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Complete Controllability Conditions for Linear Singularly Perturbed Time-Invariant Systems with Multiple Delays via Chang-Type Transformation

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Abstract: The problem of complete controllability of a linear time-invariant singularly-perturbed system with multiple commensurate non-small delays in the slow state variables is considered. An approach to the time-scale separation of the original singularly-perturbed system by means of Chang-type non-degenerate transformation, generalized for the system with delay, is used. Sufficient conditions for complete controllability of the singularly-perturbed system with delay are obtained. The conditions do not depend on a singularity parameter and are valid for all its sufficiently small values. The conditions have a parametric rank form and are expressed in terms of the controllability conditions of two systems of a lower dimension than the original one: the degenerate system and the boundary layer system.

Keywords: time delay system; multiple commensurate delays; singular perturbation; decomposition; Chang transformation; complete controllability; robust sufficient condition

1. Introduction

We consider a singularly-perturbed linear time-invariant system with a small multiplier for the derivatives and with non-small commensurate delays in the slow state variables (SPLTISD).

Singularly-perturbed controlled systems (SPS) occur as models in automatic control theory, nonlinear oscillation theory, quantum mechanics, gas dynamics, biology, chemical kinetics, and others (see, e.g., the references in [1–3]). Time-delay systems arise from inherent time delays in the components of the systems or from the deliberate introduction of time delays into the systems for control purposes. Time delays occur often in systems in engineering, biology, chemistry, physics, ecology, economics, technology, the social sphere, etc. (see, e.g., the references in [4,5]).

Controllability is one of the basic properties of controlled systems. This property is well known from the mathematical theory of systems (Kalman) as the concept of the reachability of terminal states. This means that it is possible to control a dynamic system from an arbitrary initial state to an arbitrary final state using a set of admissible controls.

Time-delay systems can be represented by delay differential equations, which belong to the class of functional differential equations and are infinite-dimensional systems [6,7]. Due to this fact, the controllability concepts for systems with delay are more diverse, and their analysis is significantly more complicated than for systems without delay. Different types of controllability for infinite-dimensional systems were studied in the literature (see, e.g., [8–10] and the references therein). There are several concepts of time delay system controllability that are a direct generalization of the concept of controllability in the Kalman theory of systems: relative controllability (equivalently, R^n or Euclidean-space controllability) (rank criterion [11]), complete controllability (formulation of the problem: Krasovsky N.N., 1963; condition [12]), and pointwise controllability [13,14]).

Controllability in the full state space (approximate, functional controllability [15,16]) is quite a restrictive concept [16]. For instance, many delay systems do not satisfy one of the necessary conditions of approximate controllability, even though they often possess other good properties such as stabilizability and spectral controllability. This suggests that from the controllability point of view, the full state space is "too big", and in the above-mentioned cases of controllability, the state (unlike the classical works of Kalman) is not based on the concept of minimality. Therefore, one should search for a "smaller space" in which a controllability would be characterized by less restrictive conditions and would be related to stabilizability and spectral controllability. The concept of F-controllability, weaker than the approximate controllability in the state space, has been introduced (see [16] and the references therein). Works on the study of systems with delay on the basis of an approach of the space of minimal states also appeared (see [17] and the references therein).

The functional controllability of systems with delay according to the approach, based on the notion of time delay systems' state minimality [17], turns out to be equivalent to the complete controllability, which solves one of the most difficult problems of controllability for systems with delay: the problem of complete damping of such systems in a finite time (total quieting, controllability to null function, null controllability). The spectral criterion of complete controllability for a system with delay (without parameter) was proven in [12,16,18] (see also the references in [17]). In [12,15,19] and a number of other papers, these conditions were associated with parametric rank Kalman-type conditions.

Studying the controllability of SPS without delay is well known (see, e.g., the reviews of [1,20] and the references therein). The controllability of SPS with delay has been studied much less (see [21–27], reviews [1,20], and the references therein).

To check a proper type of controllability for a given SPS, corresponding controllability conditions can be directly applied for any specified value of a small parameter of singular perturbation. However, the stiffness, as well as a possible high dimension of the SPS, can considerably complicate this application. Therefore, for example, with the direct use of rank controllability criteria [12,16,18], the controllability matrix of such systems has a large dimension and is ill-conditioned. Correct checking of the rank of such matrices can be complicated. Moreover, such an application depends on the value of the parameter, i.e., it is not robust with respect to this parameter, while in most of real-life problems, this value is unknown [28].

One of the approaches independent of the singular perturbation parameter controllability analysis of SPS is based on the time scale separation concept (see, e.g., [29]). Using this concept, the complete controllability of SPS without delays was analyzed in the works [29–31]. Parameter-free conditions of complete Euclidean space controllability, robust with respect to the small parameter, were obtained for linear singularly-perturbed time-invariant system with a single pointwise non-small delay in the state variables in [21], for linear SPS with point-wise and distributed small delays (of the order of the small parameter) in the state variables in [25–27,32,33], and for a linear singularly-perturbed neutral-type system with a single non-small point-wise delay in [34].

The problem of functional controllability for SPS with delay has been investigated much less. In [28], a singularly-perturbed linear time-dependent controlled system with multiple point-wise and distributed state delays was considered (the delays were small in the fast state variable and non-small in the slow state variable). It has been established that the approximate state-space controllability of two parameter-free subsystems (the slow and fast ones), associated with the original system, yields the approximate state-space controllability of the original system robustly with respect to the parameter of singular perturbation for all its sufficiently small values. In [35], the conditions of controllability in the $L_2^2[t_1 - h, t_1] \times R^2$ space for linear stationary SPS with delay was obtained on the basis of the state-space method.

One of the realizations of the time scale separation concept is Chang's transformation with a nondegenerate change of variables (for linear singularly-perturbed continuous-time varying systems without delay, introduced in [36]). Generalizations of Chang-type transformation for linear SPS slowly varying in time were proposed in [37,38] and on systems with many time scales in [39]. For SPS with delay, Chang-type transformations were constructed in [21,23,40,41]. In [40], the existence of a continuous function on the small parameter linear transformation for partial decomposition of SPS with distributed and concentrated non-small delays in slow and fast variables was proven. As a result, in the transformed system, there is a connection between fast and slow variables only through variables with delay. Constructed in [21,23], Chang-type transformation for a linear stationary SPS with a constant (non-small) delay in the state was performed by a linear operator with a finite number of delay operators and resulted in the original SPS with one delay in the state and without delay in the control to split the subsystems: slow with many delays in the state and control and heterogeneity depending on the initial conditions and fast with one delay in the state variable and with heterogeneity. In [41], the change of variables for linear time-invariant SPS of functional-differential equations with small concentrated and distributed delay in fast variables was constructed. The transformation in [42] generalizes the transformation in [36] to linear stationary SPS with concentrated delay in the slow variable. Unlike [21,23], this transformation was constructed in the form of an asymptotic series, and the obtained slow and fast subsystems did not contain inhomogeneities with the exception of control components.

In [29,43] and other papers, Chang's transformation was applied to split the original system with fast and slow variables into two independent subsystems and to obtain controllability conditions for SPS without delay. The result of the non-degenerate transformation [21,23] was used to study the relative controllability of the original SPS with delay. In [44], the sufficient conditions for complete controllability based on Chang's transformation [42] of linear stationary SPS with the single delay were obtained (without detailed proof).

In this paper, the problem of complete controllability of a linear time-invariant singularly-perturbed system with multiple commensurate non-small delays in the slow state variables on the basis of the time scale separation concept is considered. The main differences of this work from [28] are in the property under investigation (complete controllability) and in the method used for the investigation (the method of non-degenerate variable transformation is evolved in this work). The non-degenerate change of variables was developed in [42], where decoupling transformation in the form of asymptotic series was constructed for a singular perturbed system with single non-small delay. The exact separation is performed by means of non-degenerate transformation of the original system. Two much simpler subsystems than the original SPS parameter-free ones are associated with the original system. They are $O(\mu)$, close to the decoupled subsystems, the slow and fast ones. It is established that the complete controllability of the slow and fast subsystems yields the complete controllability of the original system.

