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Smashed and Twisted Wreath Products of Metagroups

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Abstract: In this article, nonassociative metagroups are studied. Different types of smashed products and smashed twisted wreath products are scrutinized. Extensions of central metagroups are studied.

Keywords: metagroup; nonassociative; product; smashed; twisted wreath

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1. Introduction

Nonassociative algebras compose a great area of algebra. In nonassociative algebra, noncommutative geometry, and quantum field theory, there frequently appear binary systems which are nonassociative generalizations of groups and related with loops, quasi-groups, Moufang loops, etc., (see References [1–4] and references therein). It was investigated and proved in the 20th century that a nontrivial geometry exists if and only if there exists a corresponding loop [1,5,6].

Octonions and generalized Cayley–Dickson algebras play very important roles in mathematics and quantum field theory [7–13]. Their structure and identities attract great attention. They are used not only in algebra and noncommutative geometry but also in noncommutative analysis, PDEs, particle physics, mathematical physics, the theory of Lie groups, algebras and their generalization, mathematical analysis, and operator theory and their applications in natural sciences including physics and quantum field theory (see References [7,11,12,14–19] and references therein).

A multiplicative law of their canonical bases is nonassociative and leads to a more general notion of a metagroup instead of a group [11,20,21]. The preposition "meta" is used to emphasize that such an algebraic object has properties milder than a group. By their axiomatics, metagroups satisfy the conditions of Equations (1)–(3) and rather mild relations (Equation (9)). They were used in References [20,21] for investigations of automorphisms, derivations, and cohomologies of nonassociative algebras. In the associative case, twisted and wreath products of groups are used for investigations not only in algebra but also in algebraic geometry, geometry, coding theory, and PDEs and their applications [22–25]. Twisted structures also naturally appear in investigations in the G-N theory of wave propagation of the components of displacement, stress, temperature distribution, and change in the volume fraction field in an isotropic homogeneous thermoelastic solid with voids subjected to thermal loading due to laser pulse [26].

In this article, nonassociative metagroups are studied. Necessary preliminary results on metagroups are described in Section 2. Quotient groups of metagroups are investigated in Theorem 1. Identities in metagroups established in Lemmas 1, 2, and 4 are applied in Sections 3 and 4.

Different types of smashed products of metagroups are investigated in Theorems 3 and 4. Besides them, direct products are also considered in Theorem 2. They provide large families of metagroups (see Remark 2).

In Section 4, smashed twisted wreath products of metagroups and particularly also of groups are scrutinized. It appears that, generally, they provide loops (see Theorem 5). If additional conditions are imposed, they give metagroups (see Theorem 6). Their metaisomorphisms are investigated in Theorem 7. In Theorem 8 and Corollary 2, smashed splitting extensions of nontrivial central metagroups are studied.

All main results of this paper are obtained for the first time. They can be used for further studies of binary systems, nonassociative algebra cohomologies, structure of nonassociative algebras, operator theory and spectral theory over Cayley–Dickson algebras, PDEs, noncommutative analysis, noncommutative geometry, mathematical physics, and their applications in the sciences (see also the conclusions).

2. Nonassociative Metagroups

To avoid misunderstandings, we give necessary definitions. A reader familiar with References [1,20,21] may skip Definition 1. For short, it will be written as a metagroup instead of a nonassociative metagroup.

Definition 1. *Let G be a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G satisfying the following conditions:*

$$\text{For each } a \text{ and } b \text{ in } G\text{, there is a unique } x \in G \text{ with } ax = b \tag{1}$$

and a unique $y \in G$ exists satisfying $ya = b$, which are denoted by

$$x = a\backslash b = Div_l(a,b) \text{ and } y = b/a = Div_r(a,b) \text{ correspondingly,} \tag{2}$$

There exists a neutral (i.e., unit) element $e_G = e \in G$:

$$eg = ge = g \text{ for each } g \in G. \tag{3}$$

If the set G with the single-valued multiplication satisfies the conditions of Equations (1) and (2), then it is called a quasi-group. If the quasi-group G satisfies also the condition of Equation (3), then it is called an algebraic loop (or in short, a loop).

The set of all elements $h \in G$ commuting and associating with G is as follows:

$$Com(G) := \{a \in G : \forall b \in G,\ ab = ba\}, \tag{4}$$

$$N_l(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ab)c = a(bc)\}, \tag{5}$$

$$N_m(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ba)c = b(ac)\}, \tag{6}$$

$$N_r(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}, \tag{7}$$

$$N(G) := N_l(G) \cap N_m(G) \cap N_r(G); \tag{8}$$

$$\mathcal{C}(G) := Com(G) \cap N(G) \text{ is called the center } \mathcal{C}(G) \text{ of } G.$$

We call G a metagroup if a set G possesses a single-valued binary operation and satisfies the conditions of Equations (1)–(3) and

$$(ab)c = t(a,b,c)a(bc) \tag{9}$$

for each a, b, and c in G, where $t(a,b,c) = t_G(a,b,c) \in \mathcal{C}(G)$. If G is a quasi-group satisfying the condition of Equation (9), then it will be called a strict quasi-group.

Then, the metagroup G will be called a central metagroup, if it satisfies also the following condition:

$$ab = \mathrm{t}_2(a,b)ba \tag{10}$$

for each a and b in G, where $\mathrm{t}_2(a,b) \in \mathcal{C}(G)$.

If H is a submetagroup (or a subloop) of the metagroup G (or the loop G) and $gH = Hg$ for each $g \in G$, then H will be called almost normal. If, in addition, $(gH)k = g(Hk)$ and $k(gH) = (kg)H$ for each g and k in G, then H will be called a normal submetagroup (or a normal subloop respectively).

Henceforward, notations $Inv_l(a) = Div_l(a,e)$ and $Inv_r(a) = Div_r(a,e)$ will be used.

Elements of a metagroup G will be denoted by small letters; subsets of G will be denoted by capital letters. If A and B are subsets in G, then $A - B$ means the difference of them: $A - B = \{a \in A : a \notin B\}$. Henceforward, maps and functions on metagroups are supposed to be single-valued unless otherwise specified.

Lemma 1. *If G is a metagroup, then for each a and $b \in G$, the following identities are fulfilled:*

$$b\backslash e = (e/b)t(e/b, b, b\backslash e) \tag{11}$$

$$(a\backslash e)b = (a\backslash b)t(e/a, a, a\backslash e)/t(e/a, a, a\backslash b); \tag{12}$$

$$b(e/a) = (b/a)t(b/a, a, a\backslash e)/t(e/a, a, a\backslash e). \tag{13}$$

Proof. The conditions of Equations (1)–(3) imply that

$$b(b\backslash a) = a,\ b\backslash(ba) = a; \tag{14}$$

$$(a/b)b = a,\ (ab)/b = a \tag{15}$$

for each a and b in G. Using the condition of Equation (9) and the identities of Equations (14) and (15), we deduce the following:

$$e/b = (e/b)(b(b\backslash e)) = (b\backslash e)/t(e/b, b, b\backslash e)$$

which leads to Equation (11).

Let $c = a\backslash b$; then, from the identities of Equations (11) and (14), it follows that

$$(a\backslash e)b = (e/a)t(e/a, a, a\backslash e)(ac) = ((e/a)a)(a\backslash b)t(e/a, a, a\backslash e)/t(e/a, a, a\backslash b)$$

which provides Equations (12).

Now, let $d = b/a$; then, the identities of Equations (11) and (15) imply that

$$b(e/a) = (da)(a\backslash e)/t(e/a, a, a\backslash e) = (b/a)t(b/a, a, a\backslash e)/t(e/a, a, a\backslash e)$$

which demonstrates Equation (13). □

Lemma 2. *Assume that G is a metagroup. Thenm for every a, a_1, a_2, and a_3 in G and p_1, p_2, and p_3 in $\mathcal{C}(G)$, we have the following:*

$$t(p_1a_1, p_2a_2, p_3a_3) = t(a_1, a_2, a_3); \tag{16}$$

$$t(a, a\backslash e, a)t(a\backslash e, a, e/a) = e. \tag{17}$$

Proof. Since $(a_1a_2)a_3 = t(a_1, a_2, a_3)a_1(a_2a_3)$ and $t(a_1, a_2, a_3) \in \mathcal{C}(G)$ for every a_1, a_2, a_3 in G, then

$$t(a_1, a_2, a_3) = ((a_1a_2)a_3)/(a_1(a_2a_3)). \tag{18}$$

Therefore, for every a_1, a_2, a_3 in G and p_1, p_2, and p_3 in $\mathcal{C}(G)$, we infer the following:

$$t(p_1a_1, p_2a_2, p_3a_3) = (((p_1a_1)(p_2a_2))(p_3a_3))/((p_1a_1)((p_2a_2)(p_3a_3)))$$
$$= ((p_1p_2p_3)((a_1a_2)a_3))/((p_1p_2p_3)(a_1(a_2a_3))) = ((a_1a_2)a_3)/(a_1(a_2a_3)),$$

since

$$b/(pa) = p^{-1}b/a \text{ and } b/p = p\backslash b = bp^{-1} \tag{19}$$

For each $p \in \mathcal{C}(G)$, a and b in G, because $\mathcal{C}(G)$ is the commutative group. Thus, $t(p_1a_1, p_2a_2, p_3a_3) = t(a_1, a_2, a_3)$.

