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Admissible Hybrid $\mathcal{Z}$-Contractions in *b*-Metric Spaces

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Received: 25 November 2019; Accepted: 19 December 2019; Published: 21 December 2019

Abstract: In this manuscript, we introduce a new notion, admissible hybrid $\mathcal{Z}$-contraction that unifies several nonlinear and linear contractions in the set-up of a *b*-metric space. In our main theorem, we discuss the existence and uniqueness result of such mappings in the context of complete *b*-metric space. The given result not only unifies the several existing results in the literature, but also extends and improves them. We express some consequences of our main theorem by using variant examples of simulation functions. As applications, the well-posedness and the Ulam–Hyers stability of the fixed point problem are also studied.

Keywords: admissible spaces; hybrid contraction; interpolative contraction; *b*-metric spaces; simulation function; interpolative contraction

MSC: 47H10; 54H25

1. Introduction

Metric fixed point theory can be settled in the intersection of two disciplines; (nonlinear) functional analysis and topology. From the fixed point researchers' aspect, the first application of the metric fixed point theory is on the solution of differential equations. However, according to the point of view of researchers in applied mathematics, metric fixed point theory is a tool in the solution of a first-order ordinary differential equation with an initial value. Indeed, fixed point theory appears, firstly, in the paper of Liouville in 1837, and, later, in the paper of Picard in 1890. In the paper of Picard, the method of the successive approaches was used to investigate the existence of the solution. In 1922, Banach reported the first metric fixed point result in the setting of complete norm space that would be called Banach space later. Examined enough and carefully, we realized that Banach's theorem is the abstraction of the successive approaches. The characterization of the nominated fixed point theorem of Banach, in the complete metric space, was reported by Caccioppoli in 1931. This can be accepted as the first generalization of Banach's theorem. After this, a huge number of papers, on the generalization and extension of Banach's fixed point theorem, has been released.

Extensions and generalizations of Banach's theorem are based on two elements: by changing the structure (abstract space) and by changing the conditions on the considered mappings. The immediate examples of these new structures are partial metric space, quasi-metric space, semi-metric space, b-metric space, etc. Among all of these, we shall consider the *b*-metric that is the most interesting and most general form of the distance. The notion of *b*-metric has been discovered by several authors, such as Bourbaki [1], Bakhtin [2], and Czerwik [3], in different periods of time. Roughly speaking, *b*-metric space is derived from metric space by relaxing the triangle inequality.

As it was mentioned before, the theory has been advanced by reporting several new fixed point results that are obtained by changing the conditions on the given mappings. As a result, in the literature, there are so many different types of metric fixed point results that cause a disturbance, conflict, and disorder. For overcoming this problem, it needs to consider new theorems that cover several different results. One of the successful results in directions was given in [4] where admissible mappings were introduced to combine different structures. Other interesting results were given in [5] in which the notion of the simulation function was defined to combine many distinct contractions. The notion of the hybrid contractions can also be considered as a result of this trend: in two recent papers [6,7], the authors introduce a new type of contraction, namely *admissible hybrid contraction*, in order to unify several linear, nonlinear and interpolative contractions in the set-up of a complete metric and b-metric spaces.

One of the main aims of this paper is to unify the several existing results in the literature by combining the interesting notions: admissible mappings, simulation functions, and hybrid contractions. Besides unifying the results, we express our results in the most generalized form: in the setting of a complete b-metric space. Next, we shall consider applications for our obtained results. In particular, we shall consider the well-posedness and the Ulam–Hyers stability of the fixed point problem. We shall give some consequences and we shall indicate how one can get more consequences from the main theorem of the paper. In the next section, we shall give some basic notions and results to provide a self-contained, easily readable paper.

2. Preliminaries

In this section, we shall collect the necessary notations, notions, and results for the sake of the completeness of the paper. We first express the definition of the b-metric, as follows.

Definition 1 (See, e.g., Bourbaki [1], Bakhtin [2], and Czerwik [3])**.** *Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric with constant s, if*

1. *d is symmetric, that is, $d(x, y) = d(y, x)$ for all x, y,*
2. *d is self-distance, that is, $d(x, y) = 0$ if and only if $x = y$,*
3. *d provides s-weighted triangle inequality, that is*

$$d(x,z) \leq s[d(x,y) + d(y,z)], \text{ for all } x,y,z \in X.$$

In this case, the triple (X, d, s) is called a b-metric space with constant s.

It is evident that the notions of b-metric and standard metric coincide in case of $s = 1$. For more details on b-metric spaces, see, e.g., [8–11] and corresponding references therein.

In what follows, we express the following immediate interesting examples of b-metric space to indicate the richness of this abstract space.