Parameter-free sufficient conditions of complete controllability of the singularly-perturbed system with non-small delay are obtained. The conditions are valid for all sufficiently small values of the parameter of singular perturbation, i.e., robustly with respect to this parameter, and have a rank form.

The paper is organized as follows. In the second section, the problem is formulated, the main definitions dependent on the parameter criterion of complete controllability of the considered system are presented. Criteria of complete controllability for the fast and slow subsystems, associated with the original one, are presented in Section 3. Section 4 is devoted to Chang-type decoupling transformation in the form of asymptotic series for singular perturbed systems with non-small delay. Section 5 is devoted to the main result. An illustrative example is presented in Section 6. The discussion and conclusions are placed in Sections 7 and 8, respectively.

The following main notations and notions are applied in the paper:

- $'$ means the transposition;
- I_n denotes the identity matrix of dimension n;
- C is the set of complex numbers;
- R is the set of real numbers;
- $p \triangleq \frac{d}{dt}$ is a differential operator;

- e^{-ph} is a delay operator: $e^{-ph}z(t) \triangleq z(t-h)$.

2. Singularly-Perturbed Linear Time-Invariant System with Delays and Its Complete Controllability: Definitions

2.1. Singularly-Perturbed Linear Time-Invariant System with Delays

Consider the singularly-perturbed linear time-invariant system with multiple commensurate delays in the slow state variables (SPLTISD):

$$\dot{x}(t) = \sum_{j=0}^{l} A_{1j}x(t-jh) + A_2y(t) + B_1u(t),\ x \in R^{n_1},\ y \in R^{n_2}, \tag{1}$$

$$\mu\dot{y}(t) = \sum_{j=0}^{l} A_{3j}x(t-jh) + A_4y(t) + B_2u(t),\ u \in R^{r},\ t \geq 0, \tag{2}$$

with the initial conditions:

$$x(0) = x_0,\ y(0) = y_0,\ x_0 \in R^{n_1},\ y_0 \in R^{n_2},\ x(\theta) = \varphi(\theta), \theta \in [-lh, 0). \tag{3}$$

Here, $0 < h$ is a given constant, x is a slow variable, y is a fast variable, u is a control, $u(t) \in U$, U is a set of piecewise continuous $t \geq 0$ vector functions, μ is a small parameter, $\mu \in (0, \mu^0], \mu^0 \ll 1$, $A_{ij}, i = 1,3, j = \overline{0,l}, A_k, k = 2,4, B_j, j = 1,2$, are constant matrices of appropriate dimensions, and $\phi(\theta)$, $\theta \in [-lh, 0)$, is a piecewise continuous n_1 vector function. Assume that det $A_4 \neq 0$.

For a given $\mu > 0$, using the notations $p \triangleq \frac{d}{dt}$, a differential operator, e^{-ph}, and a delay operator, $e^{-ph}z(t) \triangleq z(t-h)$, introduce the following matrix-valued operators that depend on the parameter:

$$\mathsf{A}\left(\mu, e^{-ph}\right) = \begin{pmatrix} \mathsf{A}_1\left(e^{-ph}\right) & A_2 \\ \frac{\mathsf{A}_3\left(e^{-ph}\right)}{\mu} & \frac{A_4}{\mu} \end{pmatrix}, \quad B(\mu) = \begin{pmatrix} B_1 \\ \frac{B_2}{\mu} \end{pmatrix}, \quad \mu > 0, \tag{4}$$

where:

$$\mathsf{A}_\mathsf{i}\left(e^{-ph}\right) \triangleq \sum_{j=0}^{l} A_{ij}e^{-jph},\ i = 1,3. \tag{5}$$

Introduce also the vector $z = (x', y')'$. Using the above notations, we can rewrite SPLTISD (1)–(2) in the equivalent operator form:

$$pz(t) = \mathsf{A}\left(\mu, e^{-ph}\right)z(t) + B(\mu)u(t),\ u \in R^r,\ t \geq 0. \tag{6}$$

From (6), we obtain the characteristic equation of the system (1)–(2):

$$w\left(\mu, \lambda, e^{-\lambda h}\right) \triangleq \det\left[\lambda I_{n_1+n_2} - \mathsf{A}\left(\mu, e^{-\lambda h}\right)\right] = 0. \tag{7}$$

For any fixed $\mu \in (0, \mu^0]$, by:

$$\sigma(\mu) = \left\{\lambda(\mu) \in \mathsf{C} : w\left(\mu, \lambda, e^{-\lambda h}\right) = 0\right\} \tag{8}$$

denote the spectrum (set of the eigenvalues) of the SPLTISD (1).

From the known properties of the delay system spectrum [6,7] follows the characterization of the SPLTISD spectrum (8) for any given $\mu \in (0, \mu^0]$.

Characterization 1. *For a given $\mu \in (0, \mu^0]$, the following statements are true:*

(a) *the spectrum $\sigma(\mu)$ of the SPLTISD (1)–(2) consists of a finite or countable set of complex numbers;*
(b) *the real part of all SPLTISD (1)–(2) eigenvalues is bounded above by some real value γ;*
(c) *any vertical strip of the complex plane with $a \le Re\, z \le b$ contains a finite number of SPLTISD eigenvalues;*
(d) *any two subsets of the set $\sigma(\mu)$ are separated on the complex plane by a vertical strip of nonzero width.*

2.2. Definition and Dependent on the μ Criterion of Split Complete Controllability

Similar to [12], let us introduce the following definition.

Definition 1. *For a given $\mu \in (0, \mu^0]$, the SPLTISD (1)–(2) is said to be completely controllable if for any fixed initial conditions (3), there exist a time moment $t_1 < +\infty$ $(t_1 > (n_1 + n_2)h)$ and a piecewise continuous control $u(t)$, $t \in [0, t_1]$, such that for this control and the corresponding solution $(x(t,\mu), y(t,\mu))$, $t \ge 0$, of the system (1)–(2) with the initial conditions (3), the following identities are valid:*

$$(x(t,\mu), y(t,\mu)) \equiv 0,\ u(t) \equiv 0,\ t \ge t_1.$$

Definition 2. *If there exists a number $\mu^* > 0$ for which SPLTISD (1)–(2) is completely controllable for any $\mu \in (0, \mu^*]$, we say that complete controllability is robust with respect to $\mu \in (0, \mu^*]$.*

For $\mu > 0, \lambda \in C$, we introduce the following matrix-valued function:

$$\mathsf{N}\left(\mu, \lambda, e^{-\lambda h}\right) \triangleq \left[\lambda I_{n_1+n_2} - \mathsf{A}\left(\mu, e^{-\lambda h}\right), B(\mu)\right]. \tag{9}$$

The following criterion of the SPLTISD complete controllability for a fixed $\mu \in (0, \mu^0]$ follows from [12].

Theorem 1. *For a given $\mu \in (0, \mu^0]$, the SPLTISD (1)–(2) is completely controllable if and only if:*

$$rank\, \mathsf{N}\left(\mu, \lambda, e^{-\lambda h}\right) = n_1 + n_2, \quad \forall \lambda \in \sigma(\mu). \tag{10}$$

2.3. Objective of the Paper

Our objective in this paper is the following. On the basis of a non-degenerate change of variables in the original system, we prove the approximation of the original SPLTISD (1)–(2) by two independent small parameter subsystems of lower dimension and obtain parametric rank-type sufficient conditions for complete controllability of the original singularly-perturbed system in terms of the complete controllability of these subsystems. The conditions do not depend on the parameter and are robust with respect to μ for all its sufficiently small values.

