$\varphi(e) = e$, we get the following result:

Corollary 2. *Equip (B, q) with an mω-distance mapping p. Assume there exists $A > 0$ such that $p(e, l) \leq A$ for all $e, l \in B$. Furthermore, assume that there exists $k \in [0, 1)$ such that for all $e, l \in B$, we have:*

$$p(f_1 e, f_1 l) \leq kp(e, l), \text{ where } k \in [0, 1)\,.$$

Furthermore, assume that:

(i) *f_1 is continuous,*
or
(ii) *if $u^* \in B$ and $u^* \neq f_1 u^*$, then:*

$$\inf \{p\,(e, u^*) + p\,(f_1 e, u^*) : e \in B\} > 0.$$

Then, f_1 has a unique fixed point in B.

Example 1. *Let $B = \{0, 1, 2, \cdots, n\}$, where $n \in \mathbb{N}$. Define $p, q : B \times B \to [0, +\infty)$ as follows:*

$$q(e, l) = \begin{cases} 0 & \text{if } e = l; \\ 3e + l & \text{if } e \neq l, \end{cases}$$

and:

$$p(e, l) = \begin{cases} 0 & \text{if } e = l; \\ \frac{1}{2}(3e + l) & \text{if } e \neq l. \end{cases}$$

Furthermore, define $f_1 : B \to B$ by:

$$f_1 e = \begin{cases} 0 & \text{if } e = 0, 1; \\ 1 & \text{if } e = 2, 3, \cdots, n, \end{cases}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by:

$$\varphi(l) = \begin{cases} 3^l - 1 & \text{if } l \in [0, n]; \\ 3^l & \text{if } l > n. \end{cases}$$

Then,

1. φ *is an almost perfect function.*
2. p *is an mω-distance function on* q.
3. q *is a quasi metric on* B.
4. (B, q) *is complete.*
5. f_1 *satisfies* (ω, φ)*-Suzuki contraction with* $k = \frac{1}{\sqrt{3}}$*, i.e.,* $\forall e, l \in B$, $j \in \mathbb{N}$*, we have:*

$$\left(1 - \frac{1}{\sqrt{3}}\right) p\left(e, f_1^j e\right) \leq p(e, l) \Longrightarrow \varphi p(f_1 e, f_1 l) \leq k \varphi p(e, l),$$

and:

$$\left(1 - \frac{1}{\sqrt{3}}\right) p\left(f_1^j e, e\right) \leq p(l, e) \Longrightarrow \varphi p(f_1 l, f_1 e) \leq k \varphi p(l, e).$$

Proof. The proofs of (1), (2), and (3) are obvious. To show that q is complete, let (l_t) be a Cauchy sequence in B. Then, for each $t, m \in \mathbb{N}$, we have:

$$\lim_{m,t \to \infty} q(l_t, l_m) = 0.$$

Therefore, we deduce that $l_t = l_m$ for all $t, m \in \{0, 1, 2, \cdots\}$, but possible for finitely many. Thus, (l_t) converges in B. Hence, (B, q) is complete. To prove (5), given $e, l \in B$, we divide our proof into the following cases: Case (1): $e = 0$. Here, we have:

$$\left(1 - \frac{1}{\sqrt{3}}\right) p(0, 0) = \left(1 - \frac{1}{\sqrt{3}}\right) p(e, f_1^j e) \leq p(0, l) \text{ where } l = 0, 1, \cdots, n.$$

If $l \in \{0, 1\}$, then:

$$\varphi p(f_1 0, f_1 l) = \varphi p(0, 0) = 0 \leq \left(\frac{1}{\sqrt{3}}\right) \varphi p(0, l).$$

If $l \in \{2, 3, \cdots, n\}$, then:

$$\varphi p(f_1 0, f_1 l) = \varphi p(0, 1) = \varphi\left(\frac{1}{2}\right) = 3^{\frac{1}{2}} - 1.$$

Therefore,

$$\varphi p(0, l) = \varphi\left(\frac{l}{2}\right) = 3^{\frac{l}{2}} - 1.$$

$$\varphi p(f_1 0, f_1 l) = 3^{\frac{1}{2}} - 1 \leq \left(\frac{1}{\sqrt{3}}\right)\left(3^{\frac{l}{2}} - 1\right).$$