From the condition in Equation (9), Lemma 1, and the identity of Equation (16), it follows that

$$t(a, a\backslash e, a) = ((a(a\backslash e))a)/(a((a\backslash e)a)) = a/[at(e/a, a, a\backslash e)] = e/t(a\backslash e, a, e/a)$$

for each $a \in G$, implying Equation (17). □

Theorem 1. *If G is a metagroup and $\mathcal{C}_0$ is a subgroup in a center $\mathcal{C}(G)$ such that $t(a, b, c) \in \mathcal{C}_0$ for each a, b, and c in G, then its quotient $G/\mathcal{C}_0$ is a group.*

Proof. As traditionally, the following notation is used:

$$AB = \{x = ab : a \in A, b \in B\}, \tag{20}$$

$$Inv_l(A) = \{x = a\backslash e : a \in A\}, \tag{21}$$

$$Inv_r(A) = \{x = e/a : a \in A\} \tag{22}$$

for subsets A and B in G. Then from the conditions of Equations (4)–(8), it follows that, for each a, b, and c in G, the following identities take place:

$((a\mathcal{C}_0)(b\mathcal{C}_0))(c\mathcal{C}_0) = (a\mathcal{C}_0)((b\mathcal{C}_0)(c\mathcal{C}_0))$ and $a\mathcal{C}_0 = \mathcal{C}_0a$. Evidently $e\mathcal{C}_0 = \mathcal{C}_0$. In view of Lemmas 1 and 2 $(a\mathcal{C}_0)\backslash e = e/(a\mathcal{C}_0)$, consequently, for each $a\mathcal{C}_0 \in G/\mathcal{C}_0$ a unique inverse $(a\mathcal{C}_0)^{-1}$ exists. Thus the quotient $G/\mathcal{C}_0$ of G by $\mathcal{C}_0$ is a group. □

Lemma 3. *Let G be a metagroup, then $Inv_r(G)$ and $Inv_l(G)$ are metagroups.*

Proof. At first, we consider $Inv_r(G)$. Let a_1 and a_2 belong to G. Then, there are unique e/a_1 and e/a_2, since the map Inv_r is single-valued (see Definition 1). Since $Inv_r \circ Inv_l(a) = a$ and $Inv_l \circ Inv_r(a) = a$ for each $a \in G$, then $Inv_r : G \to G$ and $Inv_l : G \to G$ are bijective and surjective maps.

We put $\hat{a}_1 \circ \hat{a}_2 = (e/a_2)(e/a_1)$ for each a_1 and a_2 in G, where $\hat{a}_j = Inv_r(a_j)$ for each $j \in \{1, 2\}$. This provides a single-valued map from $Inv_r(G) \times Inv_r(G)$ into $Inv_r(G)$. Then, for each a, b, x, and y in G, the equations $\hat{a} \circ \hat{x} = \hat{b}$ and $\hat{y} \circ \hat{a} = \hat{b}$ are equivalent to $(e/x)(e/a) = e/b$ and $(e/a)(e/y) = e/b$, respectively. That is, $\hat{x} = (e/b)/(e/a)$ and $\hat{y} = (e/a)\backslash(e/b)$ are unique. On the other hand, $e/e = e$ and $\hat{e} \circ \hat{b} = e/b = \hat{b} \circ \hat{e} = \hat{b}$ for each $b \in G$.

Then, we infer the following:

$$\begin{aligned} &\hat{a}_1 \circ (\hat{a}_2 \circ \hat{a}_3) = ((e/a_3)(e/a_2))(e/a_1) = \\ &t_G(e/a_3, e/a_2, e/a_1)(e/a_3)((e/a_2)(e/a_1)) = t_G(\hat{a}_3, \hat{a}_2, \hat{a}_1)(\hat{a}_1 \circ \hat{a}_2) \circ \hat{a}_3, \end{aligned}$$

Consequently, $t_{Inv_r(G)}(\hat{a}_1, \hat{a}_2, \hat{a}_3) = e/t_G(\hat{a}_3, \hat{a}_2, \hat{a}_1)$. Evidently, $Inv_r(\mathcal{C}(G)) = \mathcal{C}(G)$ and $\mathcal{C}(Inv_r(G)) = \mathcal{C}(G)$. Thus, the conditions of Equations (1)–(3) and (9) are satisfied for $Inv_r(G)$.

Similarly, putting $Inv_l(a_j) = \check{a}_j$ and $\check{a}_1 \circ \check{a}_2 = (a_2\backslash e)(a_1\backslash e)$ for each $a_j \in G$ and $j \in \{1, 2, 3\}$, the conditions of Equations (1)–(3) and (9) are verified for $Inv_l(G)$. □

Lemma 4. *Assume that G is a metagroup and that $a \in G$, $b \in G$, and $c \in G$. Then*

$$e/(ab) = (e/b)(e/a)t(e/a, a, b)/t(e/b, e/a, ab) \tag{23}$$

and

$$(ab)\backslash e = (b\backslash e)(a\backslash e)t(ab, b \setminus e, a\backslash e)/t(a, b, b\backslash e). \tag{24}$$

$$(a/(bc) = ((a/c)/b)t(a/(bc), b, c), \tag{25}$$

$$(bc)\backslash a = (c\backslash(b\backslash a))/t(b, c, (bc) \setminus a). \tag{26}$$

Proof. From Equations (9) and (15), we deduce that

$((e/b)(e/a))(ab) = t(e/b, e/a, ab)(e/b)((e/a)(ab)) = t(e/b, e/a, ab)/t(e/a, a, b)$, which implies Equation (23). Then, from Equations (9) and (14), we infer the following:

$$(ab)((b\backslash e)(a\backslash e)) = ((ab)(b\backslash e))(a\backslash e)/t(ab, b\backslash e, a\backslash e) = t(a, b, b\backslash e)/t(ab, b\backslash e, a\backslash e)$$

which implies Equation (24).

Utilizing Equations (14) and (9), we get $b(c((bc) \setminus a)) = a/t(b, c, (bc)\backslash a)$; hence, $c((bc)\backslash a) = (b \setminus a)/t(b, c, (bc)\backslash a)$, implying Equation (26).

Equations (15) and (9) imply that $((a/(bc))b)c = t(a/(bc), b, c)a$; consequently, $(a/c)t(a/(bc), b, c) = (a/(bc))b$, and hence,

$$((a/c)/b)t(a/(bc), b, c) = a/(bc).$$

□

3. Smashed Products and Smashed Twisted Products of Metagroups

Theorem 2. *Let G_j be a family of metagroups (see Definition 1), where $j \in J$, J is a set. Then, their direct product $G = \prod_{j\in J} G_j$ is a metagroup and*

$$\mathcal{C}(G) = \prod_{j\in J} \mathcal{C}(G_j). \tag{27}$$

Proof. It is given in Theorem 8 in Reference [21]. □

Remark 1. *Let A and B be two metagroups, and let $\mathcal{C}$ be a commutative group such that*

$$\mathcal{C}_m(A) \hookrightarrow \mathcal{C}, \mathcal{C}_m(B) \hookrightarrow \mathcal{C}, \mathcal{C} \hookrightarrow \mathcal{C}(A) \text{ and } \mathcal{C} \hookrightarrow \mathcal{C}(B), \tag{28}$$

where $\mathcal{C}_m(A)$ denotes a minimal subgroup in $\mathcal{C}(A)$ containing $t_A(a, b, c)$ for every a, b, and c in A.

Using direct products, it is always possible to extend either A or B to get such a case. In particular, either A or B may be a group. On $A \times B$, an equivalence relation Ξ is considered such that

$$(\gamma v, b)\Xi(v, \gamma b) \text{ and } (\gamma v, b)\Xi\gamma(v, b) \text{ and } (\gamma v, b)\Xi(v, b)\gamma \tag{29}$$

for every v in A, b in B, and γ in $\mathcal{C}$.

$$\text{Let } \phi : A \to \mathcal{A}(B) \text{ be a single-valued mapping,} \tag{30}$$

where $\mathcal{A}(B)$ denotes a family of all bijective surjective single-valued mappings of B onto B subjected to the conditions of Equations (31)–(34) given below. If $a \in A$ and $b \in B$, then it will be written shortly b^a instead of $\phi(a)b$, where $\phi(a) : B \to B$. Let also

$$\eta_{A,B,\phi} : A \times A \times B \to \mathcal{C}, \kappa_{A,B,\phi} : A \times B \times B \to \mathcal{C}$$

and

$$\xi_{A,B,\phi} : ((A \times B)/\Xi) \times ((A \times B)/\Xi) \to \mathcal{C}$$

be single-valued mappings written shortly as η, κ, and ξ correspondingly such that

$$(b^u)^v = b^{vu}\eta(v,u,b),\ e^u = e,\ b^e = b; \tag{31}$$

$$\eta(v,u,\gamma b) = \eta(v,u,b); \tag{32}$$

$$(cb)^u = c^u b^u \kappa(u,c,b); \tag{33}$$

$$\kappa(u,\gamma c,b) = \kappa(u,c,\gamma b) = \kappa(u,c,b) \tag{34}$$

and $\kappa(u,\gamma,b) = \kappa(u,b,\gamma) = e$;

$$\xi((\gamma u,c),(v,b)) = \xi((u,c),(\gamma v,b)) = \xi((u,c),(v,b))$$

and

$$\xi((\gamma,e),(v,b)) = e \text{ and } \xi((u,c),(\gamma,e)) = e \tag{35}$$

for every u and v in A, b, and c in B, γ in $\mathcal{C}$, where e denotes the neutral element in $\mathcal{C}$ and in A and B.

We put

$$(a_1,b_1)(a_2,b_2) = (a_1a_2, \xi((a_1,b_1),(a_2,b_2))b_1b_2^{a_1}) \tag{36}$$

for each of a_1 and a_2 in A and of b_1 and b_2 in B.

The Cartesian product $A \times B$ supplied with such a binary operation of Equation (36) will be denoted by $A \bigotimes^{\phi,\eta,\kappa,\xi} B$.

Then, we put

$$(a_1,b_1) \star (a_2,b_2) = (a_1a_2, \xi((a_1,b_1),(a_2,b_2))b_2^{a_1}b_1) \tag{37}$$

for each of a_1 and a_2 in A and of b_1 and b_2 in B.

The Cartesian product $A \times B$ supplied with a binary operation of Equation (37) will be denoted by $A \star^{\phi,\eta,\kappa,\xi} B$.