Example 1. *Let S be any set that has more than three elements. Suppose that S_1, S_2 are the subsets of S such that $S_1 \cap S_2 = \emptyset$ and $S = S_1 \cup S_2$ Let $s \geq 1$ be arbitrary. Consider the functional $d : X \times X \to [0, \infty)$, which is defined by:*

$$d(a,b) := \begin{cases} 0, & a = b, \\ 2s, & a, b \in S_1, \\ 1, & \text{otherwise}. \end{cases}$$

It is obvious that (X, d, s) forms a b-metric space.

Another simple, but interesting example is the following:

Example 2. *Let $X = \mathbb{R}$. The function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, defined as*

$$d(x, y) = |x - y|^2, \tag{1}$$

is a b-metric on $\mathbb{R}$ with $s = 2$. Clearly, the first two conditions are satisfied. For the third condition, we have

$$\begin{aligned} |x-y|^2 &= |x-z+z-y|^2 = |x-z|^2 + 2|x-z||z-y| + |z-y|^2 \\ &\leq 2[|x-z|^2 + |z-y|^2], \end{aligned}$$

since

$$2|x-z||z-y| \leq |x-z|^2 + |z-y|^2.$$

Thus, $(X, d, 2)$ is a b-metric space.

Example 3. *Let $X = \{a, b, c\}$ and $d : X \times X \to \mathbb{R}_0^+$ such that*

$$\begin{aligned} & d(a,b) = d(b,a) = d(a,c) = d(c,a) = 1, \\ & d(b,c) = d(c,b) = \alpha \geq 2, \\ & d(a,a) = d(b,b) = d(c,c) = 0. \end{aligned}$$

Then,

$$d(x,y) \leq \frac{\alpha}{2}\left[d(x,z) + d(z,y)\right], \text{ for } a, b, c \in X.$$

Then, $(X, d, \frac{\alpha}{2})$ is a b-metric space.

Example 4 ([8]). *Let B be a Banach space with the zero vector 0_B. Suppose that P be a cone whose interior is non-empty. Suppose also that $\preceq$ forms a partial order with respect to P.*

For a non-empty set S, we consider the functional $d : X \times X \to B$ that fulfills

$(M1)$ $0_B \preceq \delta(a, b)$,
$(M2)$ $\delta(a, b) = 0$ *if and only if* $x = y$,
$(M3)$ $\delta(a, b) \preceq \delta(a, c) + \delta(c, b)$,
$(M4)$ $\delta(a, b) = \delta(b, a)$,

for all $a, b, c \in S$. Then, δ is said to be a cone metric (or, Banach-valued metric). Furthermore, the pair (S, δ) is called a cone metric space (or Banach-valued metric space).

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to [0, \infty)$ be defined by $D(x, y) = ||d(x, y)||$, where $d : X \times X \to E$ is a cone metric space. Then, (X, D, K) forms a b-metric space.

Example 5 (See, e.g., [1]). *Let $X = L^p[0, 1]$ be the collections of all real functions $x(t)$ such that $\int_0^1 |x(t)|^p dt < \infty$, where $t \in [0, 1]$ and $0 < p < 1$. For the function $d : X \times X \to \mathbb{R}_0^+$ defined by*

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p}, \text{ for each } x, y \in L^p[0,1],$$

the ordered triple $(X, d, 2^{1/p})$ forms a b-metric space.

Example 6 (See, e.g., [1]). *Let $p \in (0, 1)$ and let*

$$X = l_p(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} \textit{ such that } \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define $d(x,y) : X \times X \to [0,\infty)$ by

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

Then, $(X, d, 2^{1/p})$ is a b-metric space.

Definition 2 ([12])**.** *A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for any $t \in [0,\infty)$.*

Example 7. *Let $\gamma : [0,\infty) \to [0,\infty)$ be a function such that*

$$\gamma(t) = ct \text{ for all } t \in [0,\infty) \text{ where } c \in (0,1).$$

Then, γ forms a comparison function.

Example 8. *Let $\beta : [0,\infty) \to [0,\infty)$ be a function such that*

$$\beta(t) = \frac{t}{1+t} \text{ for all } t \in [0,\infty).$$

Then, γ forms a comparison function.

Lemma 1 ([10])**.** *If $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \geq 1$, is also a comparison function;
(2) φ is continuous at 0;
(3) $\varphi(t) < t$, for any $t > 0$.