Case (2): $e = 1$. Here:

$$\left(1 - \frac{1}{\sqrt{3}}\right) p(e, 0) = \left(1 - \frac{1}{\sqrt{3}}\right) p(1, f_1 1) \leq p(1, l) \text{ where } l = 0, 2, 3, \cdots, n.$$

If $l = 0$, then we have $\varphi p(f1, fl) = 0$. Therefore,

$$\varphi p(f1, fl) = 0 \leq \left(\frac{1}{\sqrt{3}}\right)\left(3^{\frac{3}{2}} - 1\right).$$

If $l = 2, 3, \cdots, n$, then:

$$\varphi p(f_1 1, f_1 l) = \varphi p(0, 1) = \varphi\left(\frac{1}{2}\right) = 3^{\frac{1}{2}} - 1.$$

Now,

$$\varphi p(1,l) = \varphi\left(\frac{3+l}{2}\right) = 3^{\frac{3+l}{2}} - 1.$$

Thus,

$$\varphi p(f_1 1, f_1 l) = 3^{\frac{1}{2}} - 1 \leq \left(\frac{1}{\sqrt{3}}\right)\varphi\left(\frac{3+l}{2}\right) = \left(\frac{1}{\sqrt{3}}\right)\left(3^{\frac{3+l}{2}} - 1\right).$$

Case (3): $e \in \{2,3,\cdots,n\}$. Here,

$$\left(1-\frac{1}{\sqrt{3}}\right)p(e,1) = \left(1-\frac{1}{\sqrt{3}}\right)p(e,f_1 e) \leq p(e,l) \text{ where } l = 1,2,\cdots,n.$$

If $l = 1$, then:

$$\varphi p(fe, f1) = \varphi p(1,0) = \varphi\left(\frac{3}{2}\right) = 3^{\frac{3}{2}} - 1.$$

$$\varphi p(e,1) = \varphi\left(\frac{3e+1}{2}\right) = \begin{cases} 3^{\frac{7}{2}} - 1 & \text{if } e = 2 \\ 3^{\frac{3e+1}{2}} & \text{if } 3 \leq e \leq n. \end{cases}$$

$$\varphi p(fe, f1) = 3^{\frac{3}{2}} - 1 \leq (\frac{1}{\sqrt{3}})\varphi p(e,1).$$

If $l \in \{2,3,\cdots,n\}$, $e \in \{2,3,\cdots,n\}$ and $e \neq l$, then:

$$\varphi p(f_1 e, f_1 l) = \varphi p(1,1) = \varphi(0) = 0.$$

$$\varphi p(e,l) = \varphi\left(\frac{3e+l}{2}\right) = \begin{cases} 3^{\frac{3e+l}{2}} - 1 & \text{if } 3e+l \leq 2n \\ 3^{\frac{3e+l}{2}} & \text{if } 3e+l > 2n. \end{cases}$$

Similarly, we can show that:

$$(1-\frac{1}{\sqrt{3}})p\left(f^t e, e\right) \leq p(l,e) \Longrightarrow \varphi p(fl, fe) \leq k\varphi p(l,e).$$

Hence, f_1 satisfies (ω,φ)-Suzuki contraction. Therefore, f_1 has a unique fixed point. □

Next, we introduce the definition of a generalized (ω,φ)-Suzuki contraction.

Definition 11. *Equip (B,q) with an $m\omega$-distance mapping p. We call $f_1 : B \to B$ a generalized (ω,φ)-Suzuki contraction if there exists an ultra distance function φ and a constant $k \in [0,1)$ such that for all $e,l \in B$, $j \in \mathbb{N}$, we have:*

$$(1-k)p\left(e, f_1^j e\right) \leq p(e,l) \Longrightarrow \varphi p(f_1 e, f_1 l) \leq k \max\{\varphi p(e, f_1 e), \varphi p(l, f_1 l)\},$$

and:

$$(1-k)p\left(f_1^j e, e\right) \leq p(l,e) \Longrightarrow \varphi p(f_1 l, f_1 e) \leq k \max\{\varphi p(e, f_1 e), \varphi p(l, f_1 l)\}.$$

We introduce and prove the second result:

Theorem 2. *Equip (X,q) with an $m\omega$-distance mapping p. Assume that p is bounded with respect to the almost perfect function φ. Assume that f_1 is a generalized (ω,φ)-Suzuki contraction mapping. Furthermore, suppose that:*

(i) f_1 *is continuous,*
or

(ii) *if $u^* \in B$ and $u^* \neq f_1 u^*$, then:*

$$\inf\left\{p\left(e,u^*\right) + p\left(f_1 e, u^*\right) : e \in B\right\} > 0. \tag{13}$$

Then, f_1 has a unique fixed point in B.