Theorem 3. *Let the conditions of Remark 1 be fulfilled. Then, the Cartesian product $A \times B$ supplied with a binary operation of Equation (36) is a metagroup. Moreover, there are embeddings of A and B into $A \bigotimes^{\phi,\eta,\kappa,\xi} B = C_1$ such that B is an almost normal submetagroup in C_1. If in addition $\mathcal{C}_m(C_1) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, then B is a normal submetagroup.*

Proof. The first part of this theorem was proven in Theorem 9 in Reference [21]. Naturally, A is embedded into C_1 as $\{(a,e) : a \in A\}$ and B is embedded into C_1 as $\{(e,b) : b \in B\}$. Let $a \in A$ and $b_0 \in B$; then, $(a,b_0)B = \{(a, \xi((a,b_0),(e,b))b_0b^a) : b \in B\}$ and $B(a,b_0) = \{(a, \xi((e,b),(a,b_0))bb_0) : b \in B\}$, since $b_0^e = b_0$ by (31). From $B^a = B$, $b_0B = B$, $Bb_0 = B$, $\mathcal{C} \subset \mathcal{C}(B)$ and Equations (30) and (35), it follows that $(a,b_0)B = B(a,b_0)$, where $B^a = \{b^a : b \in B\}$. Thus, B is an almost normal submetagroup in C_1 (see Definition 1). If in addition $\mathcal{C}_m(C_1) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, then evidently B is a normal submetagroup (see also the condition of Equation (29)), since $t_{C_1}(g,b,h) \in \mathcal{C}_m(C_1)$ and $t_{C_1}(h,g,b) \in \mathcal{C}_m(C_1)$ for each g and h in G, $b \in H$. □

Theorem 4. *Suppose that the conditions of Remark 1 are satisfied. Then, the Cartesian product $A \times B$ supplied with a binary operation of Equation (37) is a metagroup. Moreover, there exist embeddings of A and B into $A \star^{\phi,\eta,\kappa,\xi} B = C_2$ such that B is an almost normal submetagroup in C_2. If additionally $\mathcal{C}_m(C_2) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, then B is a normal submetagroup.*

Proof. The conditions of Remark 1 imply that the binary operation of Equation (37) is single-valued.

We consider the following formulas:

$I_1 = ((a_1, b_1) \star (a_2, b_2)) \star (a_3, b_3)$ and $I_2 = (a_1, b_1) \star ((a_2, b_2) \star (a_3, b_3))$, where a_1, a_2, and a_3 are in A and where b_1, b_2, and b_3 are in B. Utilizing Equations (31)–(35) and (37), we get the following:

$$I_1 = ((a_1a_2)a_3, \xi((a_1a_2, b_2^{a_1}b_1), (a_3, b_3))\xi((a_1, b_1), (a_2, b_2))b_3^{a_1a_2}(b_2^{a_1}b_1))$$

and

$$I_2 = (a_1(a_2a_3), \xi((a_1, b_1), (a_2a_3, b_3^{a_2}b_2))[\xi((a_2, b_2), (a_3, b_3))]^{a_1}(b_3^{a_1a_2}b_2^{a_1})b_1\kappa(a_1, b_3^{a_2}, b_2)\eta(a_1, a_2, b_3)).$$

Therefore

$$I_1 = t((a_1, b_1), (a_2, b_2), (a_3, b_3))I_2 \tag{38}$$

with

$$\begin{array}{l} t((a_1, b_1), (a_2, b_2), (a_3, b_3)) = t_A(a_1, a_2, a_3)\xi((a_1, b_1), (a_2, b_2)) \\ \xi((a_1a_2, b_2^{a_1}b_1), (a_3, b_3))/\{t_B(b_3^{a_1a_2}, b_2^{a_1}, b_1) \\ \xi((a_1, b_1), (a_2a_3, b_3^{a_2}b_2))[\xi((a_2, b_2), (a_3, b_3))]^{a_1}\kappa(a_1, b_3^{a_2}, b_2)\eta(a_1, a_2, b_3))\}, \end{array} \tag{39}$$

Consequently, $t((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in \mathcal{C}$ for each $a_j \in A$, $b_j \in B$, $j \in \{1, 2, 3\}$. We denote

$$t((a_1, b_1), (a_2, b_2), (a_3, b_3))$$

in more details by

$$t_{A\star^{\phi,\eta,\kappa,\xi}B}((a_1, b_1), (a_2, b_2), (a_3, b_3))$$

(see Equation (39)).

Evidently, Equation (3) is a consequence of Equations (35) and (37).

Note that, if $\gamma \in \mathcal{C}$, then

$$\begin{array}{l} \gamma((a_1, b_1) \star (a_2, b_2)) = (\gamma a_1a_2, \xi((a_1, b_1), (a_2, b_2))b_2^{a_1}b_1) = \\ (a_1a_2, b_2^{a_1}b_1)\gamma\xi((a_1, b_1), (a_2, b_2)) = ((a_1, b_1) \star (a_2, b_2))\gamma. \end{array}$$

Therefore, $\gamma \in \mathcal{C}(A \star^{\phi,\eta,\kappa,\xi} B)$. Consequently, $\mathcal{C} \subseteq \mathcal{C}(A \star^{\phi,\eta,\kappa,\xi} B)$.

Then, we seek a solution of the following equation:

$$(a_1, b_1) \star (a, b) = (e, e), \tag{40}$$

where $a \in A$, $b \in B$.

From Equations (2) and (37), it follows that

$$a_1 = e/a \tag{41}$$

Consequently, $\xi((e/a, b_1), (a, b))b^{(e/a)}b_1 = e$. Therefore, Equations (1) and (35) imply that

$$b_1 = [\xi((e/a, b^{(e/a)}), (a, b))b^{(e/a)}]\backslash e. \tag{42}$$

Thus, $a_1 \in A$ and $b_1 \in B$ prescribed by Equations (41) and (42) provide a unique solution of Equation (40).

Analogously for the following equation

$$(a, b)(a_2, b_2) = (e, e), \tag{43}$$

where $a \in A$, $b \in B$, we deduce that

$$a_2 = a\backslash e. \tag{44}$$

Consequently, $\xi((a,b),(a\backslash e,b_2))b_2^a b = e$, and hence, $b_2^a = e/[\xi((a,b),(a\backslash e,b_2))b]$. From Equations (31) and (32), it follows that $(b_2^a)^{e/a} = \eta(e/a,a,b_2)b_2$; consequently,

$$b_2 = (e/b)^{e/a}/\{[(\xi((a,b),(a\backslash e,(e/b)^{e/a}))]^{e/a}\eta(e/a,a,(e/b)^{e/a})\}. \tag{45}$$

Thus, a unique solution of Equation (43) is given by Equations (44) and (45).

Then, we have $(a_1,b_1) = (e,e)/(a,b)$ and $(a_2,b_2) = (a,b)\backslash(e,e)$ and get the following:

$$(a,b) \setminus (c,d) = ((a,b)\backslash(e,e))(c,d)$$

$$t((e,e)/(a,b),(a,b),((a,b)\backslash(e,e))(c,d))/t((e,e)/(a,b),(a,b),(a,b)\backslash(e,e)); \tag{46}$$

$$(c,d)/(a,b) = (c,d)((e,e)/(a,b))$$
$$t((e,e)/(a,b),(a,b),(a,b)\backslash(e,e))/t((c,d)(e/(a,b)),(a,b),(a,b)\backslash(e,e)) \tag{47}$$

and $e_G = (e,e)$, where $G = A \star^{\phi,\eta,\kappa,\xi} B$. This means that the properties of Equations (1)–(3) and (9) are fulfilled for $A \star^{\phi,\eta,\kappa,\xi} B$.

Evidently, there are embeddings of A and B into C_2 as (A,e) and (e,B), respectively. Suppose that $a \in A$ and $b_0 \in B$, then

$(a,b_0) \star B = \{(a,\xi((a,b_0),(e,b))b^a b_0) : b \in B\}$ and

$B \star (a,b_0) = \{(a,\xi((e,b),(a,b_0))b_0 b) : b \in B\}$.

Therefore, $(a,b_0) \star B = B \star (a,b_0)$ by the conditions of Equations (30) and (35), since $B^a = B$ and $\mathcal{C} \subset \mathcal{C}(B)$. Thus, B is an almost normal submetagroup in C_2 (see Definition 1). If additionally $\mathcal{C}_m(C_2) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, then apparently B is a normal submetagroup (see also the condition of Equation (29)), since $t_{C_2}(g,b,h) \in \mathcal{C}_m(C_2)$ and $t_{C_2}(h,g,b) \in \mathcal{C}_m(C_2)$ for each g and h in G, $b \in B$. $\square$

Definition 2. *We call the metagroup $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ provided by Theorem 3 (or $A \star^{\phi,\eta,\kappa,\xi} B$ by Theorem 4) a smashed product (or a smashed twisted product correspondingly) of metagroups A and B with smashing factors ϕ, η, κ, and ξ.*

Remark 2. *From Theorems 2–4, it follows that, taking nontrivial η, κ, and ξ and starting even from groups with nontrivial $\mathcal{C}(G_j)$ or $\mathcal{C}(A)$, it is possible to construct new metagroups with nontrivial $\mathcal{C}(G)$ and ranges $t_G(G,G,G)$ of t_G that may be infinite.*

With suitable smashing factors ϕ, η, κ, and ξ and with nontrivial metagroups or groups A and B, it is easy to get examples of metagroups in which $e/a \neq a\backslash e$ for an infinite family of elements a in $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ or in $A \star^{\phi,\eta,\kappa,\xi} B$. Evidently, smashed products and smashed twisted products (see Definition 2) are nonassociative generalizations of semidirect products. Combining Theorems 3 and 4 with Lemmas 3 and 4 provides other types of smashed products by taking $\hat{b}_1 \circ \hat{b}_2^{a_1}$ instead of $b_1 b_2^{a_1}$ or $\check{b}_2^{a_1} \circ \check{b}_1$ instead of $b_2^{a_1} b_1$ on the right sides of Equations (36) and (37), correspondingly, etc.

4. Smashed Twisted Wreath Products of Metagroups

Lemma 5. *Let D be a metagroup and A be a submetagroup in D. Then, there exists a subset V in D such that D is a disjoint union of vA, where $v \in V$, that is,*

$$D = \bigcup_{v \in V} vA \tag{48}$$

and

$$(\forall v_1 \in V,\ \forall v_2 \in V,\ v_1 \neq v_2) \Rightarrow (v_1 A \cap v_2 A) = \emptyset). \tag{49}$$

Proof. The cases $A = \{e\}$ and $A = D$ are trivial. Let $A \neq \{e\}$ and $A \neq D$, and let $\mathcal{C}(D)$ be a center of D. From the conditions of Equations (4)–(8), it follows that $z \in \mathcal{C}(D) \cap A$ implies $z \in \mathcal{C}(A)$.