Definition 3 ([12])**.** *A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a c-comparison function if*

(1) φ is increasing;
(2) there exists $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

Remark 1. *Note that γ in Example 7 is also c-comparison function. On the other hand, β in Example 8 is not a c-comparison function.*

It is evident that the c-comparison function is not useful to work in the setting of b-metric space due to the third axiom, s-weighted triangle inequality. In the setting of b-metric space, we should involve the b-metric constant "s" in our analysis. That is why the b-comparison function was suggested by Berinde [10]. Indeed, the idea is so simple. In order to investigate fixed point results in the class of b-metric spaces, the notion of c-comparison function was extended to the b-comparison function by involving the b-metric constant "s".

In what follows, we remind readers about the formal definition of the b-comparison function:

Definition 4 ([10])**.** *Let $s \geq 1$ be a real number. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a b-comparison function if the following conditions are fulfilled:*

(1) φ is monotone increasing;
(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0,\infty)$.

Example 9. *Let $s \geq 1$ be a real number and $\gamma : [0,\infty) \to [0,\infty)$ be a function such that*

$$\gamma(t) = ct \text{ for all } t \in [0,\infty) \text{ where } c \in (0, \frac{1}{s}).$$

Then, γ forms a comparison function.

The following lemma is very important in the proof of our results.

Lemma 2 ([10])**.** *If $\varphi : [0,\infty) \to [0,\infty)$ is a $b-$comparison function, then we have the following conclusions:*

(1) the series $\sum\limits_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in [0,\infty)$;
(2) the function $S_b : [0,\infty) \to [0,\infty)$ defined by $S_b(t) = \sum\limits_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0,\infty)$, is increasing and continuous at 0.

Remark 2. *Due to the Lemma 1.2., any b-comparison function is a comparison function.*

Let $\alpha : X \times X \to [0,\infty)$ be a function. We say that a mapping $f : X \to X$ is α-orbital admissible ([13]) if

$$\alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2(x)) \geq 1.$$

An α-orbital admissible mapping f is called triangular α-orbital admissible ([13]) if

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y, fy) \geq 1 \Rightarrow \alpha(x, fy) \geq 1, \text{ for every } x, y \in X.$$

Lemma 3. *Let (X,d) be a b-metric space with constant $s \geq 1$, and let $f : X \to X$ be triangular α-orbital admissible mapping having the property that there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$. Then,*

$$\alpha(x_n, x_m) \geq 1, \quad \textit{for all } n, m \in \mathbb{N},$$

where the sequence $(x_n)_{n\in\mathbb{N}}$ is defined by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$.

Very recently, an interesting auxiliary function, to unify the different type contraction, was defined by Khojasteh [5] under the name of *simulation function*.

Definition 5 ([5])**.** *A* simulation function *is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) *$\zeta(t,s) < s - t$ for all $t, s > 0$;*
(ζ_2) *if $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{2}$$

In the original definition, given in [5], there was an additional but a superfluous condition $\zeta(0,0) = 0$. We underline the observation that a function $\zeta(t,s) := ks - t$, where $k \in [0,1)$ for all $s, t \in [0,\infty)$, is an instantaneous example of a simulation function. For further and more interesting examples, we refer e.g., [5,14–18] and relate references therein.

A self-mapping f, defined on a metric space (X,d), is called a $\mathcal{Z}$*-contraction* with respect to $\zeta \in \mathcal{Z}$ [5], if

$$\zeta(d(fx,fy), d(x,y)) \geq 0 \qquad \text{for all } x, y \in X. \tag{3}$$

The following is the main results of [5]:

Theorem 1. *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

As it is mentioned above, the immediate example $\zeta(t,s) := ks - t$ implies the outstanding Banach contraction mapping principle.

Definition 6 (cf. [7]). *Let (X,d) be a b-metric space with constant $s \geq 1$. A self-mapping f is called an admissible hybrid contraction, if there exist $\varphi : [0,\infty) \to [0,\infty)$ a b-comparison function and $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)d(fx,fy) \leq \varphi\left(\mathcal{R}_f^q(x,y)\right), \tag{4}$$

where $q \geq 0$ and $\lambda_i \geq 0, i = 1,2,3,4,5$ such that $\sum_{i=1}^5 \lambda_i = 1$ and

$$\mathcal{R}_f^q d(x,y) = \begin{cases} [N(x,y)]^{1/q}, & \text{for } q > 0, x,y \in X, \\ P(x,y), & \text{for } q = 0, x,y \in X. \end{cases} \tag{5}$$

where

$$\begin{aligned} N(x,y) := \ & \lambda_1 d^q(x,y) + \lambda_2 d^q(x,fx) + \lambda_3 d^q(y,fy) \\ & + \lambda_4 \left(\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^q + \lambda_5 \left(\frac{d(y,fx)(1+d(x,fy))}{1+d(x,y)}\right)^q, \end{aligned}$$

and

$$\begin{aligned} P(x,y) := \ & d^{\lambda_1}(x,y) \cdot d^{\lambda_2}(x,fx) \cdot d^{\lambda_3}(y,fy) \\ & \cdot \left(\frac{d(y,fy)(1+d(x,fx))}{1+d(x,y)}\right)^{\lambda_4} \cdot \left(\frac{d(x,fy)+d(y,fx)}{2s}\right)^{\lambda_5}. \end{aligned}$$