3. Subsystems of SPLTISD

3.1. Slow and Fast Subsystems of SPLTISD

With the $n_1 + n_2$-dimensional system (1)–(3) is associated two independent μ subsystems: the slow and the fast ones. The slow subsystem, the degenerate system (DS), has the form:

$$\dot{x}_s(t) = \sum_{j=0}^{l} A_{sj} x_s(t - jh) + B_s u_s(t),\ x_s \in R^{n_1},\ t \ge 0, \tag{11}$$

$$x_s(0) = x_0, x_s(\theta) = \phi(\theta), \theta \in [-lh, 0), \tag{12}$$

where:

$$A_{sj} \triangleq A_{1j} - A_2 A_4^{-1} A_{3j},\ j = \overline{0,l},\ B_s \triangleq B_1 - A_2 A_4^{-1} B_2, \tag{13}$$

and $u_s(t) \in R^r$ (u_s is a control). Introducing the following matrix-valued operator:

$$\mathsf{A}_s\left(e^{-ph}\right) \triangleq \mathsf{A}_1\left(e^{-ph}\right) - A_2 A_4^{-1} \mathsf{A}_3\left(e^{-ph}\right), \tag{14}$$

we can rewrite the degenerate system (11) in the operator form:

$$px_s(t) = \mathsf{A}_s\left(e^{-ph}\right) x_s(t) + B_s u_s(t), \; x_s \in R^{n_1}, \; t \geq 0, \tag{15}$$

$$x_s(0) = x_0, x_s(\theta) = \phi(\theta), \theta \in [-lh, 0). \tag{16}$$

The degenerate system (11) is a linear stationary n_1-dimensional system with multiple commensurate delays. It is obtained from (1)–(2) by setting there formally $\mu = 0$, expressing $y_s(t) = A_4^{-1}\left[\mathsf{A}_3\left(e^{-ph}\right) x_s(t) + B_2 u_s(t)\right]$ from (2) and substituting it into (1).

The characteristic equation for the DS (11) is:

$$w_s\left(\lambda, e^{-\lambda h}\right) \triangleq \det\left[\lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right)\right] = 0, \tag{17}$$

the spectrum of the DS (11):

$$\sigma_s = \left\{\lambda \in C : w_s\left(\lambda, e^{-\lambda h}\right) = 0\right\} \tag{18}$$

is a finite or countable set of complex numbers.

Since the DS (11) is a system with delay, properties similar to the properties from the characterization 1 are valid for the spectrum (18).

The fast subsystem, the boundary layer system (BLS), has the form:

$$\frac{dy_f(\tau)}{d\tau} = A_4 y_f(\tau) + B_2 u_f(\tau), \; y_f \in R^{n_2}, \; \tau = \frac{t}{\mu} \geq 0, \tag{19}$$

$$y_f(0) = y_0 - A_4^{-1}\left[\mathsf{A}_3\left(e^{-\lambda h}\right)\phi(0) + B_2 u_s(0)\right]. \tag{20}$$

Here, $y_f(\tau) = y(\mu\tau) - y_s(\mu\tau)$, $u_f(\tau) = u(\mu\tau) - u_s(\mu\tau)$.

The boundary layer system (19) is a linear stationary n_2-dimensional system without delay and is derived from Equation (2) for the fast state variable y in the following way: (i) the terms containing the slow state variable x are removed from Equation (2); (ii) the transformation of variables $t = \mu\tau$, $y(\mu\tau) = y_f(\tau)$, $u(\mu\tau) = u_f(\tau)$ is done in the resulting equation, where τ, y_f and u_f are new independent variables (the stretched time), state and control, respectively.

The characteristic equation for the BLS (19) is:

$$w_f(\lambda) \triangleq \det\left[\lambda I_{n_2} - A_4\right] = 0. \tag{21}$$

The spectrum of the BLS (19) is the finite set of complex numbers:

$$\sigma_f = \left\{\lambda \in C : w_f(\lambda) = 0\right\}. \tag{22}$$

Similar to [12,45], we introduce the following definitions.

Definition 3. *The DS (11) is said to be completely controllable if for any fixed initial conditions (16), there exists a time moment $t_1 < +\infty$ and a piecewise continuous control $u_s(t)$, $t \in [0, t_1]$ such that for this control and corresponding solution $x_s(t), t \geq 0$, of the system (11) with the initial conditions (16), the following identities are true:*

$$x_s(t) \equiv 0, \; u_s(t) \equiv 0, \; t \geq t_1.$$

Definition 4. *The BLS (19) is said to be completely controllable if for any fixed initial conditions (20), there exist a time moment* $\tau_1 < +\infty$ *and a piecewise continuous control* $u_f(\tau)$, $\tau \in [0, \tau_1]$ *such that for this control and corresponding solution* $y_f(\tau), \tau \geq 0$, *of the system (14) with the initial conditions (20), the following equality is true:*

$$y_f(\tau_1) = 0.$$

Note that the subsystems (11) and (19) have smaller dimensions than the original SPLTISD (1)–(2) and do not depend on a small parameter μ.

The main objective of the article is to obtain the conditions of complete controllability of the SPLTISD (1)–(2) (Definition 1) in terms of complete controllability of its subsystems (11) and (19) (Definitions 3 and 4), robust with respect to μ for all its sufficiently small values (Definition 2).

3.2. Controllability of Subsystems

Define the matrix-valued functions:

$$\mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) \triangleq \left[\lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right), B_s\right],\ \lambda \in C, \tag{23}$$

$$N_f(\lambda) \triangleq \left[\lambda I_{n_2} - A_4, B_2\right],\ \lambda \in C. \tag{24}$$

Applying the conditions of complete controllability from [12] to DS (11) and BLS (19), we obtain that the following theorems are valid.

Theorem 2. *The DS (11) is completely controllable if and only if the following condition is valid:*

$$rank\, \mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) = n_1 \quad \forall \lambda \in \sigma_s. \tag{25}$$

Theorem 3. *The BLS (19) is completely controllable if and only if the following condition is valid:*

$$rank\, N_f(\lambda) = n_2 \quad \forall \lambda \in \sigma_f. \tag{26}$$

Along with Conditions (25) and (26), we formulate some more applicable conditions for the complete controllability of the subsystems, which simplify the procedure for checking this property. To do this, we define the matrix-valued function:

$$\mathsf{P}_\mathsf{s}(z) \triangleq \left[B_s, \mathsf{A}_\mathsf{s}(z) B_s, ..., \mathsf{A}_\mathsf{s}{}^{n_1-1}(z) B_s\right],\ z \in C, \tag{27}$$

and the matrix:

$$\mathsf{P}_\mathsf{f} \triangleq \left[B_2, \mathsf{A}_4 B_2, ..., \mathsf{A}_4{}^{n_2-1} B_2\right]. \tag{28}$$

The following theorem follows from the application to the subsystems (11) and (19) of the results from [19] about the connection of the ranks of matrices (27) and (23).

Theorem 4. *Let for some* $\lambda \in C$:

$$rank\ \mathsf{P}_s\left(e^{-\lambda h}\right) = n_1. \tag{29}$$

Then, for this λ:

$$rank\, \mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) = n_1. \tag{30}$$

If we apply the well-known criterion of the controllability of a linear stationary system [45] to the BLS (19), we obtain the following theorem:

Theorem 5. *The following equality:*

$$rank\,\mathsf{N}_f(\lambda) = n_2\ \forall\lambda \in C \tag{31}$$

holds if and only if:

$$rank\,\mathsf{P}_f = n_2. \tag{32}$$

In order to formulate the conditions of subsystems' controllability, that do not require the computation of all eigenvalues from (18) and (22), let us define the following set of complex numbers:

$$Z_s = \{z \in C:\ rank\,\mathsf{P_s}(z) < n_1\} \tag{33}$$

the set of numbers for which the matrix $\mathsf{P_s}(z)$ does not have full rank by rows.