Assume that $b \in D$ and $z \in \mathcal{C}(D)$ are such that $zbA \cap bA \neq \emptyset$. It is equivalent to $(\exists s_1 \in A, \exists s_2 \in A, zbs_1 = bs_2)$. From Equation (14), it follows that $(zbs_1 = bs_2) \Leftrightarrow (zs_1 = s_2) \Leftrightarrow (z = s_2/s_1 \in A)$ because $z \in \mathcal{C}(D)$. Thus,

$$(\exists b \in D,\ \exists z \in \mathcal{C}(D),\ zbA \cap bA = \emptyset) \Leftrightarrow (\exists b \in D,\ \exists z \in \mathcal{C}(D) - A). \tag{50}$$

Suppose now that $b_1 \in D$, $b_2 \in D$ and $b_1 A \cap b_2 A \neq \emptyset$. This is equivalent to $(\exists s_1 \in A, \exists s_2 \in A, b_1 s_1 = b_2 s_2)$. By the identity of Equation (15), the latter is equivalent to $b_1 = (b_2 s_2)/s_1$. On the other hand,

$$\begin{aligned}(b_2 s_2)/s_1 &= (b_2 s_2)(e/s_1)t(e/s_1, s_1, s_1 \setminus e)/t((b_2 s_2)/s_1, s_1, s_1 \backslash e) \\ &= b_2(s_2(e/s_1))t(b_2, s_2, e/s_1)t(e/s_1, s_1, s_1 \setminus e)/t((b_2 s_2)/s_1, s_1, s_1 \backslash e)\end{aligned}$$

by Equations (9), (13), and (15). Together with (50) this gives the equivalence:

$$(\exists b_1 \in D,\ \exists b_2 \in D,\ b_1 A \cap b_2 A \neq \emptyset) \Leftrightarrow (\exists b_1 \in D,\ \exists b_2 \in D,\ \exists s \in A, \exists z \in \mathcal{C}(D) - A,\ b_1 = z b_2 s). \tag{51}$$

Let Y be a family of subsets K in D such that $k_1 A \cap k_2 A = \emptyset$ for each $k_1 \neq k_2$ in K. Let Y be directed by inclusion. Then, $Y \neq \emptyset$, since $A \subset D$ and $A \neq D$. Therefore, from Equations (50) and (51) and the Kuratowski-Zorn lemma (see Reference [27]), the assertion of this lemma follows, since a maximal element V in Y gives Equations (48) and (49). □

Definition 3. *A set V from Lemma 5 is called a right transversal (or complete set of right coset representatives) of A in D.*

The following corollary is an immediate consequence of Lemma 5.

Corollary 1. *Let D be a metagroup, A be a submetagroup in D, and V be a right transversal of A in D. Then,*

$$\forall a \in D,\ \exists_1 s \in A,\ \exists_1 b \in V,\ a = sb \text{ for a given triple } (A, D, V). \tag{52}$$

Remark 3. *We denote b in the decomposition of Equation* (52) *by* $b = \tau(a) = a^\tau$ *and* $s = \psi(a) = a^\psi$*, where* τ *and* ψ *are the shortened notations of* $\tau_{A,D,V}$ *and* $\psi_{A,D,V}$*, respectively. That is, there are single-valued maps*

$$\tau : D \to V \text{ and } \psi : D \to A. \tag{53}$$

These maps are idempotent $\tau(\tau(a)) = \tau(a)$ *and* $\psi(\psi(a)) = \psi(a)$ *for each* $a \in D$.

$$If\ b = a^\tau,\ \text{then we denote } e/b \text{ by } a^{e/\tau} \text{ and } b\backslash e \text{ by } a^{\tau\backslash e}. \tag{54}$$

According to Equation (2), $s = a/b$*; hence,* $a^\psi = a/a^\tau$*. From Equation* (13) *,it follows that* $a/b = a(e/b)t(e/b, b, b \setminus e)/t(a/b, b, b\backslash e)$*; consequently, by Lemma 2,*

$$s = aa^{e/\tau} t(a^{e/\tau}, a^\tau, a^{\tau\backslash e})/t(aa^{e/\tau}, a^\tau, a^{\tau\backslash e}). \tag{55}$$

Notice that the metagroup need not be power-associative. Then, e/s *and* $s\backslash e$ *can be calculated with the help of the identity of Equation* (11)*. Suppose that a and y belong to D,* $s = a^\psi$, $b = a^\tau$, $s_2 = y^\psi$*, and* $b_2 = y^\tau$*. Then,* $(a^\tau y) = b(s_2 b_2)$*. According to Equation* (52) *there exists a unique decomposition* $b(s_2 b_2) = s_3 b_3$*, where* $s_3 \in A$, $b_3 \in V$*; hence,* $(a^\tau y)^\tau = b_3$*. On the other hand, by Equation* (9) $ay = s(b(s_2 b_2))t(s, b, y) =$

$(ss_3)b_3t(s,b,y)/t(s,s_3,b_3)$. *We denote a subgroup* $\mathcal{C}(D) \cap A$ *in* $\mathcal{C}(D)$ *by* $\mathcal{C}_A(D)$ *or shortly* $\mathcal{C}_A$*, when* D *is specified. From Lemma 2 and Equation* (51)*, it follows that*

$$\mathcal{C}(D)^\tau \text{ is isomorphic with } \mathcal{C}(D)/\mathcal{C}_A, \tag{56}$$

where $\mathcal{C}(D)^\tau = \{a^\tau : a \in \mathcal{C}(D)\}$.

Let $\mathcal{C}_m(A)$ *be a minimal subgroup in* $\mathcal{C}(A)$ *generated by a set* $\{t_A(a,b,c) : a \in A,\ b \in A,\ c \in A\}$. *From Equation* (9)*, it follows that* $\mathcal{C}_m(A) \subset \mathcal{C}_A(D)$ *and* $A\mathcal{C}(D)$ *is a submetagroup in* D. *By virtue of Theorem 1,* $(A\mathcal{C}(D))/\mathcal{C}_A(D)$ *and* $A/\mathcal{C}_A(D)$ *are groups such that* $A/\mathcal{C}_A(D) \hookrightarrow (A\mathcal{C}(D))/\mathcal{C}_A(D)$. *For each* $d \in D$*, there exists a unique decomposition*

$$d = d^\psi d^\tau \tag{57}$$

by Equation (53)*. Take in particular* $\gamma \in \mathcal{C}(D)$*; then,* $\gamma = \gamma^\psi \gamma^\tau$*, where* $\gamma^\psi \in \mathcal{C}_A(D)$*,* $\gamma^\tau \in V$*. Therefore,* $\mathcal{C}(D)/\mathcal{C}_A(D) \subset V$ *and there exists a subset* V_0 *in* V *such that* $(\mathcal{C}(D)/\mathcal{C}_A(D))V_0 = V$*, since* $\mathcal{C}(D)/\mathcal{C}_A(D)$ *is a subgroup in* $(A\mathcal{C}(D))/\mathcal{C}_A(D)$ *(see Equation* (56)*). Equation* (57) *implies that* $(d^\tau)^\psi = e$ *and* $(d^\psi)^\tau = e$ *for each* $d \in D$*. Using this, we subsequently deduce that*

$$(d^\psi \gamma)^\psi = d^\psi \gamma^\psi, \tag{58}$$

$$(d^\psi \gamma)^\tau = \gamma^\tau, \tag{59}$$

$$(d^\tau \gamma)^\psi = \gamma^\psi, \tag{60}$$

$$(d^\tau \gamma)^\tau = d^\tau \gamma^\tau \tag{61}$$

for each $d \in D$ *and* $\gamma \in \mathcal{C}(D)$*. Hence,*

$(d\gamma) = (d\gamma)^\psi (d\gamma)^\tau = (d^\psi d^\tau)(\gamma^\psi \gamma^\tau) = (d^\psi \gamma^\psi)(d^\tau \gamma^\tau) = (d^\psi \gamma)^\psi (d^\tau \gamma)^\tau$,

where $d^\psi \gamma^\psi \in A$ *and* $d^\tau \gamma^\tau \in V$*. From a uniqueness of this representation, it follows that*

$$(d\gamma)^\psi = d^\psi \gamma^\psi \tag{62}$$

and

$$(d\gamma)^\tau = d^\tau \gamma^\tau \text{ for each } d \in D \text{ and } \gamma \in \mathcal{C}(D). \tag{63}$$

Using Equation (63) *we infer that*

$$(a^\tau y)^\tau = (ay)^\tau [t_D(a^\psi, (a^\tau y)^\psi, (ay)^\tau)/t_D(a^\psi, a^\tau, y)]^\tau. \tag{64}$$

On the other hand, if $\gamma \in \mathcal{C}(D)$*, then* $\gamma^\psi = \gamma/\gamma^\tau$ *and Equations* (64) *and* (60) *imply particularly that*

$$(a^\tau \gamma)^\tau = (a\gamma)^\tau \text{ for each } a \in D \text{ and } \gamma \in \mathcal{C}(D), \tag{65}$$

since $t_D(a,d,\gamma) = e$ *for each* a *and* d *in* D *and* $\gamma \in \mathcal{C}(D)$*. Then, from* $s = a^\psi$*,* $a^\tau y = s_3 b_3$*, it follows that* $a^\psi (a^\tau y)^\psi = ss_3$ *and* $(ay)^\psi = [(ss_3)b_3 t_D(s,b,y)/t_D(s,s_3,b_3)]^\psi$*; consequently, by Lemma 2 and Equation* (62)*,*

$$a^\psi (a^\tau y)^\psi = (ay)^\psi [t_D(a^\psi, (a^\tau y)^\psi, (ay)^\tau)/t_D(a^\psi, a^\tau, y)]^\psi \tag{66}$$

for each a *and* y *in* D*. Particularly,*

$$a^\psi (a^\tau \gamma)^\psi = (a\gamma)^\psi \text{ for each } a \in D \text{ and } \gamma \in \mathcal{C}(D). \tag{67}$$

From Equations (64) *and* (65)*, it follows that the metagroup* D *acts on* V *transitively by right shift operators* R_y*, where* $R_y a = ay$ *for each* a *and* y *in* D*. Therefore, we put*

$$(a^\tau)^{[c]} := (a^\tau c)^\tau \text{ for each } a \text{ and } c \text{ in } D. \tag{68}$$

Then from Equations (64), (65), (68), and (9) and Lemma 2, we deduce that, for each a, c, and d in D

$$(a^\tau)^{[cd]} = ((a^\tau)^{[c]})^{[d]}[t_D((a^\tau c)^\psi, (a^\tau c)^\tau, d)/(t_D((a^\tau c)^\psi, ((a^\tau c)^\tau d)^\psi, ((a^\tau c)d)^\tau)t_D(a^\tau, c, d))]^\tau. \quad (69)$$