Definition 7. *Let (X,d) be a b-metric space with constant $s \geq 1$. A mapping $f : X \to X$ is called admissible hybrid $\mathcal{Z}$-contraction mapping if there is $\varphi : [0,\infty) \to [0,\infty)$ a b-comparison function, $\alpha : X \times X \to [0,\infty)$ and $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\alpha(x,y)d(fx,fy), \varphi\left(\mathcal{R}_f^q(x,y)\right)\right) \geq 0, \text{ for all } x,y \in X, \tag{6}$$

where $\mathcal{R}_f^q(x,y)$ is as above.

3. Existence and Uniqueness Results

Theorem 2. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $f : X \to X$ be an admissible hybrid $\mathcal{Z}$-contraction. Suppose also that:*

- (i) *f is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
- (iii) *either, f is continuous or*
- (iv) *f^2 is continuous and $\alpha(fx,x) \geq 1$ for any $x \in Fix_{f^2}(X)$.*

Then, f has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Starting from here, we recursively construct the sequence $(x_n)_{n\in\mathbb{N}}$, as $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$. Supposing that there exists some $m \in \mathbb{N}$ such that $fx_m = x_{m+1} = x_m$, we find that x_m is a fixed point of f and the proof is finished. Thus, we can presume, from now on, that $x_n \neq x_{n-1}$ for any $n \in \mathbb{N}$. Under the assumption (i), f is an admissible hybrid $\mathcal{Z}$-contraction, if we consider in (6) $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} 0 \leq \ & \zeta(\alpha(x_{n-1},x_n)d(f(x_{n-1}), f(x_n)), \varphi(\mathcal{R}_f^q(x_{n-1},x_n))) \\ & < \varphi(\mathcal{R}_f^q(x_{n-1},x_n)) - \alpha(x_{n-1},x_n)d(f(x_{n-1}), f(x_n)), \end{aligned}$$

which is equivalent to

$$\alpha(x_{n-1},x_n)d(f(x_{n-1}),f(x_n)) \leq \varphi(\mathcal{R}_f^q(x_{n-1},x_n)). \tag{7}$$

Taking into account that f is triangular α-orbital admissible, from (ii) and Lemma 1.3., we have $\alpha(x_{n-1},x_n) \geq 1$. In this way, the above inequality becomes

$$d(x_n,x_{n+1}) \leq \alpha(x_{n-1},x_n)d(f(x_{n-1}),f(x_n)) < \varphi(\mathcal{R}_f^q(x_{n-1},x_n)). \tag{8}$$

Case 1. For the case $q > 0$, we have

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1},x_n) &= [\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},f(x_{n-1})) + \lambda_3 d^q(x_n,f(x_n)) + \\
&\quad + \lambda_4 \left(\tfrac{d(x_n,f(x_n))(1+d(x_{n-1},f(x_{n-1})))}{1+d(x_{n-1},x_n)}\right)^q + \lambda_5 \left(\tfrac{d(x_n,f(x_{n-1}))(1+d(x_{n-1},f(x_n)))}{1+d(x_{n-1},x_n)}\right)^q\Big]^{\frac{1}{q}} \\
&= [\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},x_n) + \lambda_3 d^q(x_n,x_{n+1}) + \\
&\quad + \lambda_4 \left(\tfrac{d(x_n,x_{n+1})(1+d(x_{n-1},x_n))}{1+d(x_{n-1},x_n)}\right)^q + \lambda_5 \left(\tfrac{d(x_n,x_n)(1+d(x_{n-1},x_{n+1}))}{1+d(x_{n-1},x_n)}\right)^q\Big]^{\frac{1}{q}} \\
&= \left[\lambda_1 d^q(x_{n-1},x_n) + \lambda_2 d^q(x_{n-1},x_n) + \lambda_3 d^q(x_n,x_{n+1}) + \lambda_4 \left(d(x_n,x_{n+1})\right)^q\right]^{\frac{1}{q}} \\
&= [(\lambda_1+\lambda_2)d^q(x_{n-1},x_n) + (\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q},
\end{aligned}$$

and from (8) we get

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \alpha(x_{n-1},x_n)d(f(x_{n-1}),f(x_n)) < \varphi(\mathcal{R}_f^q(x_{n-1},x_n)) \\
&= \varphi([(\lambda_1+\lambda_2)d^q(x_{n-1},x_n) + (\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q}).
\end{aligned} \tag{9}$$