Proof. Start with $l_0 \in B$ to construct (l_n) in B inductively by putting $l_{t+1} = f_1 l_t$ for all $t \in \mathbb{N} \cup \{0\}$. Given $t, m \in \mathbb{N} \cup \{0\}$ with $t < m$, let $m = t + j$ with $j \in \mathbb{N}$. We note that:

$$\begin{aligned}(1-k)p(l_{t-1}, l_{m-1}) &= (1-k)p(l_{t-1}, f_1^j l_{t-1}) \\ &\leq p(l_{t-1}, l_{m-1}).\end{aligned}$$

Since f_1 is a generalized (ω, φ)-Suzuki contraction, we have:

$$\begin{aligned}\varphi p(l_t, l_m) &= \varphi p(f_1 l_{t-1}, f_1 l_{m-1}) \\ &\leq k \max\{\varphi p(l_{t-1}, f_1 l_{t-1}), \varphi p(l_{m-1}, f_1 l_{m-1})\} \\ &= k \max\{\varphi p(l_{t-1}, l_t)), \varphi p(l_{m-1}, l_m)\}.\end{aligned} \tag{14}$$

Now,

$$\begin{aligned}(1-k)p(l_{t-2}, l_{t-1}) &= (1-k)p(l_{t-2}, f_1 l_{t-2}) \\ &\leq p(l_{t-2}, l_{t-1}).\end{aligned}$$

Therefore, we get that:

$$\begin{aligned}\varphi p(l_{t-1}, l_t) &= \varphi p(f_1 l_{t-2}, f_1 l_{t-1}) \\ &\leq k \max\{\varphi p(l_{t-2}, f_1 l_{t-2}), \varphi p(l_{t-1}, f_1 l_{t-1})\} \\ &= k \max\{\varphi p(l_{t-2}, l_{t-1}), \varphi p(l_{t-1}, l_t)\}.\end{aligned} \tag{15}$$

Since $k < 1$, we get that:

$$\varphi p(l_{t-1}, l_t) \leq k \varphi p(l_{t-2}, l_{t-1}). \tag{16}$$

Repeating (16) t-times, we get that:

$$\varphi p(l_{t-1}, l_t) \leq k^{t-1} \varphi p(l_0, l_1). \tag{17}$$

Similarly, we get that that:

$$\varphi p(l_{m-1}, l_m) \leq k^{m-1} \varphi p(l_0, l_1). \tag{18}$$

Using Equations (14), (17), and (18), we get:

$$\varphi p(l_t, l_m) \leq k \max\{k^{t-1} \varphi p(l_0, l_1), k^{m-1} \varphi p(l_0, l_1)\}. \tag{19}$$

Since $t < m$, we get that:

$$\varphi p(l_t, l_m) \leq k^t \varphi p(l_0, l_1). \tag{20}$$

The boundedness property of p with respect to φ implies that:

$$\varphi p(l_t, l_m) \leq k^t A \text{ for some integer } A \geq 0. \tag{21}$$

By letting $t, m \to \infty$, we get that:

$$\lim_{t,m \to \infty} \varphi p(l_t, l_m) = 0. \tag{22}$$

Thus,

$$\lim_{t,m \to \infty} p(l_t, l_m) = 0. \tag{23}$$

Since $t < m$, Lemma 1 implies that (l_t) is right Cauchy. In a similar manner, we can show that (l_t) is left Cauchy. Hence, (l_t) is Cauchy. The completeness of q ensures that there exists $l^* \in B$ such that