In particular, $(a^\tau)^{[e]} = a^\tau$ for each $a \in D$. Next, we put $e^\tau = b_*$. It is convenient to choose $b_* = e$. Hence, $b_*^{[s]} = (e^\tau)^{[s]} = (e^\tau s)^\tau = s^\tau = e = e^\tau$ for each $s \in A$. Thus, the submetagroup A is the stabilizer of e and Equation (68) implies that

$$e^{[s]} = e \text{ and } e^{[q]} = q \text{ for each } s \in A \text{ and } q \in V. \quad (70)$$

Remark 4. *Let B and D be metagroups, A be a submetagroup in D, and V be a right transversal of A in D. Let also the conditions of Equations (28)–(35) be satisfied for A and B. By Theorem 2, there exists a metagroup*

$$F = B^V, \text{ where } B^V = \prod_{v\in V} B_v,\ B_v = B \text{ for each } v \in V. \quad (71)$$

It contains a submetagroup
$F^* = \{f \in F :\ card(\sigma(f)) < \aleph_0\}$,
where $\sigma(f) = \{v \in V :\ f(v) \neq e\}$ is a support of $f \in F$ and $card(\Omega)$ denotes the cardinality of a set Ω.
Let $T_h f = f^h$ for each $f \in F$ and $h : V \to A$. We put
$\hat{S}_d(T_h f J) = T_{hS_d^{-1}} f S_d J$,
where $J : V \times F \to B,\ J(f, v) = fJv,\ S_d Jv = Jv^{[d\backslash e]}$ for each $d \in D$, $f \in F$ and $v \in V$. Then, for each $f \in F$, $d \in D$ we put

$$f^{\{d\}} = \hat{S}_d(T_{g_d} f E), \quad (72)$$

where

$$s(d, v) = e/(v/d)^\psi,\ g_d(v) = s(d, v),$$
$$fEv = f(v) \text{ for each } v \in V \quad (73)$$

(see also Equations (52) and (68)). Hence,

$$f^{\{e\}} = f, \quad (74)$$

since $v^{e\backslash e} = v$ and $s(e, v) = e$.

Lemma 6. *Let the conditions of Remark 4 be satisfied. Then, for each of f and f_1 in F and of d and d_1 in D, $v \in V$,*

$$(ff_1)^{\{d\}}(v) = \kappa(s(d, v), f(v^{[d\backslash e]}), f_1(v^{[d\backslash e]}))f^{\{d\}}(v)f_1^{\{d\}}(v) \quad (75)$$

and

$$f^{\{dd_1\}}(v) = \{[(f^{\{d_1\}})^{\{d\}}]^{w_2(d,d_1,v)}(vw_1(d, d_1, v))\}w_3(d, d_1, v), \quad (76)$$

where $w_j = w_j(d, d_1, v) \in \mathcal{C}(D)$, $j \in \{1, 2, 3\}$, $w_1^\tau = w_1$.

Proof. Equations (72) and (33) imply the identity of Equation (75).

Let $v \in V$, d and d_1 belong to D, and $f \in F$; then, from Equations (72) and (73), it follows that

$$f^{\{dd_1\}}(v) = f^{s(dd_1,v)}(v^{[(dd_1)\backslash e]}) \quad (77)$$

and

$$(f^{\{d_1\}})^{\{d\}}(v) = (f^{s(d_1,v)})^{s(d,v^{[d_1\backslash e]})}((v^{[d_1\backslash e]})^{[d\backslash e]}). \quad (78)$$

From Equations (24), (69), (58), (61), (11), and (13) and Lemma 2, we deduce that

$$(dd_1)\backslash e = (d_1\backslash e)(d \backslash e)t_D(dd_1, d_1\backslash e, d\backslash e)/t_D(d, d_1, d_1\backslash e)$$

and

$$v^{[(dd_1)\backslash e]} = (v^{[d_1\backslash e]})^{[d\backslash e]} w_1(d, d_1, v), \tag{79}$$

where $w_1(d, d_1, v) = \gamma^\tau$, where

$$\gamma = t_D(dd_1, d_1\backslash e, d \setminus e) t_D((v/d_1)^\psi, (v/d_1)^\tau, d\backslash e)$$
$$/[t_D(d, d_1, d_1\backslash e) t_D(v, e/d_1, e/d) t_D((v/d_1)^\psi, ((v/d_1)^\tau/d)^\psi, (v/(dd_1))^\tau)].$$

Then Equations (73), (66), (25), (13), and (62) and Lemma 2 imply that

$$s(dd_1, v) = e/([(v/d_1)(e/d)]^\psi \gamma_1^\psi, \tag{80}$$

where $\gamma_1 = t_D(v/(dd_1), d, d_1) t_D(e/d, d, d \setminus e)/t_D(v/(dd_1), d, d\backslash e)$ by (13), (25) and Lemma 2;

$$[(v/d_1)(e/d)]^\psi = (v/d_1)^\psi [(v/d_1)^\tau (e/d)]^\psi \gamma_2^\psi, \tag{81}$$

where

$$\gamma_2 = t_D((v/d_1)^\psi, (v/d_1)^\tau, e/d)/t_D((v/d_1)^\psi, ((v/d_1)^\tau (e/d))^\psi, ((v/d_1)(e/d))^\tau).$$

Note that

$$s(dd_1, v) = (e/[(v/d_1)^\tau (e/d)]^\psi)(e/(v/d_1)^\psi)\gamma_3/\{\gamma_1^\psi \gamma_2^\psi\} \tag{82}$$

by Equations (81) and (23), where

$$\gamma_3 = t_D(e/(v/d_1)^\psi, (v/d_1)^\psi, [(v/d_1)^\tau (e/d)]^\psi)$$
$$/t_D(e/[(v/d_1)^\tau (e/d)]^\psi, e/(v/d_1)^\psi, (v/d_1)^\psi [(v/d_1)^\tau (e/d)]^\psi).$$

Then,

$$(v/d_1)^\tau = [v(d_1\backslash e)]^\tau \gamma_4^\tau = v^{[d_1\backslash e]} \gamma_4^\tau$$

by Equations (11), (13), (63), and (68), where $\gamma_4 \in \mathcal{C}_m(D)$. Hence,

$$[(v/d_1)^\tau (e/d)]^\psi = [v^{[d_1\backslash e]}(e/d)]^\psi \gamma_5^\psi,$$

since

$$(\gamma_4^\tau)^\psi = e, \tag{83}$$

where $\gamma_5 = t_D(v^{[d_1\backslash e]}/d, d, d\backslash e)/t_D(e/d, d, d \setminus e)$.

Thus, the identities of Equations (80)–(83) imply that

$$s(dd_1, v) = s(d, v^{[d_1\backslash e]}) s(d_1, v) w_2(d, d_1, v), \tag{84}$$

where

$$w_2(d, d_1, v) = \gamma_3/(\gamma_1^\psi \gamma_2^\psi \gamma_5^\psi),\ w_2(d, d_1, v) \in \mathcal{C}(D).$$

By Lemmas 1 and 2 and Equation (73), representations of γ_j simplify:

$$\gamma_2 = t_D(e/s(d_1, v), v^{[d_1\backslash e]}, e/d)/t_D(e/s(d_1, v), e/s(d, v^{[d_1\backslash e]}), v^{[(dd_1)\backslash e]}),$$
$$\gamma_3 = t_D(s(d_1, v), e/s(d_1, v), e/s(d, v^{[d_1\backslash e]}))$$
$$/t_D(s(d, v^{[d_1\backslash e]}), s(d_1, v), e/(s(d, v^{[d_1\backslash e]}) s(d_1, v))).$$

Therefore, $w_2(d, d_1, v) \in \mathcal{C}(D) \cap A$ for each d and d_1 in D and $v \in V$, since $s(d, d_1, v)$, $s(d, v^{[d_1\backslash e]})$, and $s(d_1, v)$ belong to A. Then, from Equations (77), (84), (31), (32), we infer that

$$f^{\{dd_1\}}(v) = \{[(f^{s(d_1,v)})^{s(d,v^{[d_1\backslash e]})}]^{w_2(d,d_1,v)}((v^{[d_1\backslash e]})^{[d\backslash e]} w_1(d, d_1, v))\} w_3(d, d_1, v), \tag{85}$$

where $w_3(d,d_1,v) = e/[\eta(s(d,v^{[d_1\backslash e]})w_2, s(d_1,v), b)\eta(s(d,v^{[d_1\backslash e]}), w_2, b^{s(d_1,v)})]$,

$w_2 = w_2(d,d_1,v)$, $b = f((v^{[d_1\backslash e]})^{[d\backslash e]}w_1(d,d_1,v))$. Equations (68) and (61) imply that $(v\gamma)^{[a]} = v^{[a]}\gamma^\tau$ for each $v \in V$ and $\gamma \in \mathcal{C}(D)$, $a \in D$; consequently, $((vw_1)^{[d_1\backslash e]})^{[d\backslash e]} = (v^{[d_1\backslash e]})^{[d\backslash e]}w_1$, and hence, $vw_1 \in V$ for each $v \in V$, d and d_1 in D, $w_1 = w_1(d,d_1,v)$ by Equation (79), since $w_1 = \gamma^\tau$. Thus Equation (76) follows from Equations (78) and (85). □

Definition 4. *Suppose that the conditions of Remark 4 are satisfied and on the Cartesian product $C = D \times F$ (or $C^* = D \times F^*$) a binary operation is given by the following formula:*

$$(d_1, f_1)(d, f) = (d_1 d, \xi((d_1^\psi, f_1), (d^\psi, f)) f_1 f^{\{d_1\}}), \tag{86}$$

where $\xi((d_1^\psi, f_1), (d^\psi, f))(v) = \xi((d_1^\psi, f_1(v)), (d^\psi, f(v)))$ for every d and d_1 in D, f and f_1 in F (or F^* respectively), and $v \in V$.