Suppose that $d(x_{n-1},x_n) \leq d(x_n,x_{n+1})$. Since φ is a nondecreasing function, Equation (9) can be estimated as follows:

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \alpha(x_{n-1},x_n)d(f(x_{n-1}),f(x_n)) \\
&\leq \varphi([(\lambda_1+\lambda_2)d^q(x_{n-1},x_n) + (\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q}) \\
&\quad \text{due to assumption } d(x_{n-1},x_n) \leq d(x_n,x_{n+1}) \text{ we get} \\
&\leq \varphi([\lambda_1+\lambda_2+\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q}) \\
&\quad \text{when we rearrange it, we get} \\
&= \varphi((\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/q}d(x_n,x_{n+1})) \\
&\quad \text{on account of the fact that } \varphi(t) < t, \text{ we find} \\
&< (\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/q}d(x_n,x_{n+1}) \\
&\text{since } \lambda_1+\lambda_2+\lambda_3+\lambda_4 \leq 1, \text{ we obtain} \\
&\leq d(x_n,x_{n+1}),
\end{aligned}$$

which is a contradiction. Therefore, for every $n \in \mathbb{N}$, we have

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n),$$

in which case the inequality (8) yields

$$\begin{aligned} d(x_n, x_{n+1}) &\le \varphi([(\lambda_1+\lambda_2)d^q(x_{n-1},x_n)+(\lambda_3+\lambda_4)d^q(x_n,x_{n+1})]^{1/q}) \\ &< \varphi\left((\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/q} d(x_{n-1},x_n)\right) \\ &\le \varphi\left(d(x_{n-1},x_n)\right) \le \varphi^2\left(d(x_{n-2},x_{n-1})\right) \le ... \le \varphi^n\left(d(x_0,x_1)\right). \end{aligned} \tag{10}$$

Now let $m, p \in \mathbb{N}$ such that $p > m$. Using the triangle inequality and (10), we have

$$\begin{aligned} d(x_m, x_p) &\le s d(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + \ldots + s^{p-m} \cdot d(x_{p-1}, x_p) \\ &\le s\varphi^m((d(x_0,x_1)) + s^2\varphi^{m+1}(d(x_0,x_1)) + \ldots + s^{p-m+1}\varphi^p(d(x_0,x_1)) \\ &= \tfrac{1}{s^{m-1}}\left(s^m\varphi^m((d(x_0,x_1)) + s^{m+1}\varphi^{m+1}(d(x_0,x_1)) + \ldots + s^p\varphi^p(d(x_0,x_1))\right) \\ &= \tfrac{1}{s^{m-1}} \sum_{j=m}^{p} s^j \varphi^j((d(x_0,x_1)). \end{aligned}$$

Since φ is a b-comparison function, the series $\sum_{j=0}^{\infty} \varphi^j(d(x_0,x_1))$ is convergent. Denoting by $\mathcal{S}_n = \sum_{j=0}^{n} \varphi^j(d(x_0,x_1))$, the above inequality becomes

$$d(x_m, x_p) \le \frac{1}{s^{m-1}}\left(\mathcal{S}_{p-1} - \mathcal{S}_{m-1}\right),$$

and as $m, p \to \infty$ we get

$$d(x_m, x_p) \to 0, \tag{11}$$

which tells us that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence on a complete b-metric space, so there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n x^*) = 0. \tag{12}$$

We shall prove that x^* is a fixed point of f. If f is continuous, (due to assumption (iii))

$$d\left(x^*, f\left(x^*\right)\right) = \lim_{n\to\infty} d\left(x_n, f\left(x_n\right)\right) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

so we get that $f\left(x^*\right) = x^*$, that is, x^* is a fixed point of f.

Suppose now that f^2 is continuous. It follows that $f^2\left(x^*\right) = \lim_{n\to\infty} f^2\left(x_n\right) = x^*$. We shall prove that $f\left(x^*\right) = x^*$. Supposing that, on the contrary, $f\left(x^*\right) \ne x^*$, we have from (6)

$$\begin{aligned} 0 &\le \zeta(\alpha(f\left(x^*\right), x^*)d(f^2\left(x^*\right), f\left(x^*\right)), \varphi(\mathcal{R}_f^q(f\left(x^*\right), x^*))) \\ &= \varphi(\mathcal{R}_f^q(f\left(x^*\right), x^*)) - \alpha(f\left(x^*\right), x^*)d(f^2\left(x^*\right), f\left(x^*\right)), \end{aligned}$$