(l_t) converges to l^*. If f_1 is continuous, then $(l_{t+1}) = (f_1 l_t)$ converges to $f_1 l^*$. The uniqueness of the limit implies that $f_1 l^* = l^*$. Given $\varepsilon > 0$. Since $\lim_{t,m\to\infty} p(l_t, l_m)) = 0$, there exists $n_0 \in \mathbb{N}$ such that $p(l_t, l_m) \leq \frac{\epsilon}{2}$ for all $t, m \geq n_0$. The lower semi continuity of p implies that:

$$p(l_t, l^*) \leq \lim_{i\to\infty} \inf p(l_t, l_i) \leq \frac{\varepsilon}{2} \text{ for all } t \geq n_0.$$

Assume that $l^* \neq f_1 l^*$, then by (13), we have:

$$\begin{aligned} &\inf\{p(e, l^*) + p(f_1 e, l^*) : e \in B\} \\ &\qquad \leq \inf\{p(l_m, l^*) + p(f_1 l_t, l^*) : t \in \mathbb{N}\} \\ &\qquad = \inf\{p(l_t, l^*) + p(l_{t+1}, l^*) : n \in \mathbb{N}\} \leq \varepsilon, \end{aligned}$$

a contradiction. Therefore, $l^* = f_1 l^*$. Assume $z^* \in B$ such that $f_1 z^* = z^*$. First, we prove that $p(z^*, z^*) = 0$. Since:

$$(1-k)p(z^*, f_1^j z^*) = (1-k)p(z^*, z^*) \leq p(z^*, z^*),$$

then:

$$\varphi p(z^*, z^*) = \varphi p(f_1 z^*, f_1 z^*) \leq k\varphi p(z^*, z^*).$$

Since $k < 1$ and φ is an almost perfect function, then $p(z^*, z^*) = 0$. Therefore,

$$(1-k)p(z^*, f_1{}^t z^*) = (1-k)p(z^*, z^*) \leq p(z^*, l^*).$$

Therefore,

$$\begin{aligned} \varphi p(z^*, l^*) &= \varphi p(f_1 z^*, f_1 l^*) \\ &\leq k \max\{\varphi p(z^*, f_1 z^*), \varphi p(l^*, f_1 l^*)\} \\ &= k \max\{\varphi(p(z^*, z^*)), \varphi(p(v^*, v^*))\} \\ &= 0. \end{aligned}$$

The definition of φ informs us that $p(z^*, l^*) = 0$. The definition of p implies that $q(z^*, l^*) = 0$. Hence: $z^* = l^*$. □

Corollary 3. *Equip (B, q) with an mω-distance mapping p. Assume p is bounded with respect to the almost perfect function φ. Suppose that for all $e, l \in B$, we have:*

$$\varphi p(f_1 e, f_1 l) \leq k \max\{\varphi p(e, f_1 e), \varphi p(l, f_1 l)\}, \text{ where } k \in [0, 1)\,. \tag{24}$$

Furthermore, assume that:

(i) *f is continuous;*
or
(ii) *if $u^* \in B$ and $u^* \neq f_1 u^*$, then:*

$$\inf\{p(e, u^*) + p(f_1 e, u^*) : e \in B\} > 0.$$

Then, f_1 has a unique fixed point in B.

Corollary 4. *Equip (B, q) with an mω-distance mapping p. Assume that there exists $A > 0$ such that $p(e, l) \leq A$ for all $e, l \in B$. Furthermore, assume that for all $e, l \in B$, we have:*

$$p(f_1 e, f_1 l) \leq \alpha(p(e, f_1 e) + p(l, f_1 l)), \text{ where } 0 \leq \alpha < \frac{1}{2}.$$

Assume that:

(i) f_1 *is continuous;*
or

(ii) *if* $u^* \in B$ *and* $u^* \neq f_1 u^*$, *then:*

$$\inf\{p(e,u^*) + p(f_1 e, u^*) : e \in B\} > 0.$$

Then, f_1 *has a unique fixed point in B.*

Proof. Define the almost perfect function φ via $\varphi(e) = e$ in Corollary 3. Then:

$$\begin{aligned}
\varphi(p(f_1 e, f_1 l)) &= p(f_1 e, f_1 l) \\
&\leq \lambda\big(p(e, f_1 e) + p(l, f_1 l)\big) \\
&\leq 2\lambda \max\big\{p(e, f_1 e), p(l, f_1)\big\} \\
&= 2\lambda \max\big\{\varphi(p(e, f_1 e)), \varphi(p(l, f_1 l))\big\}.
\end{aligned}$$

□

3. Application

In this section, we utilize Corollaries 1 and 4 to give some applications of our work.