Theorem 5. *Let C, C^*, D, F, and F^* be the same as in Definition 4. Then, C and C^* are loops and there are natural embeddings $D \hookrightarrow C$, $F \hookrightarrow C$, $D \hookrightarrow C^*$, and $F^* \hookrightarrow C^*$ such that F (or F^*) is an almost normal subloop in C (or C^* respectively).*

Proof. The operation of Equation (86) is single-valued. Let $a = (d, f)$ and $b = (d_0, f_0)$, where d and d_0 are in D and where f and f_0 are in F (or F^*).

The equation $ay = b$ is equivalent to $dd_2 = d_0$ and

$$\xi((d^\psi, f), (d_2^\psi, f_2)) f f_2^{\{d\}} = f_0,$$

where $d_2 \in D$, $f_2 \in F$ (or $f_2 \in F^*$ respectively), $y = (d_2, f_2)$, $\xi((d^\psi, f), (d_2^\psi, f_2))(v) = \xi((d^\psi, f(v)), (d_2^\psi, f_2(v)))$ for each $v \in V$. Therefore, $d_2 = d\backslash d_0$, $f_2^{\{d\}} = [\xi((d^\psi, f), ((d\backslash d_0)^\psi, f_2))f]\backslash f_0$ by Equation (1) and Theorem 2. On the other hand, $f_2^{\{e\}} = f_2$ by Equation (74) and $f_2(v) = \{[(f_2^{\{d\}})^{\{d_3\}}]^{w_2}(vw_1)\}w_3$ by Equation (76), where $w_j = w_j(d, d_3, v)$, $j \in \{1,2,3\}$, $d_3 = d\backslash e$, and $dd_3 = e$ by Equation (14). Thus, using Equation (35), we get that

$$y = (d\backslash d_0, \{\{([\xi((d^\psi, f), ((d\backslash d_0)^\psi, [(f\backslash f_0)^{\{d\backslash e\}}]^{w_2} w_3))f] \setminus f_0)^{\{d\backslash e\}}\}^{w_2}(vw_1)\}w_3)$$

belongs to C (or C^* respectively), giving Equation (1).

Then, we seek a solution $x \in C$ (or $x \in C^*$ respectively) of the equation $xa = b$. It is equivalent to two equations: $d_1 d = d_0$ and

$$\xi((d_1^\psi, f_1(v)), (d, f(v))) f_1(v) f^{\{d_1\}}(v) = f_0(v)$$

for each $v \in V$, where $d_1 \in D$, $f_1 \in F$ (or $f_1 \in F^*$ respectively), and $x = (d_1, f_1)$. Therefore, $d_1 = d_0/d$ and $f_1(v) = f_0(v)/[\xi(((d_0/d)^\psi, f_1(v)), (d, f(v))) f^{\{d_0/d\}}(v)]$. Thus,

$$x = (d_0/d, f_0/[\xi(((d_0/d)^\psi, f_1), (d, f)) f^{\{d_0/d\}}])$$

belongs to C (or C^* respectively), giving Equation (2).

Moreover, $(e,e)(d,f) = (d,f)$ and $(d,f)(e,e) = (d,f)$ for each $d \in D$, $f \in F$ (or $f \in F^*$ respectively) by Equations (35) and (86). Therefore, the condition of Equation (3) is also satisfied. Thus, C and C^* are loops.

Evidently $D \ni d \mapsto (d,e)$ and $F \ni f \mapsto (e,f) \in C$ (or $F^* \ni f \mapsto (e,f) \in C^*$ respectively) provide embeddings of D and F (or D and F^* respectively) into C (or C^* respectively).

It remains to verify that F (or F^* respectively) is an almost normal subloop in C (or C^* respectively). Assume that $d_1 \in D$, $f_1 \in F$. Then,

$$(d_1, f_1)F = \{(d_1, \xi((d_1^{\psi}, f_1), (e, f))f_1 f^{\{d_1\}}) : f \in F\}$$

and

$$F(d_1, f_1) = \{(d_1, \xi((e, f), (d_1^{\psi}, f_1))f f_1) : f \in F\}.$$

Using the embedding $\mathcal{C}^V \hookrightarrow F$ and Equation (35), we infer that $(d_1, f_1)F = F(d_1, f_1)$, since $F^{\{d_1\}} = F$ by Equation (68), Lemma 5, and Equation (30). It can be verified similarly that F^* is the almost normal subloop in C^*. □

Definition 5. *The product Equation* (86) *in the loop C (or C^*) of Theorem 5 is called a smashed twisted wreath product of D and F (or a restricted smashed twisted wreath product of D and F^* respectively) with smashing factors ϕ, η, κ, and ξ, and it will be denoted by $C = D\Delta^{\phi,\eta,\kappa,\xi}F$ (or $C^* = D\Delta^{\phi,\eta,\kappa,\xi}F^*$ respectively). The loop C (or C^*) is also called a smashed splitting extension of F (or of F^* respectively) by D.*

Theorem 6. *Let the conditions of Remark 4 be satisfied and $\mathcal{C}_m(D) \subseteq \mathcal{C}$, where $\mathcal{C}$ is as in Equation* (28). *Then, C and C^* supplied with the binary operation of Equation* (86) *are metagroups.*

Proof. In view of Theorem 5, C and C^* are loops. To each element b in B, there corresponds an element $\{b(v) : \forall v \in V,\ b(v) = b\}$ in F which can be denoted by b also. From the conditions of Equations (29)–(35), we deduce that

$$\gamma^a = \gamma \text{ and } f^{\gamma} = f \text{ for every } \gamma \in \mathcal{C} \text{ and } a \in A. \tag{87}$$

Hence, Equations (87) and (86) imply that $(\mathcal{C}(A), \mathcal{C}(F)) \subseteq \mathcal{C}(C)$. On the other hand, $w_1 = \gamma^{\tau}$ with $\gamma \in \mathcal{C}_m(D)$ and $w_2 = \gamma_3/(\gamma_1^{\psi}\gamma_2^{\psi}\gamma_5^{\psi})$ with $\gamma_1,...,\gamma_5$ in $\mathcal{C}_m(D)$ (see Equation (84)); hence, the condition $\mathcal{C}_m(D) \subset \mathcal{C}$ implies that Equation (76) simplifies to

$$f^{\{dd_1\}}(v) = (f^{\{d\}}(v))^{\{d_1\}} w_3(d, d_1, v) \tag{88}$$

for each $f \in F$, $v \in V$, and d and d_1 in D, since $\mathcal{C} \subseteq \mathcal{C}(A)$ by Equation (28). Next, we consider the following products:

$$I_1 = ((d_2, f_2)(d_1, f_1))(d, f) = ((d_2 d_1, \xi((d_2^{\psi}, f_2), (d_1^{\psi}, f_1))f_2 f_1^{\{d_2\}})(d, f) \tag{89}$$

and

$$I_2 = (d_2, f_2)((d_1, f_1)(d, f)) = (d_2, f_2)(d_1 d, \xi((d_1^{\psi}, f_1), (d^{\psi}, f))f_1 f^{\{d_1\}}). \tag{90}$$

Then, Equations (86), (90), and (33)–(35) imply that

$$I_2 = (d_2(d_1 d), \xi((d_1^{\psi}, f_1), (d^{\psi}, f))\xi((d_2^{\psi}, f_2), ((d_1 d)^{\psi}, f_1 f^{\{d_1\}}))\kappa(s(d_2, v), f_1(v^{[d_2\setminus e]}), f^{\{d_1\}}(v^{[d_2\setminus e]}))f_2(v)[f_1^{\{d_2\}}(v)(f^{\{d_1\}})^{\{d_2\}}(v)]. \tag{91}$$

From Equations (88), (89), (76), and (35), we infer that

$$I_1 = ((d_2 d_1)d, \xi((d_1^{\psi}, f_1), (d^{\psi}, f))\xi(((d_2 d_1)^{\psi}, f_2 f_1^{\{d_2\}}), (d^{\psi}, f))(f_2 f_1^{\{d_2\}})(f^{\{d_1\}})^{\{d_2\}} w_3, \tag{92}$$

where $w_3 = w_3(d_1, d_2, v)$. Therefore, from Equations (91) and (92), we infer that

$$I_1 = t_C((d_2, f_2), (d_1, f_1), (d, f))I_2, \tag{93}$$

where

$$t_C((d_2,f_2),(d_1,f_1),(d,f)) = t_D(d_2,d_1,d)t_B(f_2,f_1,f^{\{d_2d_1\}})\xi((d_1^{\psi},f_1),(d^{\psi},f))$$
$$\xi((d_2^{\psi},f_2),((d_1d)^{\psi},f_1f^{\{d_1\}}))\kappa(s(d_2,v),f_1(v^{[d_2\backslash e]}),f^{\{d_1\}}(v^{[d_2\backslash e]}))$$

$$/[\xi((d_2^{\psi},f_2),(d_1^{\psi},f_1))\xi(((d_2d_1)^{\psi},f_2f_1^{\{d_2\}}),(d^{\psi},f))w_3(d_1,d_2,v)]; \quad (94)$$

$$t_B(f_2,f_1,f)(v) = t_B(f_2(v),f_1(v),f(v)); \quad (95)$$

$$\xi((d_2^{\psi},f_2),(d_1^{\psi},f_1))(v) = \xi((d_2^{\psi},f_2(v)),(d_1^{\psi},f_1(v))) \quad (96)$$

for every f, f_1, f_2 in F, d, d_1, d_2 in D, and $v \in V$. Then from Equation (93), $\mathcal{C}(F) = (\mathcal{C}(B))^V$ (see Theorem 2) and Equation (28), it follows that the loops C and C^* satisfy the condition of Equation (9), since $(\mathcal{C},\mathcal{C}^V) \subseteq \mathcal{C}(C)$. Thus, C and C^* are metagroups. □

Remark 5. *Generally, if $A \neq \{e\}$ and $A \neq D$, B, ϕ, η, κ, and ξ are nontrivial, where A, B, and D are metagroups or particularly may be groups, then the loops C and C^* of Theorem 5 can be non-metagroups. If Equation* (35) *drops the conditions $\xi((e,e),(v,b)) = e$ and $\xi((v,b),(e,e)) = e$ for each $v \in V$ and $b \in B$, then the proofs of Theorems 3–5 demonstrate that C_1 and C_2 are strict quasi-groups and that C and C^* are quasi-groups.*

Definition 6. *Let P_1 and P_2 be two loops with centers $\mathcal{C}(P_1)$ and $\mathcal{C}(P_2)$. Let also*

$$\mu(a,b) = \nu(a,b)\mu(a)\mu(b) \quad (97)$$

for each a and b in P_1, where $\nu(a,b) \in \mathcal{C}(P_2)$. Then, μ will be called a metamorphism of P_1 into P_2. If in addition μ is surjective and bijective, then it will be called a metaisomorphism and it will be said that P_1 is metaisomorphic to P_2.