which implies

$$\begin{aligned}
d(x^*, f(x^*)) &= d(f^2(x^*), f(x^*)) \leq \alpha(f(x^*), x^*) d(f(x^*), x^*) \\
&\text{since } \varphi(t) < t, \text{ we get} \\
&\leq \varphi(\mathcal{R}_f^q(f(x^*), x^*)) < \mathcal{R}_f^q(f(x^*), x^*); \text{ due to (5), we have} \\
&= \Big[\lambda_1 d^q(f(x^*), x^*) + \lambda_2 d^q(x^*, f(x^*)) + \lambda_3 d^q(f(x^*), f^2(x^*)) + \\
&\lambda_4 \left(\tfrac{d(x^*, f(x^*))(1+d(f(x^*), f^2(x^*)))}{1+d(x^*, f(x^*))}\right)^q + \lambda_5 \left(\tfrac{d(f(x^*), f(x^*))(1+d(x^*, f^2(x^*)))}{1+d(x^*, f(x^*))}\right)^q\Big]^{\frac{1}{q}} \\
&= \Big[\lambda_1 d^q(f(x^*), x^*) + \lambda_2 d^q(x^*, f(x^*)) + \lambda_3 d^q(f(x^*), x^*) + \\
&+\lambda_4 \left(\tfrac{d(x^*, f(x^*))(1+d(f(x^*), x^*))}{1+d(x^*, f(x^*))}\right)^q + \lambda_5 \left(\tfrac{d(f(x^*), f(x^*))(1+d(x^*, x^*))}{1+d(x^*, f(x^*))}\right)^q\Big]^{\frac{1}{q}} \\
&= [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) d^q(x^*, f(x^*))]^{\frac{1}{q}} \\
&= [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)]^{\frac{1}{q}} d(x^*, f(x^*)) \\
&\leq d(x^*, f(x^*)).
\end{aligned}$$

This is a contradiction, so that $f(x^*) = x^*$.

Case 2. For the case $q = 0$, if we consider $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1}, x_n) &= d^{\lambda_1}(x_{n-1}, x_n) \cdot d^{\lambda_2}(x_{n-1}, f(x_{n-1})) \cdot d^{\lambda_3}(x_n, f(x_n)) \cdot \\
&\cdot \left[\tfrac{d(x_n, f(x_n))(1+d(x_{n-1}, f x_{n-1}))}{1+d(x_{n-1}, x_n)}\right]^{\lambda_4} \cdot \left[\tfrac{d(x_{n-1}, f(x_n)) + d(x_n, f x_{n-1}))}{2s}\right]^{\lambda_5} \\
&= d^{\lambda_1}(x_{n-1}, x_n) \cdot d^{\lambda_2}(x_{n-1}, x_n) \cdot d^{\lambda_3}(x_n, x_{n+1}) \cdot \\
&\cdot \left[\tfrac{d(x_n, x_{n+1})(1+d(x_{n-1}, x_n))}{1+d(x_{n-1}, x_n)}\right]^{\lambda_4} \cdot \left[\tfrac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n))}{2s}\right]^{\lambda_5} \\
&= d^{\lambda_1}(x_{n-1}, x_n) \cdot d^{\lambda_2}(x_{n-1}, x_n) \cdot d^{\lambda_3}(x_n, x_{n+1}) \cdot d^{\lambda_4}(x_n, x_{n+1}) \cdot \left[\tfrac{d(x_{n-1}, x_{n+1})}{2s}\right]^{\lambda_5}.
\end{aligned}$$

Employing the triangle inequality, we have

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1}, x_n) &\leq d^{\lambda_1}(x_{n-1}, x_n) \cdot d^{\lambda_2}(x_{n-1}, x_n) \cdot d^{\lambda_3}(x_n, x_{n+1}) \cdot d^{\lambda_4}(x_n, x_{n+1}) \\
&\cdot \left[\tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right]^{\lambda_5}.
\end{aligned} \tag{13}$$

Using the following inequality

$$\left(\frac{a+b}{2}\right)^k \leq \frac{a^k + b^k}{2}, \text{ for all } a, b, k > 0,$$

(13) becomes

$$\begin{aligned}
\mathcal{R}_f^q(x_{n-1}, x_n) &\leq d^{\lambda_1}(x_{n-1}, x_n) \cdot d^{\lambda_2}(x_{n-1}, x_n) \cdot d^{\lambda_3}(x_n, x_{n+1}) \\
&\cdot d^{\lambda_4}(x_n, x_{n+1}) \cdot \tfrac{d^{\lambda_5}(x_{n-1}, x_n) + d^{\lambda_5}(x_n, x_{n+1})}{2},
\end{aligned}$$

and, from (6),

$$\begin{aligned}
0 \leq\ & \zeta(\alpha(x_{n-1}, x_n) d(f(x_{n-1}), f(x_n)), \varphi(\mathcal{R}_f^q(x_{n-1}, x_n))) \\
& < \varphi(\mathcal{R}_f^q(x_{n-1}, x_n)) - \alpha(x_{n-1}, x_n) d(f(x_{n-1}), f(x_n)),
\end{aligned}$$

which yields that

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(f(x_{n-1}), f(x_n)) \leq \varphi(\mathcal{R}_f^q(x_{n-1}, x_n)). \tag{14}$$