Let $\pi_s(z)$ be the greatest common divisor of all minors of order n_1 of the function matrix $P_s(z)$ (27). Taking into account (5), (14), (13), and (27), we have that $\pi_s(z)$ is a polynomial of degree l_s no higher than n_1 and can be represented as:

$$\pi_s(z) = \sum_{i=0}^{l_s} k_i z,\ k_{l_s} \neq 0. \tag{34}$$

Since the set Z_s (33) coincides with the set of all roots of the polynomial $\pi_s(z)$, then:

$$Z_s = \{z \in C : \pi_s(z) = 0\}$$

and the set of Z_s contains the finite numbers of elements.

Along with (33), let us define the set of complex numbers:

$$\Lambda_s = \left\{\lambda \in C : e^{-\lambda h} = z,\, z \in Z_s\right\}, \tag{35}$$

associated with Z_s. The elements $z \in Z_s$, $\lambda_k \in \Lambda_s$, are connected by the relation:

$$h\,\lambda_k = \ln|z| + i\,(\arg z + 2\pi k)\,,\ z \in Z_s,\ i = \sqrt{-1},\ k = 0, \pm 1, \pm 2, \ldots. \tag{36}$$

Since the set Z_s consists of a finite number of elements, from the connection (36) between the sets Λ_s and Z_s, the validity follows:

Characterization 2. *There are real numbers $\alpha, \gamma,\ \alpha < \gamma$, such that $\alpha < Re\,\lambda \leq \gamma\ \forall\lambda \in \Lambda_s$.*

Define also the set:

$$\Omega_s \triangleq \sigma_s \cap \Lambda_s.$$

By virtue of the connection (36) between the elements of the sets (35) and (33) from Theorem 4, it follows that the conditions (25) of complete controllability of DS (11) are sufficient to check only for $\lambda \in \Omega_s$, and the following theorem holds [19].

Theorem 6. *Let:*

(1) there exists $z \in C$ that:

$$rank\,\mathsf{P}_s(z) = n_1; \tag{37}$$

(2)

$$rank\,\mathsf{N}_s(\lambda, z) = n_1\ \forall\lambda \in \Omega_s,\ z \in Z_s. \tag{38}$$

Then, the DS (11) is completely controllable.

From [45] follows:

Theorem 7. *The BLS (19) is completely controllable if and only if the following condition is satisfied:*

$$rank\, P_f = n_2. \tag{39}$$

4. Decoupling Transformation for the SPLTISD

Similar to [42] for asymptotic decomposition of the SPLTISD, we introduce the change of variables:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathsf{G}\left(\mu, e^{-ph}\right) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}, \quad \xi(t) \in R^{n_1},\ \eta(t) \in R^{n_2},\ t \in T, \tag{40}$$

$$\mathsf{G}\left(\mu, e^{-ph}\right) = \begin{pmatrix} I_{n_1} & \mu\,\mathsf{H}\left(\mu, e^{-ph}\right) \\ -\mathsf{L}\left(\mu, e^{-ph}\right) & I_{n_2} - \mu\,\mathsf{L}\left(\mu, e^{-ph}\right)\mathsf{H}\left(\mu, e^{-ph}\right) \end{pmatrix}, \tag{41}$$

where $\mathsf{H}\left(\mu, e^{-ph}\right)$ and $\mathsf{L}\left(\mu, e^{-ph}\right)$ are the matrix-valued operators, depending on the parameter μ. They are the solutions of the following matrix-valued functional equations (in order to reduce the records, where this does not lead to ambiguous understanding, we omit the arguments $\left(\mu, e^{-\lambda h}\right)$ of the matrix-valued operators $\mathsf{H}\left(\mu, e^{-ph}\right)$, $\mathsf{L}\left(\mu, e^{-ph}\right)$):

$$\begin{aligned} &\mathsf{A}_3\left(e^{-ph}\right) - A_4\mathsf{L} + \mu\,\mathsf{L}\,\mathsf{A}_1\left(e^{-ph}\right) - \mu\,\mathsf{L}\,A_2\mathsf{L} = 0, \\ &\mu\left(\mathsf{A}_1\left(e^{-ph}\right) - A_2\mathsf{L}\right)\mathsf{H} - \mathsf{H}\left(A_4 + \mu\,\mathsf{L}\,A_2\right) + A_2 = 0. \end{aligned} \tag{42}$$

Notice that:

$$\det\,\mathsf{G}\left(\mu, e^{-ph}\right) \equiv 1, \quad \mathsf{G}^{-1}\left(\mu, e^{-ph}\right) = \begin{pmatrix} I_{n_1} - \mu\,\mathsf{HL} & -\mu\,\mathsf{H} \\ \mathsf{L} & I_{n_2} \end{pmatrix}. \tag{43}$$

By $O(\mu)$, we denote any vector function $f(t, \mu)$, $t \in [t_1, t_2]$, with the following property: there exist positive constants μ^* and c such that the Euclidean norm $|f(t, \mu)|$ satisfies the inequality $|f(t, \mu)| \le c\mu$ for all $\mu \in (0, \mu^*]$ and all $t \in [t_1, t_2]$.

Lemma 1. *Suppose that* $\det A_4 \neq 0$*. Then, there exists a* $\mu^* > 0$ *such that for all* $\mu \in [0, \mu^*]$*, there is a continuous function depending on the* μ *solution* $\mathsf{L}\left(\mu, e^{-ph}\right)$*,* $\mathsf{H}\left(\mu, e^{-ph}\right)$ *of Equation (42) that could be represented in asymptotic series form:*

$$\begin{aligned} \mathsf{L}\left(\mu, e^{-ph}\right) &= \sum_{i=0}^{k} \mu^i \mathsf{L}^i\left(e^{-ph}\right) + O\left(\mu^{k+1}\right), \\ \mathsf{H}\left(\mu, e^{-ph}\right) &= \sum_{i=0}^{k} \mu^i \mathsf{H}^i\left(e^{-ph}\right) + O\left(\mu^{k}\right), \end{aligned} \tag{44}$$

where:

$$\begin{aligned} &\mathsf{L}^0\left(e^{-ph}\right) = A_4^{-1}\mathsf{A}_3\left(e^{-ph}\right),\ \mathsf{L}^1\left(e^{-ph}\right) = A_4^{-2}\mathsf{A}_3\left(e^{-ph}\right)\mathsf{A}_0\left(e^{-ph}\right), \\ &\mathsf{A}_0\left(e^{-ph}\right) = \mathsf{A}_1\left(e^{-ph}\right) - A_2A_4^{-1}\mathsf{A}_3\left(e^{-ph}\right),\ H^0 = A_2A_4^{-1}. \end{aligned} \tag{45}$$

Proof. It is easy to prove the decomposition (44) according to the scheme of the proof of [29]. Continuity is proven as in [40]. For the SPLTISD with a simple delay, see [42]. □

The next corollary follows from Lemma 1 if we substitute (44) into (42) and equate the coefficients of equal powers of μ in the resulting equations.

Corollary 1. *Let* $\det A_4 \neq 0$*. A solution of matrix equations (42) can be found with any degree of accuracy in the form of (44), where terms of the asymptotic series (44) can be found according to the following iterative scheme (in order to reduce the records, we omit the arguments* $e^{-\lambda h}$ *of the matrix-valued operators* $\mathsf{L}^k\left(e^{-ph}\right)$, $\mathsf{H}^k\left(e^{-ph}\right)$*):*

$$\begin{aligned} &\mathsf{L}^{k+1} = A_4^{-1}\left(\mathsf{L}^k\mathsf{A}_1\left(e^{-ph}\right) - \sum_{j=0}^{k}\mathsf{L}^{k-j}A_2\mathsf{L}^j\right),\ \mathsf{L}^0\left(e^{-ph}\right) = A_4^{-1}\mathsf{A}_3\left(e^{-ph}\right),\\ &\mathsf{H}^{k+1} = A_4^{-1}\left(\mathsf{A}_1\left(e^{-ph}\right)\mathsf{H}^k - A_2\sum_{i=0}^{k}\mathsf{L}^i\mathsf{H}^{k-i} - \sum_{i=0}^{k}\mathsf{H}^i\mathsf{L}^{k-i}A_2\right),\ \mathsf{H}^0 = A_2A_4^{-1}. \end{aligned} \tag{46}$$