Theorem 7. *Suppose that A, B, and D are metagroups and that $A \subset D$, V_1, and V_2 are right transversals of A in D, $F_j = B^{V_j}$,*

$$P_j = D\Delta^{\phi,\eta,\kappa,\xi}F_j,\ P_j^* = D\Delta^{\phi,\eta,\kappa,\xi}F_j^*,\ j \in \{1,2\}.$$

Then, P_1 is metaisomorphic to P_2 and P_1^* to P_2^*.

Proof. By virtue of Theorem 5, P_j and P_j^* are loops, where $j \in \{1,2\}$, $\mathcal{C}^{V_j} \subset \mathcal{C}(P_j)$. From Equations (62) and (73), it follows that

$$s_j(\delta d,v) = s_j(d,v/\delta) = \delta^{\psi_j}s_j(d,v) \quad (98)$$

for each $d \in D$, $v \in V_j$, and $\delta \in \mathcal{C}(D)$, where s_j, $v^{[a]_j}$, d^{τ_j}, and d^{ψ_j} correspond to V_j, $j \in \{1,2\}$. Then, Equations (68) and (63) imply that

$$v^{[\delta/d]_j} = v^{[e/d]}\delta^{\tau_j} \quad (99)$$

for each $d \in D$, $v \in V_j$, and $\delta \in \mathcal{C}(D)$, $j \in \{1,2\}$. Therefore, from the identities of Equations (98), (99), and (84) and Lemma 2, we infer that

$$w_2(\delta d,\delta_1d_1,\delta_2v) = w_2(d,d_1,v) \quad (100)$$

for each of d and d_1 in D, δ, δ_1 and δ_2 in $\mathcal{C}(D)$, and $v \in V$.

For each of $f \in F_1$ and $v \in V_2$, we put

$$\mu f(v) = f^{e/v^{\psi_1}}(v^{\tau_1}). \quad (101)$$

From Lemma 5, it follows that $V_2^{\tau_1} = V_1$ and $v_1^{\tau_1} \neq v_2^{\tau_1}$ for each $v_1 \neq v_2$ in V_2, where $V_2^{\tau_1} = \{v^{\tau_1} : v \in V_2\}$. Then, Equations (87), (62), (100), and (101) and Lemma 1 imply that

$$f^{\{d\}\mu} = f^{\mu\{d\}} \tag{102}$$

for each $f \in F_1$, $d \in D$, where $f^{\mu\{d\}} = (\mu f)^{\{d\}}$, $f^{\{d\}\mu} = \mu(f^{\{d\}})$ (see also Equation (72)). From the identity of Equation (102) and the conditions of Equations (33) and (34), we infer that

$$\mu((d_1, f_1)(d, f))(v) = \kappa(e/v^{\psi_1}, f_1(v^{\tau_1}), f^{\{d_1\}}(v^{\tau_1}))(\mu(d_1, f_1))(\mu(d, f)) \tag{103}$$

for each of d and d_1 in D, f and f_1 in F_1, and $v \in V_2$, where $\mu(d, f) = (d, \mu f)$, $(d, f)(v) = (d, f(v))$. Hence,

$$\nu((d_1, f_1), (d, f))(v) = \kappa(e/v^{\psi_1}, f_1(v^{\tau_1}), f^{\{d_1\}}(v^{\tau_1})) \in \mathcal{C} \tag{104}$$

for each $v \in V_2$ (see also Equations (28) and (30)). Thus, P_1 is metaisomorphic to P_2 and P_1^* to P_2^*. □

Theorem 8. *Suppose that D is a nontrivial metagroup. Then, there exists a smashed splitting extension C^* of a nontrivial central metagroup H by D such that $[H, C^*]\mathcal{C}(H) = H$, where $[a, b] = (e/a)((e/b)(ab))$ for each a and b in C^*.*

Proof. Let d_0 be an arbitrary fixed element in $D - \mathcal{C}(D)$. Assume that A is a submetagroup in D such that A is generated by d_0 and a subgroup $\mathcal{C}_0$ contained in a center $\mathcal{C}(D)$ of D, $\mathcal{C}_m(D) \subseteq \mathcal{C}_0 \subseteq \mathcal{C}(D)$, where $\mathcal{C}_m(D)$ is a minimal subgroup in a center $\mathcal{C}(D)$ of D such that $t_D(a, b, c) \in \mathcal{C}_m(D)$ for each of a, b, and c in D. Therefore,

$$a^k a^n = p(k, n, a) a^{k+n} \tag{105}$$

for each $a \in A$, k, and n in $\mathcal{C} = \{0, -1, 1, -2, 2, ...\}$, where the following notation is used: $a^2 = aa$, $a^{n+1} = a^n a$ and $a^{-n} = e/a^n$, and $a^0 = e$ for each $n \in \mathbf{N}$ and $p(k, n, a) \in \mathcal{C}_m(A)$. Hence, in particular, A is a central metagroup. Then, $d_0\mathcal{C}_m(A)$ is a cyclic element in the quotient group $A/\mathcal{C}_m(A)$ (see Theorem 1). Then, we choose a central metagroup B generated by an element b_0 and a commutative group $\mathcal{C}_1$ such that $b_0 \notin \mathcal{C}_1$, $\mathcal{C}_m(D) \hookrightarrow \mathcal{C}_1$ and $\mathcal{C}(A) \hookrightarrow \mathcal{C}_1$ and the quotient group $B/\mathcal{C}_m(B)$ is of finite order $l > 1$. Then, let $\phi : A \to \mathcal{A}(B)$ satisfy the condition of Equation (30) and be such that

$$\phi(d_0) b_0 = b_0^2. \tag{106}$$

To satisfy the condition of Equation (106), a natural number l can be chosen as a divisor of $2^{|d_0\mathcal{C}_m(A)|} - 1$ if the order $|d_0\mathcal{C}_m(A)|$ of $d_0\mathcal{C}_m(A)$ in $A/\mathcal{C}_m(A)$ is positive; otherwise, l can be taken as any fixed odd number $l > 1$ if $A/\mathcal{C}_m(A)$ is infinite.

Then, we take a right transversal V of A in D so that A is represented in V by e. Let Ξ, η, κ, and ξ be chosen to satisfy the conditions of Equations (29)–(35), where $\mathcal{C}_m(B) \hookrightarrow \mathcal{C}$, $\mathcal{C}_m(A) \hookrightarrow \mathcal{C}$, $\mathcal{C}_0 \hookrightarrow \mathcal{C}$, and $\mathcal{C}_1 \hookrightarrow \mathcal{C}$. With these data, according to Theorem 6, C^* is a metagroup, since $\mathcal{C}_m(D) \hookrightarrow \mathcal{C}_1$ and $\mathcal{C}_m(D) \hookrightarrow \mathcal{C}_0$. That is, C^* is a smashed splitting extension of the central metagroup F^* by D.

Apparently, there exists $f_0 \in F^*$ such that $f_0(e) = b_0$, $f_0(v) = e$ for each $v \in V - \{e\}$. Therefore, $f_0^{\{v\}}(v) = b_0$ for each $v \in V$, since $s(v, v) = e$, $v^{[v\backslash e]} = [v(v\backslash e)]^\tau = e$.

Let $v_1 \neq v_2$ belong to V. Then, $(v_2(v_1\backslash e))^\tau = v_3 \in V$. Assume that $v_3 = e$. The latter is equivalent to $v_2(v_1\backslash e) = a \in A$. From Equation (13), it follows that $v_2 = a/(v_1\backslash e) = \gamma a v_1$, where $\gamma = t_D(v_1, v_1\backslash e, v_1)/t_D(av_1, v_1\backslash e, v_1)$ by Equation (11) and Lemma 2, since $e/(v_1\backslash e) = v_1$. Hence, $v_2 = v_2^\tau = (\gamma a v_1)^\tau = \gamma^\tau v_1$ by Equation (63), and consequently, $(v_2(v_1\backslash e))^\tau = \gamma^\tau = e$, contradicting the supposition $v_1 \neq v_2$. Thus, $v_3 \neq e$, and consequently, $f_0^{\{v_1\}}(v_2) = e^{s(v_1, v_2)} = e$ by Equation (31). This implies that $\{f_0^{\{v\}} : v \in V\}\mathcal{C}(F^*)$ generates F^*.

Evidently, $[v(d_0\backslash e)]^\tau \neq e$ for each $v \in V - \{e\}$, since $d_0\backslash e \in A$ and the following conditions $s \in D$, $sq \in A$, and $q \in A$ imply that $s \in A$ because A is the submetagroup in D. Note that

$e/d = (d \setminus e)/t_A(e/d, d, d\backslash e)$ for each $d \in A$ by Equation (11); consequently, $s(d,e) = dt_A(e/d, d, d\backslash e)$. On the other hand, $t_A(a,b,c) \in \mathcal{C}$ for each of a, b, and c in A and

$$f_0^{\gamma} = f_0 \; for \; each \; \gamma \in \mathcal{C} \tag{107}$$

by Equation (87); hence, $f_0^{\{d_0\}}(e) = \phi(d_0)b_0 = b_0^2$, and consequently,

$$f_0^{\{d_0\}} = f_0^2,$$

since

$$f_0^{\{d_0\}}(v) = e \; for \; each \; v \in V - \{e\}. \tag{108}$$

Therefore, we deduce using Equation (107) that

$$[(e, f_0), (e/d_0, e)] = (e, wf_0), \tag{109}$$

where

$$w = \xi((e, f_0), (e/d_0, e))\xi((d_0, e), (e/d_0, f_0))$$
$$\xi((e, e/f_0), (e, (f_0)^2))/t_{F^*}(e/f_0, f_0, f_0), \tag{110}$$

$$t_{F^*}(f, g, h)(v) = t_B(f(v), g(v), h(v)) \;\; for \; each \;\; v \in V, f, g \; and \; h \;\; in \;\; F^*. \tag{111}$$