Supposing that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, since φ is a nondecreasing function, we have

$$d(x_n, x_{n+1}) < d^{\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5}(x_n, x_{n+1}) = d(x_n, x_{n+1}),$$

which is a contradiction. Then, from (14), inductively, we obtain

$$d(x_n, x_{n+1}) \leq \varphi(\mathcal{R}_f^q(x_{n-1}, x_n)) < \varphi^n(d(x_0, x_1)). \tag{15}$$

By using the same arguments as the case $q > 0$, we shall easily obtain that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a complete metric space and thus there exists x^* such that $\lim_{n\to\infty} x_n = x^*$.

We claim that x^* is a fixed point of f.

Under the assumption that f is continuous, we have

$$d\left(x^*, f\left(x^*\right)\right) = \lim_{n\to\infty} d\left(x_n, f\left(x_n\right)\right) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and together with the uniqueness of limit, $f\left(x^*\right) = x^*$. In addition, if f^2 is continuous, as in **case 1**, we have that $f^2\left(x^*\right) = x^*$ and suppose that $f\left(x^*\right) \neq x^*$. Then, we get

$$\begin{aligned} 0 &\leq \zeta(\alpha(f\left(x^*\right), x^*)d(f^2\left(x^*\right), f\left(x^*\right)), \varphi(\mathcal{R}_f^q(f^2\left(x^*\right), f\left(x^*\right))) \\ &= \varphi(\mathcal{R}_f^q(f^2\left(x^*\right), f\left(x^*\right))) - \alpha(f\left(x^*\right), x^*)d(f^2\left(x^*\right), f\left(x^*\right)), \end{aligned}$$

which implies

$$\begin{aligned} d(x^*, f\left(x^*\right)) &= d(f^2\left(x^*\right), f\left(x^*\right)) \\ &\leq \alpha(f\left(x^*\right), x^*)d(f^2\left(x^*\right), f\left(x^*\right)) \\ &\leq \varphi(\mathcal{R}_f^q(f^2\left(x^*\right), f\left(x^*\right)) = \varphi(\mathcal{R}_f^q(x^*, f\left(x^*\right)), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_f^q(x^*, f\left(x^*\right)) &= d^{\lambda_1+\lambda_2+\lambda_3}(x^*, f\left(x^*\right)) \cdot \left[\frac{d(x^*, f(x^*))(1+d(x^*, f(x^*))}{1+d(x^*, f(x^*))}\right]^{\lambda_4} \cdot \left[\frac{d(x^*, x^*)+d(f(x^*), f(x^*))}{2s}\right]^{\lambda_5} \\ &= d^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}(x^*, f\left(x^*\right)) < d(x^*, f\left(x^*\right)). \end{aligned}$$

Hence, we have

$$d(x^*, f\left(x^*\right)) \leq \varphi(\mathcal{R}_f^q(x^*, f\left(x^*\right)) < \varphi(d(x^*, f\left(x^*\right)) < d(x^*, f\left(x^*\right)),$$

which is a contradiction. □

Theorem 3. *In the hypothesis of Theorem 2, if we assume supplementary that*

$$\alpha(x^*, y^*) \geq 1,$$

for any $x^*, y^* \in Fix_f(X)$, then the fixed point of f is unique.

Proof. Let $y^* \in X$ be another fixed point of f. Suppose that $x^* \neq y^*$. In the case that $q > 0$, using (6), we have:

$$\begin{aligned} 0 &\leq \zeta(\alpha\left(x^*, y^*\right) d(f\left(x^*\right), f\left(y^*\right)), \varphi(\mathcal{R}_f^q(x^*, y^*))) \\ &< \varphi(\mathcal{R}_f^q(x^*, y^*)) - \alpha\left(x^*, y^*\right) d(f\left(x^*\right), f\left(y^*\right)), \end{aligned}$$

which yields that

$$\begin{aligned} d(x^*,y^*) &\leq \alpha\left(x^*,y^*\right) d(f\left(x^*\right), f\left(y^*\right)) \leq \varphi(\mathcal{R}_f^q(x^*,y^*)) < \mathcal{R}_f^q(x^*,y^*) \\ &= \Big[\lambda_1 d(x^*,y^*) + \lambda_2 d^q(x^*, f\left(x^*\right)) + \lambda_3 d^q(y^*, f\left(y^*\right)) + \lambda_4 \left(\tfrac{d(y^*,f(y^*))(1+d(x^*,f(x^*))}{1+d(x^*,y^*)}\right)^q + \\ &\lambda_5 \left(\tfrac{d(y^*,f(x^*))(1+d(x^*,f(y^*))}{1+d(x^*,y^*)}\right)^q\Big]^{\frac{1}{q}} \\ &= \left(\lambda_1 + \lambda_5\right)^{\frac{1}{q}} d(x^*,y^*) < d(x^*,y^*), \end{aligned}$$

which is a contradiction.