Theorem 3. *For any positive integer n, the equation:*

$$nx^n - x^{n-1} + 4nx - 2 = 0$$

has a unique solution in [0, 1].

Proof. Let $B = [0,1]$. Define $q : B \times B \to \mathbb{R}^+$ by $q(x,y) = |x - y|$. Then, (B,q) is a complete quasi metric space. Furthermore, define $p : B \times B \to [0,\infty)$ by $p(x,y) = |x-y|$. Then, p is an $m\omega$-distance mapping. Now, equip (B,q) with p.
Define $f_1 : B \to B$ by:

$$f_1(x) = \frac{x^{n-1}+2}{n(x^{n-1}+4)}.$$

Furthermore, define $\varphi : [0,\infty) \to [0,\infty)$ by:

$$\varphi(a) = \begin{cases} a^2 & \text{if } a \in [0,1]; \\ a^2 + \frac{1}{2} & \text{if } a > 1. \end{cases}$$

Note that φ is an almost perfect function and p is bounded with respect to φ. For $x, y \in B$, we have:

$$\begin{aligned}
\varphi p(f_1 x, f_1 y) &= \frac{1}{n^2}\left|\frac{x^{n-1}+2}{x^{n-1}+4} - \frac{y^{n-1}+2}{y^{n-1}+4}\right|^2 \\
&= \frac{1}{n^2}\left|\frac{2x^{n-1} - 2y^{n-1}}{(x^{n-1}+4)(y^{n-1}+4)}\right|^2 \\
&\leq \frac{4(n-1)^2}{n^2}\left(\frac{1}{(x^2+4)^2(y^2+4)^2}\right)\Big|x-y\Big|^2 \\
&\leq \frac{(n-1)^2}{64n^2}\Big|x-y\Big|^2 \\
&= \frac{(n-1)^2}{64n^2}\varphi p(x,y).
\end{aligned}$$

By taking $k = \frac{(n-1)^2}{64n^2}$ and noting that f_1 is continuous, we conclude that f_1 satisfies all conditions of Corollary 1. Thus, f_1 has a unique fixed point. Note that the unique fixed point of f_1 is the unique solution of:

$$nx^n - x^{n-1} + 4nx - 2 = 0.$$

□

Example 2. *The equation:*

$$1000x^{1000} - x^{999} + 4000x - 2 = 0$$

has a unique solution in [0, 1].

Proof. It follows from Theorem 3 by taking $n = 1000$. □

Let Y be the set of non-decreasing functions $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ such that τ is Lebesgue integrable for all compact sets in $\mathbb{R}^+$ and:

$$\int_0^{\mu} \tau(\nu)d\nu > 0 \text{ where } \mu > 0.$$

Theorem 4. *Equip (B, q) with an $m\omega$-distance mapping p. Assume that there exists $A > 0$ such that $p(e, l) \leq A$ for all $e, l \in B$. Furthermore, suppose the following condition:*

(i) f_1 *is continuous.*
(ii) *There exists $\tau \in Y$ and $\alpha \in [0, 1/2)$ such that for all $e, l \in B$, we have:*

$$\int_0^{p(f_1e, f_1l)} \tau(\nu)d\nu \leq \alpha\left(\int_0^{p(e, f_1e)} \tau(\nu)d\nu + \int_0^{p(l, f_1l)} \tau(\nu)d\nu\right).$$

Then, f_1 has a unique fixed point in B.

Proof. Let $\varphi = \int_0^t \tau(\nu)d\nu$. Then, φ is an almost perfect function. Corollary 4 ensures that f_1 has a unique fixed point in B. □

4. Conclusions

The notions of (ω, φ)-Suzuki contraction and generalized (ω, φ)-Suzuki contraction are introduced. According to these nations many fixed point results are investigated. Some applications are introduced on the obtained results.

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