Thus, $w = w(v) \in \mathcal{C}$ for each $v \in V$ and $f_0 \in [F^*, C^*]$, since $\mathcal{C}^V \cap F^* \subset \mathcal{C}(F^*)$. Hence, $F^* \subseteq [F^*, C^*]\mathcal{C}(F^*)$, since $F^* \hookrightarrow C^*$ and $\mathcal{C}(C^*) \cap F^* \subseteq \mathcal{C}(F^*)$. On the other hand, $\mathcal{C}_m(A) \hookrightarrow \mathcal{C}$, $\mathcal{C}_m(B) \hookrightarrow \mathcal{C}$, $\mathcal{C}_m(D) \hookrightarrow \mathcal{C}_j$, and $\mathcal{C}_j \hookrightarrow \mathcal{C}$ for each $j \in \{0, 1\}$. Therefore, Equations (107), (108), and (88) imply that $cF^* = F^*c$ and $c[F^*, C^*]\mathcal{C}(F^*) = [F^*, C^*]\mathcal{C}(F^*)c$ for each $c \in C^*$. Hence, $[F^*, C^*]\mathcal{C}(F^*) \subseteq F^*$. Taking $H = F^*$, we get the assertion of this theorem. □

Corollary 2. *Let the conditions of Theorem 8 be satisfied and D be generated by $\mathcal{C}_m(D)$ and at least two elements d_1, d_2,... such that $d_1 \neq e$ and $[d_2\backslash e, d_1\backslash e] = e$. Then, the smashed splitting extension C^* can be generated by $\mathcal{C}(F^*)$ and elements c_1, c_2,... such that $d_j\backslash e \in F^*c_j$ for each j.*

Proof. We take $d_0 = d_1$ in the proof of Theorem 8; thus, $c_1 = (d_1\backslash e, e)$, $c_2 = (d_2\backslash e, f_0)$, and $c_j = (d_j\backslash e, e)$ for each $j \geq 3$. Therefore Equations (66), (108), and (35) imply that

$$[c_2, c_1] = (e, pf_0), \text{ where}$$

$$p = \xi((d_2\backslash e, f_0), (d_1\backslash e, e))\xi((d_1, e), ((d_2\backslash e)(d_1 \setminus e), f_0))$$

$$\xi((e, e)/(d_2\backslash e, f_0), (d_2 \setminus e, (f_0)^2))/t_{F^*}(e/f_0, f_0, f_0), \tag{112}$$

since $[d_2\backslash e, d_1\backslash e] = e$ and $e/(d_2\backslash e) = d_2$. Thus, the submetagroup of C^* which is generated by $\mathcal{C}_m(D)$ and $\{c_j : j\}$ contains the metagroup D and (e, pf_0). Therefore, the following set $\{f^{\{d\}} : d \in D\}\mathcal{C}(F^*)$ generates the central metagroup F^*, since $V \subset D$ and $\{f^{\{v\}} : v \in V\}\mathcal{C}(F^*)$ generate F^*. Notice that $\mathcal{C}_m(D) \hookrightarrow \mathcal{C}(F^*)$. Hence, $\{c_j : j\}\mathcal{C}(F^*)$ generates C^*. □

Example 1. *Assume that A is a unital algebra over a commutative associative unital ring F supplied with a scalar involution $a \mapsto \bar{a}$ so that its norm N and trace T maps have values in F and fulfil conditions:*

$$a\bar{a} = N(a)1 \text{ with } N(a) \in F, \tag{113}$$

$$a + \bar{a} = T(a)1 \text{ with } T(a) \in F, \tag{114}$$

$$T(ab) = T(ba) \tag{115}$$

for each a and b in A.

We remind that, if a scalar $f \in F$ *satisfies the condition* $\forall a \in A\ fa = 0 \Rightarrow a = 0$*, then such element* f *is called cancelable. For such a cancelable scalar* f*, the Cayley–Dickson doubling procedure induces a new algebra* $C(A, f)$ *over F such that*

$$C(A, f) = A \oplus Al, \tag{116}$$

$$(a + bl)(c + dl) = (ac - f\bar{d}b) + (da + b\bar{c})l \tag{117}$$

and

$$\overline{(a + bl)} = \bar{a} - bl \tag{118}$$

for each a and b in A. Such an element l is called a doubling generator. From Equations (113)–(115)*, it follows that* $\forall a \in A, \forall b \in A\ \ T(a) = T(a + bl)$ *and* $N(a + bl) = N(a) + fN(b)$*. Apparently, the algebra A is embedded into* $C(A, f)$ *as* $A \ni a \mapsto (a, 0)$*, where* $(a, b) = a + bl$*. It is put by induction* $A_n(f_{(n)}) = C(A_{n-1}, f_n)$*, where* $A_0 = A$*,* $f_1 = f$*,* $n = 1, 2, ...,$ *and* $f_{(n)} = (f_1, ..., f_n)$*. Then,* $A_n(f_{(n)})$ *is a generalized Cayley–Dickson algebra, when F is not a field, or a Cayley–Dickson algebra, when F is a field.*

There is an algebra $A_\infty(f) := \bigcup_{n=1}^{\infty} A_n(f_{(n)})$*, where* $f = (f_n : n \in \mathbf{N})$*. In the case of* $char(F) \neq 2$*, let* $Im(z) = z - T(z)/2$ *be the imaginary part of a Cayley–Dickson number z and, hence,* $N(a) := N_2(a, \bar{a})/2$*, where* $N_2(a, b) := T(a\bar{b})$*.*

If the doubling procedure starts from $A = F1 =: A_0$*, then* $A_1 = C(A, f_1)$ *is a* $*$*-extension of F. If* A_1 *has a basis* $\{1, u\}$ *over F with the multiplication table* $u^2 = u + w$*, where* $w \in F$ *and* $4w + 1 \neq 0$*, with the involution* $\bar{1} = 1$*,* $\bar{u} = 1 - u$*, then* A_2 *is the generalized quaternion algebra and* A_3 *is the generalized octonion (Cayley–Dickson) algebra.*

Particularly, for $F = \mathbf{R}$ *and* $f_n = 1$ *for each n by* $\mathcal{A}_r$ *the real Cayley-Dickson algebra with generators* $i_0, ..., i_{2^r-1}$ *will be denoted such that* $i_0 = 1$*,* $i_j^2 = -1$ *for each* $j \geq 1$*, and* $i_j i_k = -i_k i_j$ *for each* $j \neq k \geq 1$*. Note that the Cayley–Dickson algebra* $\mathcal{A}_r$ *for each* $r \geq 3$ *is nonassociative, for example,* $(i_1 i_2) i_4 = -i_1(i_2 i_4)$*, etc. Moreover, for each* $r \geq 4$*, the Cayley–Dickson algebra* $\mathcal{A}_r$ *is nonalternative (see References [7,11,12]). Frequently,* $\bar{a}$ *is also denoted by* a^* *or* $\tilde{a}$*.*

Then, $G_r = \{i_j, \ -i_j : \ j = 0, 1, ..., 2^r - 1\}$ *is a finite metagroup for each* $3 \leq r < \infty$*. Equation* (117) *is an example of the smashed product.*

Then, one can take a Cayley–Dickson algebra A_n *over a commutative associative unital ring* $\mathcal{R}$ *of characteristic different from two such that* $A_0 = \mathcal{R}$*,* $n \geq 2$*. There are basic generators* $i_0, i_1, ..., i_{2^n-1}$*, where* $i_0 = 1$*. Choose* $\mathbf{\Psi}$ *as a multiplicative subgroup contained in the ring* $\mathcal{R}$ *such that* $f_j \in \mathbf{\Psi}$ *for each* $j = 0, ..., n$*. Put* $G_n = \{i_0, i_1, ..., i_{2^n-1}\} \times \mathbf{\Psi}$*. Then,* G_n *is a central metagroup because, in this case,* $\mathbf{\Psi}$ *is commutative.*

Example 2. *More generally, suppose that H is a group such that* $\mathbf{\Psi} \subset H$*, with relations* $hi_k = i_k h$ *and* $(hg)i_k = h(gi_k)$ *for each* $k = 0, 1, ..., 2^n - 1$ *and each h and g in H. Then,* $G_n = \{i_0, i_1, ..., i_{2^n-1}\} \times H$ *is also a metagroup. If the group H is noncommutative, then the latter metagroup can be noncentral (see the condition of Equation* (10) *in Definition 1). Utilizing the notation of Example 1, we get that the Cayley–Dickson algebra* $\mathcal{A}_\infty$ *over the real field* $\mathbf{R}$ *with* $f_n = 1$ *for each n provides a pattern of a metagroup* $G_\infty = \{i_j, \ -i_j : \ 0 \leq j \in \mathbf{Z}\}$*, where* $\mathbf{Z}$ *denotes the ring of integers.*

Example 3. *Certainly, in general, metagroups need not be central. On the other hand, if a metagroup is associative, then it is a group [1]. Apparently, each group is a metagroup also. For a group G, its associativity evidently means that* $t_G(a, b, c) = e$ *[1].*

From the given metagroups, new metagroups can be constructed using their direct, semidirect products, smashed products, and smashed twisted wreath products. Therefore, there are abundant families of noncentral metagroups and also of central metagroups different from groups.

Equations (39)*,* (46)*,* (47)*,* (85)*,* (86)*, and* (94)*–*(96) *provide examples of metagroups with complicated nonassociative noncommutative structures. The presented above theorems also permit to construct different examples of nonassociative quasi-groups and loops.*

5. Conclusions

The results of this article can be used for further studies of metagroups, quasi-groups, loops, and noncommutative manifolds related with them. Besides applications of metagroups, loops, and quasi-groups outlined in the introduction, it is interesting to mention possible applications in mathematical coding theory and classification of information flows and their technological implementations [28–30] because, frequently, codes are based on binary systems. Moreover, twisted products are used for creating complicated codes [22]. In view of this, to study creating more complicated codes with the help of smashed twisted products of metagroups, Equations (86) and (94)–(96) provide additional options in the nonassociative case in comparison with the associative case.

Wreath products of groups are used for studies of varieties [24], so it will be interesting to investigate noncommutative varieties using metagroups. Then, twisted products are utilized for investigations of Lie groups and semi-Riemann manifolds [23,25]. Therefore, we will study their nonassociative metagroup analogs that can be used in noncommutative geometry and quantum field theory [16,31–35] because Lie groups and manifolds are actively used in these areas.

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