By using the SPLTISD (1)–(2) matrix parameters and the matrix-valued functions $\mathsf{L}\left(\mu, e^{-ph}\right)$, $\mathsf{H}\left(\mu, e^{-ph}\right)$, we introduce the matrix-valued functions:

$$\begin{aligned} &\mathsf{A}_\xi\left(\mu, e^{-ph}\right) \triangleq \mathsf{A}_1\left(e^{-ph}\right) - A_2\mathsf{L}\left(\mu, e^{-ph}\right),\\ &\mathsf{B}_\xi\left(\mu, e^{-ph}\right) \triangleq B_1 - \mathsf{H}\left(\mu, e^{-ph}\right)B_2 - \mu\mathsf{H}\left(\mu, e^{-ph}\right)\mathsf{L}\left(\mu, e^{-ph}\right)B_1,\\ &\mathsf{A}_\eta\left(\mu, e^{-ph}\right) \triangleq A_4 + \mu\,\mathsf{L}\left(\mu, e^{-ph}\right)A_2,\\ &\mathsf{B}_\eta\left(\mu, e^{-ph}\right) \triangleq B_2 + \mu\mathsf{L}\left(\mu, e^{-ph}\right)B_1. \end{aligned} \tag{47}$$

Note here that due to Lemma 1, similar to [40], it is easy to prove that matrices from (47) continuously depend on μ for $[0, \mu^*]$.

From (14), (13), (44), and (45), we have:

$$\begin{aligned} \mathsf{A}_\xi\left(\mu, e^{-ph}\right) &= \mathsf{A}_s\left(e^{-\lambda h}\right) + O\left(\mu\right),\\ \mathsf{B}_\xi\left(\mu, e^{-ph}\right) &= B_s + O\left(\mu\right),\\ \mathsf{A}_\eta\left(\mu, e^{-ph}\right) &= A_4 + O\left(\mu\right),\\ \mathsf{B}_\eta\left(\mu, e^{-ph}\right) &= B_2 + O\left(\mu\right). \end{aligned} \tag{48}$$

As a result of the application to the system (1)–(2) of the transformation (40), taking into account (43) and (47), the SPLTISD (1)–(2) goes into the equivalent system with separated motions:

$$\dot{\xi}\left(t\right) = \mathsf{A}_\xi\left(\mu, e^{-ph}\right)\xi\left(t\right) + \mathsf{B}_\xi\left(\mu, e^{-ph}\right)u\left(t\right), \xi\left(t\right) \in R^{n_1}, \tag{49}$$

$$\mu\dot{\eta}\left(t\right) = \mathsf{A}_\eta\left(\mu, e^{-ph}\right)\eta\left(t\right) + \mathsf{B}_\eta\left(\mu, e^{-ph}\right)u\left(t\right), \eta\left(t\right) \in R^{n_2},\ t > 0. \tag{50}$$

Due to (48), the decoupled system (49) and (50) is $O\left(\mu\right)$-close to the DS (11) and the BLS (19).

The decomposition (49) and (50) allows us to prove the separation (at sufficiently small μ) of the SPLTISD spectrum $\sigma\left(\mu\right)$ (8) into two disjoint parts with "slow" and "fast" eigenvalues, as well as the approximation of the SPLTISD spectrum $\sigma\left(\mu\right)$ (8) elements by the eigenvalues of σ_s (18) and σ_f (22).

Let us define:

$$w_\xi\left(\mu, \lambda, e^{-\lambda h}\right) \triangleq \left[\lambda I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda h}\right)\right]. \tag{51}$$

Due to (48) and (17), we have:

$$w_\xi\left(\mu, \lambda, e^{-\lambda h}\right) = w_s\left(\lambda, e^{-\lambda h}\right) + O\left(\mu\right). \tag{52}$$

Theorem 8. *For sufficiently small* $\mu \in \left(0, \mu^0\right]$*, the spectrum* $\sigma\left(\mu\right)$ *(8) of the SPLTISD (1)–(2) is separated into two disjoint parts:*

$$\sigma\left(\mu\right) = \sigma_x\left(\mu\right) \cup \sigma_y\left(\mu\right),\quad \sigma_x\left(\mu\right) \cap \sigma_y\left(\mu\right) = \varnothing. \tag{53}$$

The "slow" part:

$$\sigma_x(\mu) = \left\{\lambda \in C : \det\left[\lambda I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda h}\right)\right] = 0\right\} \tag{54}$$

consists of elements that for sufficiently small μ are the functions $\lambda(\mu)$ that continuously depend on μ and tend to the elements of the DS (11) spectrum (18) as $\mu \to 0$:

$$\lim_{\mu \to 0} \lambda_i(\mu) = \lambda_{si} \in \sigma_s. \tag{55}$$

The fast part:

$$\sigma_y(\mu) = \left\{\lambda \in C : \det\left[\lambda I_{n_2} - \mathsf{A}_\eta\left(\mu, e^{-\lambda h}\right)\right] = 0\right\} \tag{56}$$

consists of n_2 elements that tend to infinity, with the rate μ^{-1}, and are of the form $\frac{\lambda_i(\mu)}{\mu}$, where:

$$\lim_{\mu \to 0} \lambda_i(\mu) = \lambda_{fi} \in \sigma_f. \tag{57}$$

If in the spectrum σ_s (18) of the DS (11), there are not multiple values and in the spectrum σ_f (22) of BLS (19) there are not multiple values (it is allowed $\sigma_s \cap \sigma_f \neq \emptyset$), then the eigenvalues of the SPLTISD (1)–(2) are approximated as:

$$\lambda_i(\mu) = \lambda_{si} + O(\mu), \ \lambda_{si} \in \sigma_s, \ \forall \lambda_i(\mu) \in \sigma_x, \tag{58}$$

$$\lambda_i(\mu) = \frac{\lambda_{fi} + O(\mu)}{\mu}, \ \lambda_{fi} \in \sigma_f, \ \forall \lambda_i(\mu) \in \sigma_y. \tag{59}$$

Proof. The separation (53) and the representation (58) and (59) of the SPLTISD spectrum can be proven according to the scheme from [29]. The continuity of $\lambda(\mu) \in \sigma_x(\mu)$ follows from the continuous dependence of the roots of a quasi-polynomial with respect to its coefficients and Lemma 1. □

Note that from Theorem 8 follows the continuity and, therefore, boundedness on $\mu \in [0, \mu^0]$ the functions $\lambda(\mu) \in \sigma_x(\mu)$.

Let us define:

$$N_\xi\left(\mu, \lambda, e^{-\lambda h}\right) \triangleq \left[\ \lambda I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda h}\right), \quad B_\xi\left(\mu, e^{-\lambda h}\right)\ \right]. \tag{60}$$

Due to (48) and (23), the following equality is true:

$$N_\xi\left(\mu, \lambda, e^{-\lambda h}\right) = N_s\left(\lambda, e^{-\lambda h}\right) + O(\mu), \tag{61}$$

For $\mu \geq 0, z \in C$, we introduce the following matrix function:

$$\mathsf{P}_\xi(\mu, z) \triangleq \left[B_\xi(\mu), \mathsf{A}_\xi(\mu, z)\, B_\xi(\mu), ..., \mathsf{A}_\xi^{n_1+n_2-1}(\mu, z)\, B_\xi(\mu)\right]. \tag{62}$$

For a given $\mu \geq 0$, define the following set of complex numbers:

$$Z_\xi(\mu) = \left\{z \in C : \ rank\ \mathsf{P}_\xi(\mu, z) < n_1\right\}. \tag{63}$$

Let $\pi_\xi(\mu, z)$ be the greatest common divisor of all minors of the order n_1 of the function matrix $P_\xi(\mu, z)$. Taking into account (48), (27), (62), and (34), $\pi_\xi(\mu, z)$ can be represented as:

$$\pi_\xi(\mu, z) = \sum_{i=0}^{l_s} k_i(\mu)\, z, \ k_i(\mu) = k_i + O(\mu). \tag{64}$$

Since the set Z_ξ (63) coincides with the set of roots of the polynomial $\pi_\xi(\mu,z)$ (64), then:

$$Z_\xi = \{z \in C : \pi_\xi(\mu,z) = 0\}.$$

By virtue of the continuous dependence of the roots of a polynomial with respect to its coefficients for sufficiently small $\mu > 0$, the elements $z(\mu) \in Z_\xi(\mu)$ are continuous and, therefore, bounded functions of μ.

For a given $\mu \in [0,\mu^0]$, define the set of complex numbers:

$$\Lambda_\xi(\mu) = \left\{\lambda \in C : e^{-\lambda h} = z,\, z \in Z_\xi(\mu)\right\},$$

associated with $Z(\mu)$. The elements $z(\mu) \in Z_\xi(\mu)$, $\lambda_k \in \Lambda_\xi(\mu)$, are connected by:

$$h\,\lambda_k(\mu) = \ln|z(\mu)| + i\left(\arg z(\mu) + 2\pi k\right),\, z \in Z_\xi(\mu),\, i = \sqrt{-1},\, k = 0, \pm 1, \pm 2, \ldots. \quad (65)$$

By virtue of the connection (65) and the continuity of the functions $z(\mu) \in Z_\xi(\mu)$, the functions $Re\lambda_k(\mu) = \frac{1}{h}\ln|z(\mu)|$ are also continuous for sufficiently small $\mu > 0$.

5. Complete Controllability of a Singularly-Perturbed Linear Time-Invariant System with Delays

5.1. Auxiliary Results

Lemma 2. *Let δ and α be two real numbers, $\delta < \alpha$, and:*

$$w_s\left(\lambda, e^{-\lambda h}\right) \neq 0\ \forall \lambda \in C:\ \delta \leq Re\,\lambda \leq \alpha. \quad (66)$$

Then, there exists a positive number μ^*, such that for all $\mu \in [0,\mu^*]$, the following inequalities hold:

$$w_\xi\left(\mu, \lambda, e^{-\lambda h}\right) \neq 0\ \forall \lambda \in C:\ \delta \leq Re\,\lambda \leq \alpha. \quad (67)$$

Proof. (By contradiction) Let the statement of the lemma be wrong. Then, two sequences $\{\mu_i\}$ and $\{\lambda_i\}$, $i = 1,2,\ldots$ exist such that:

(a) $\mu_i > 0$, $i = 1,2,\ldots$;
(b) $\lim_{i\to+\infty}\mu_i = 0$;
(c) $\delta \leq Re\,\lambda_i \leq \alpha$, $i = 1,2,\ldots$
(d)

$$w_\xi\left(\mu_i, \lambda_i, e^{-\lambda_i h}\right) = 0,\ i = 1,2,\ldots \quad (68)$$

Two cases can be distinguished: (i) the sequence $\{\lambda_i\}$, $i = 1,2,\ldots$ is bounded; (ii) the sequence $\{\lambda_i\}$, $i = 1,2,\ldots$ is unbounded. Begin with Case (i). In this case, there exists a convergent subsequence of $\{\lambda_i\}$. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with $\{\lambda_i\}$. Let $\bar{\lambda} \stackrel{\Delta}{=} \lim_{i\to\pm\infty}\lambda_i$. Due to Assumption (c),

$$\delta \leq Re\,\bar{\lambda} \leq \alpha. \quad (69)$$

Calculating the limit of (68) for $i \to \infty$, we obtain that $w_s\left(\bar{\lambda}, e^{-\bar{\lambda}h}\right) = 0$. The latter along with (69) contradicts the assumption (66) of the lemma.

Proceed to Case (ii). In this case, there exists a subsequence of λ_i, which tends to infinity. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with λ_i. Then, $\lim_{i\to+\infty}\lambda_i = \pm\infty$.

Using (51) and (48), Equation (68) can be rewritten in the form:

$$(-1)^{n_1}\lambda_i^{n_1} + \lambda_i^{n_1-1}f_1(\lambda_i,\mu_i) + \cdots + f_{n_1}(\lambda_i,\mu_i) = 0, \quad (70)$$

where $f_i(\lambda_i, \mu_i)$, $j = 1, \dots, n_1$ are some functions of (λ_i, μ_i). The functions $f_i(\lambda_i, \mu_i)$ are bounded uniformly with respect to μ_i for all sufficiently large i.

Dividing Equation (70) by $\lambda_i^{n_1}$ and calculating the limit of the resulting one for $i \to +\infty$ yield the contradiction $(-1)^{n_1} = 0$. This contradiction and the contradiction obtained in Case (i) imply that the statement of the lemma holds. □

Lemma 3. *Let $\alpha \in R$ such that $\forall \lambda \in \Lambda_s : Re\,\lambda > \alpha$. Then, there exists a positive number $\bar{\mu}$ such that $Re\,\lambda(\mu) \geq \alpha$, $\forall \lambda(\mu) \in \Lambda_\xi(\mu)$, for all $\mu \in (0, \bar{\mu}]$.*

Proof. Let the statement of the lemma be wrong. Then, three sequences $\{\mu_i\}$, $\{\lambda_i\}$, and $\{z_i\}$, exist such that:

(a) $\mu_i > 0, i = 1, 2, \dots;$

(b) $\lim_{i\to+\infty} \mu_i = 0;$

(c) $\lambda_i \in \Lambda_\xi(\mu_i)$,

(d) $Re\,\lambda_i < \alpha,\ i = 1, 2, \dots,$

(e) $Re\,\lambda_i = ln|z_i|, i = 1, 2, \dots$

(f)

$$\pi_\xi(\mu_i, z_i) = 0. \tag{71}$$

Since the sequence $\{z_i\}$ is bounded, there exists a convergent subsequence of $\{z_i\}$. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with $\{z_i\}$. Let $\bar{z} \triangleq \lim_{i\to\pm\infty} z_i$, $Re\bar{\lambda} = ln|\bar{z}|$, $\bar{z} \in Z_s$. Due to Assumptions (c) and (d), in view of the continuous dependence of $Re\lambda(\mu) \in \Lambda_\xi$ on μ for $\mu \in [0, \mu^0]$, the following inequalities are satisfied:

$$Re\,\bar{\lambda} \leq \alpha,\ \bar{\lambda} \in \Lambda_s. \tag{72}$$

Calculating the limit of (71) for $i \to \infty$, we obtain that $\pi_s(\bar{z}) = 0$. The latter along with (72) contradicts the assumption $\forall \lambda \in \Lambda_s : Re\lambda > \alpha$ of the lemma. This contradiction implies that the statement of the lemma holds. □

Lemma 4. *Let $\alpha \in R$ such that $\forall \lambda \in \sigma_s : Re\lambda > \alpha$. Then, there exists a positive number $\hat{\mu}$ such that $Re\lambda(\mu) \geq \alpha$, $\forall \lambda(\mu) \in \sigma_x(\mu)$, for all $\mu \in (0, \hat{\mu}]$.*

Proof. Let the statement of the lemma be wrong. Then, two sequences $\{\mu_i\}$, $\{\lambda_i\}$ exist such that:

(a) $\mu_i > 0, i = 1, 2, \dots;$

(b) $\lim_{i\to+\infty} \mu_i = 0;$

(c) $\lambda_i \in \sigma_x(\mu_i)$,

(d) $Re\,\lambda_i < \alpha,\ i = 1, 2, \dots$

Let us note that the sequence $\{Re\lambda_i\}, i = 1, 2, \dots$ is bounded. Therefore, there exists a convergent subsequence of $\{Re\lambda_i\}$. For the sake of simplicity (but without loss of generality), we assume that this corresponding subsequence coincides with $\{\lambda_i\}$. Let $Re\bar{\lambda} \triangleq \lim_{i\to\pm\infty} Re\lambda_i, \bar{\lambda} \in \sigma_s$. Due to Assumptions (c) and (d), in view of the continuous dependence of $\lambda(\mu) \in \sigma_x$ on μ for $\mu \in [0, \mu^0]$:

$$Re\,\bar{\lambda} \leq \alpha,\ \bar{\lambda} \in \Lambda_s. \tag{73}$$

Calculating the limit of (71) for $i \to \infty$, we obtain that $rankP_s(\bar{z}) < n_1$. The latter along with (69) contradicts the assumption (66) of the lemma. □

Lemma 5. *Let:*

$$rank\,\mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) = n_1 \quad \forall \lambda \in \sigma_s, \tag{74}$$

Then, there exists a positive number μ^*, such that for all $\mu \in (0, \mu^*]$:

$$rank\, \mathsf{N}_\xi \left(\mu, \lambda, e^{-\lambda h}\right) = n_1 \quad \forall \lambda \in \sigma_x(\mu). \tag{75}$$

Proof. From the fact that σ_s is a countable set, the characterizations 2, it follows that there exist real numbers $\delta, \alpha, \gamma,\ \delta < \alpha < \gamma$, such that:

(a) $Re\,\lambda \le \gamma\ \forall \lambda \in \sigma_s$;

(b) $\alpha < Re\,\lambda \le \gamma\ \forall \lambda \in \Lambda_s$;

(c) $w_s\left(\lambda, e^{-\lambda h}\right) \ne 0\ \forall \lambda \in C : \delta \le Re\,\lambda \le \alpha$.

By $\sigma_s^{<\delta},\ \sigma_s^{>\alpha}$, denote the following subset of σ_s:

$$\sigma_s^{<\delta} = \{\lambda \in \sigma_s :\ Re\,\lambda < \delta\},\ \sigma_s^{>\alpha} = \{\lambda \in \sigma_s :\ \alpha < Re\,\lambda \le \gamma\}.$$

Therefore,

$$\sigma_s = \sigma_s^{<\delta} \cup \sigma_s^{>\alpha}.$$

In view of Theorem 4, (33) and (35), the condition (74) can be violated only for $\lambda \in \Lambda_{s}$,. Similarly, we can prove that for a given $\mu > 0$, the condition (75) can be violated only for $\lambda \in \Lambda_\xi$. Due to Lemmas 4 and 3 for all sufficiently small $\mu > 0$, it is true that $Re\lambda(\mu) \ge \alpha,\ \forall \lambda(\mu) \in \sigma_x^{>\alpha}(\mu)$, and $\forall \lambda(\mu) \in \Lambda_\xi(\mu)$. Due to Lemma 2 for all sufficiently small $\mu > 0$, it is true that $Re\lambda(\mu) \le \delta,\ \forall \lambda(\mu) \in \sigma_x^{<\delta}(\mu)$. Therefore, $\sigma_x^{<\delta}(\mu) \cup \Lambda_\xi(\mu) = \emptyset$, and the condition (75) is true $\forall \lambda \in \sigma_x^{<\delta}(\mu)$ for all sufficiently small $\mu > 0$.

Then, similar to [29] (p. 75), it is proven that if for some λ, the condition (74) is true, then for the same λ, the condition (75) is true for all sufficiently small $\mu > 0$.

Due to the above-mentioned property (a), the characterization 1 (c), and the continuity of the functions $\lambda(\mu) \in \sigma_\xi(\mu)$, it is possible to choose such μ^*, such that (74) follows (75). □

5.2. Split Controllability: Parameter-Free Sufficient Conditions

Theorem 9. *Let the DS (11) be completely controllable, i.e., the conditions (37) and (38) are fulfilled, and the BLS is completely controllable, i.e., the condition (39) is fulfilled. Then, there exists a $\mu^* \in (0, \mu^0]$ such that the SPLTISD (1)–(2) is completely controllable for all $\mu \in (0, \mu^*]$.*

Proof. For a given $\mu \in (0, \mu^0]$, the SPLTISD (1)–(2) is completely controllable if and only if:

$$rank\, \mathsf{N}\left(\mu, \lambda, e^{-\lambda h}\right) = n_1 + n_2, \quad \forall \lambda \in \sigma(\mu). \tag{76}$$

Consider the matrix-valued function:

$$\mathsf{N}_{\xi\eta}\left(\mu, \lambda, e^{-\lambda h}\right) \triangleq \mathsf{G}^{-1} N\left(\mu, \lambda, e^{-\lambda h}\right) diag\,\{\mathsf{G}, E_r\}, \tag{77}$$

that by virtue of continuity with μ of the matrices (48), (9) for sufficiently small $\mu > 0$ can be extended by continuity at $\mu = 0$.

By using (9), (43), and (47), it is easy to make sure that for $\mu \ge 0$:

$$\mathsf{N}_{\xi\eta}\left(\mu, \lambda, e^{-\lambda h}\right) = \begin{pmatrix} \lambda I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda h}\right) & 0_{n_1 \times n_2} & B_\xi\left(\mu, e^{-\lambda h}\right) \\ 0_{n_2 \times n_1} & \mu\lambda I_{n_2} - \mathsf{A}_\eta\left(\mu, e^{-\lambda h}\right) & B_\eta\left(\mu, e^{-\lambda h}\right) \end{pmatrix}. \tag{78}$$

Due to the invariance of the spectrum and preserving the matrix rank under nondegenerate transformations, it is determined from (76) that the SPLTISD (1)–(2) is completely controllable at a fixed $\mu > 0$ if and only if:

$$rank\, \mathsf{N}_{\xi\eta}\left(\mu, \lambda, e^{-\lambda h}\right) = n_1 + n_2, \quad \forall \lambda \in \sigma(\mu). \tag{79}$$

Due to (48) and (78), the condition (79) has a view: $\forall\lambda \in \sigma(\mu)$:

$$rank \begin{pmatrix} \lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right) + O(\mu) & 0_{n_1\times n_2} & B_s + O(\mu) \\ 0_{n_2\times n_1} & \mu\lambda I_{n_2} - A_4 + O(\mu) & B_2 + O(\mu) \end{pmatrix} = n_1 + n_2. \quad (80)$$

Let us show that the condition (80) is fulfilled for all sufficiently small $\mu > 0$ if DS (11) and BLS (19) are completely controllable.

Let DS (11) be completely controllable, i.e., (25) is true. Then, by Lemma 5, the following condition:

$$rank\, \mathsf{N}_{\xi}\left(\mu, \lambda, e^{-\lambda h}\right) = n_1, \quad \forall\lambda \in \sigma_x(\mu) \quad (81)$$

is true for all sufficiently small $\mu > 0$.

Since for all sufficiently small $\mu > 0$, the sets $\sigma_x(\mu)$ (54) and $\sigma_y(\mu)$ (56) have no elements in common, then:

$$rank\left[\mu\lambda I_{n_2} - \mathsf{A}_{\eta}\left(\mu, e^{-\lambda h}\right)\right] = n_2, \forall\lambda = \lambda(\mu) \in \sigma_x(\mu) = \sigma(\mu) \setminus \sigma_y(\mu) \quad (82)$$

for all sufficiently small $\mu > 0$. From the conditions (81) and (82), it follows that (80) is true for all $\lambda \in \sigma_x(\mu)$ for all sufficiently small $\mu > 0$.

Let BLS (19) is completely controllable, i.e., (26) is true. Then, since for all $\lambda(\mu) = \frac{1}{\mu}\lambda_i(\mu) \in \sigma_y(\mu)$, it is true that $\lambda_i(\mu) \underset{\mu\to 0}{\to} \lambda_{fi} \in \sigma_f$ (see Theorem 8), and due to the finiteness of the set σ_f (22) in view of the preservation of the full rank of a matrix under small regular perturbation, we have that the condition:

$$rank \begin{pmatrix} \mu\lambda I_{n_2} - A_4 + O(\mu) & B_2 + O(\mu) \end{pmatrix} = n_2, \quad \forall\lambda(\mu) \in \sigma_y(\mu)$$

is true for all sufficiently small $\mu > 0$. Since for all sufficiently small $\mu > 0$, the sets $\sigma_x(\mu)$ and $\sigma_x(\mu)$ have no elements in common, then:

$$rank\left[\lambda I_{n_1} - \mathsf{A}_{\xi}\left(\mu, e^{-\lambda h}\right)\right] = n_2, \ \forall\lambda = \lambda(\mu) \in \sigma_y(\mu) = \sigma(\mu) \setminus \sigma_x(\mu)$$

for all sufficiently small $\mu > 0$. From the last conditions, it follows that the condition (80) is true for all sufficiently small $\mu > 0$ for all $\lambda(\mu) \in \sigma_y(\mu)$.

Combining the above results, we have that if DS (11) is completely controllable and BLS (19) is completely controllable, then the condition (80) is fulfilled for all sufficiently small $\mu > 0$ for all $\lambda(\mu) \in \sigma(\mu)$.

Applying Theorems 6 and 7, we are convinced of the validity of the statement of Theorem 9. □

According to the proven theorem, the complete controllability of the slow and fast subsystems yields the complete controllability of the original system for all sufficiently small values of the parameter of singular perturbation.

Note that the conditions (37)–(39) do not depend on the small parameter; they have a ranked form; they are expressed through the matrix parameters of the SPLTISD and guarantee the preservation of its complete controllability for all sufficiently small values of the parameter $\mu > 0$. A similar statement for SPS without delay was proven in [29] and with a single delay in [44].

It is not difficult to verify the condition (37). In addition, for a given $\mu \in (0, \mu^0]$, the condition (38) may be violated only for λ_k that also are the roots of the DS (11) characteristic Equation (18), so the verification of this condition (38) is necessary only for the λ_k view of (36) that comprise the roots of the polynomial $w_s(\lambda, z)$ for $z \in Z_s$.

6. Example

In this section, we consider an illustrative example. Consider the following system, a particular case of the SPLTISD (1)–(2),

$$\begin{aligned} &\dot{x}_1(t) = -x_1(t) + 2x_2(t) - x_2(t-1) - y(t), \\ &\dot{x}_2(t) = -x_1(t) + 2x_2(t) + x_1(t-1) - x_2(t-1), \\ &\mu\dot{y}(t) = -x_1(t-1) - y(t) + u(t), \end{aligned} \tag{83}$$

with the parameters $n_1 = 2,\ n_2 = r = 1,\ l = 1,\ h = 1$ and matrices:

$$\begin{aligned} &A_{10} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},\ A_{11} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &A_{30} = \begin{pmatrix} 0 & 0 \end{pmatrix},\ A_{31} = \begin{pmatrix} -1 & 0 \end{pmatrix},\ A_4 = (-1),\ B_2 = (1). \end{aligned} \tag{84}$$

The characteristic Equation (7) for the system (83) is:

$$w\left(\mu, \lambda, e^{-\lambda}\right) = \frac{1}{\mu}\left(\lambda(\lambda-1)(1+\mu\lambda) - \mu\lambda\mathrm{e}^{-\lambda}\left(2 - \mathrm{e}^{-\lambda} - \lambda\right)\right) = 0$$

and for sufficient small $\mu > 0$ has the roots (8):

$$\sigma(\mu) = \left\{0, 1 + O(\mu), -\frac{1}{\mu} + O(\mu)\right\}.$$

The DS (11) for SPLTISD (83) has the form:

$$\begin{aligned} &\dot{x}_{s1}(t) = -x_{s1}(t) + x_{s1}(t-1) + 2x_{s2}(t) - x_{s2}(t-1) - u_s(t), \\ &\dot{x}_{s2}(t) = -x_{s1}(t) + x_{s1}(t-1) + 2x_{s2}(t) - x_{s2}(t-1), \end{aligned} \tag{85}$$

and the matrix parameters (13) for DS (85) have the form:

$$A_{s0} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},\ A_{s1} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\ B_s = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The BLS (19) for SPLTISD (83) has the form:

$$\frac{dy_f(\tau)}{d\tau} = -y_f(\tau) + u_f(\tau). \tag{86}$$

The characteristic Equation (17) for the DS (85) is:

$$w_s = \lambda^2 - \lambda + 2e^{-\lambda}(1 - e^{-\lambda}) = 0,$$

and the characteristic Equation (21) for the BLS (86) is:

$$w_f = \lambda + 1 = 0.$$

The spectra (18) of the DS (85) and the BLS (86) for (83): $\sigma_s = \{0, 1\}, \sigma_f = \{-1\}$.

Since the matrices (23) and (24) for (1)–(2) and (84):

$$N_s\left(\lambda, e^{-\lambda h}\right) = \begin{bmatrix} \lambda + 1 - e^{-\lambda} & -1 - e^{-\lambda} & -1 \\ 2 - e^{-\lambda} & \lambda - 2 + e^{-\lambda} & 0 \end{bmatrix},\quad N_f(\lambda) = [\lambda + 1, 1]$$

have a full rank for all $\lambda \in \sigma_s$, all $\lambda \in \sigma_f$, respectively, then according to Theorem 2, the DS for (83) is completely controllable and the BLS for (1)–(2) and (84) is completely controllable.

Then, with according to Theorem 9, there exists a $\mu^* > 0$ such that the SPLTISD (1)–(2) and (84) is completely controllable for all $\mu \in (0, \mu^*]$.

Let us show the validity of the conditions (37)–(39) for (1)–(2) and (84). We have:

$$P_s(z) = \begin{bmatrix} -1 & 1-z \\ 0 & 1-z \end{bmatrix}, \quad \pi_s(z) = -1 + z, \quad Z_s = \{1\}, \quad P_f = [1, \ -1].$$

It is obvious that (37) and (39) for SPLTISD (83) are valid. Since among the roots of the polynomial $w_s(\lambda, 1) = \lambda(\lambda - 1)$, there are no numbers of the form $\lambda_k = \ln|1| + i \cdot 2\pi k$, $i = \sqrt{-1}$, $k = 0, \pm 1, \pm 2, \ldots$, so $\Omega_s = \emptyset$, then we conclude that (38) is also fulfilled.

Thus, all the conditions of Theorem 9 are fulfilled for the system (83). This confirms the above conclusion about the complete robust controllability of the SPLTISD (83) with respect to $\mu > 0$ for all sufficiently small values of this parameter.

7. Discussion

The rank condition of complete controllability [12] is also known as a condition of spectral controllability and observability [46], and related to various structural properties of the system, for example realization, modal control, completeness, etc. Therefore, the results of this work can be used to obtain the conditions of similar properties for a singularly-perturbed system (1)–(2), robust with respect to a small parameter and expressed in the form of rank parametric conditions for systems of lower dimensions than the original system.

For complete controllable systems with delay, we can design static feedback controllers, providing an arbitrary finite spectrum of a closed system [47,48]. In particular, by choosing a spectrum, a closed system can be made asymptotically stable. Based on the decoupling transformation for the original singularly-perturbed system, it is possible to construct a stabilizing feedback in the form of a composite regulator that combines the stabilizing regulators of its slow and fast subsystems of lower dimensions (see, e.g., [49]).

8. Conclusions

In this paper, a singularly-perturbed linear time-invariant controlled system with multiple commensurate time delays in the slow state variables was considered. For this system, the complete controllability, robust with respect to a small parameter μ, was studied. This study is based on the Chang-type transformation of the original system, which decouples the original singularly-perturbed system into two $O(\mu)$-close to μ-free subsystems, slow and fast subsystems of smaller dimensions than the original.

Based on the above-mentioned Chang-type transformation of the original singularly-perturbed system, μ-free verifiable parametric rank-type sufficient conditions for the complete controllability of this system were established. These conditions, being μ-free, provide the complete controllability of the original singularly-perturbed system with delay for all sufficiently small values of $\mu > 0$, i.e., robustly with respect to this parameter of singular perturbation.

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