In the case that $q = 0$, if we suppose that $x^* \neq y^*$, then we obtain that $0 < d(x^*,y^*) < 0$, which is a contradiction.

Thus, $x^* = y^*$, so that f possesses exactly one fixed point. □

Example 10. *Let $X = [0,2]$, $d : X \times X \to [0,\infty)$, $d(x,y) = |x-y|^2$ for all $x,y \in X$. Consider that the mapping $f : X \to X$ is defined by $f(x) = \begin{cases} 1/2, & if x \in [0,1] \\ x/2, & if x \in (1,2] \end{cases}$ and the function $\alpha(x,y) = \begin{cases} 2, & if x,y \in [0,1], \\ 1, & if x = 0, y = 2 \\ 0, & otherwise. \end{cases}$ and the b-comparison function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(t) = \frac{t}{2}, \zeta\left(t,s\right) = \frac{1}{2}s - t$,*

We can easily observe that:

1. *(X,d) is a complete b-metric space with the constant $s = 2$;*
2. *f triangular α-orbital admissible;*
3. *for $x_0 \in [0,1], f\left(x_0\right) = \frac{1}{2} \in [0,1]$ and hence $\alpha\left(x_0, f\left(x_0\right)\right) = 2 > 1$;*
4. *f is continuous;*
5. *$f^2\left(x\right) = \frac{1}{2}$ is continuous. Moreover, for $x = \frac{1}{2} \in Fix_{f^2}\left(X\right)$, we have $\alpha\left(f\left(\frac{1}{2}\right).\frac{1}{2}\right) = \alpha\left(\frac{1}{2}, \frac{1}{2}\right) = 2 > 1$;*
6. *$\zeta\left(\alpha(x,y)d(fx,fy), \varphi\left(\mathcal{R}_f^q(x,y)\right)\right) \geq 0$.*

If $x,y \in [0,1]$, then $fx = fy = \frac{1}{2}$ and hence $d\left(fx,fy\right) = 0$. We have

$$\zeta\left(0, \varphi\left(\mathcal{R}_f^q(x,y)\right)\right) = \frac{1}{2}\varphi\left(\mathcal{R}_f^q(x,y)\right) \geq 0, \textit{ for all } x,y \in [0,1],$$

and hence

$$\zeta\left(\alpha(x,y)d(fx,fy), \varphi\left(\mathcal{R}_f^q(x,y)\right)\right) \geq 0, \textit{ for all } x,y \in [0,1].$$

If $x = 0$ and $y = 2$, then if we consider $q = 2, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \frac{1}{5}$, then we have

$$\begin{aligned} \zeta\left(\alpha(0,2)d(f\left(0\right), f\left(2\right)), \varphi\left(\mathcal{R}_f^q(0,2)\right)\right) &= \tfrac{1}{2}\varphi\left(\mathcal{R}_f^q(0,2)\right) - \alpha(0,2)d(f\left(0\right), f\left(2\right)) =<= \\ &= \tfrac{1}{4}\Big[\tfrac{1}{5}d^2(0,2) + \tfrac{1}{5}d^2(0, f\left(0\right)) + \tfrac{1}{5}d^2(2, f\left(2\right)) + \\ &+ \tfrac{1}{5}\left(\tfrac{d(2,f(2))(1+d(0,f(0)))}{1+d(0,2)}\right)^2 + \tfrac{1}{5}\left(\tfrac{d(2,f(0))(1+d(0,f(2)))}{1+d(0,2)}\right)^2\Big]^{\frac{1}{2}} \\ &-\alpha(0,2)d\left(\tfrac{1}{2}, 1\right) \\ &= \tfrac{1}{4}\left[\tfrac{1}{5}\left(16 + \tfrac{1}{16} + 1 + \tfrac{1}{16} + \tfrac{81}{100}\right)\right]^{\frac{1}{2}} \\ &= \tfrac{1}{4}\left(\tfrac{3587}{1000}\right)^{\frac{1}{2}} - \tfrac{1}{4} \geq 0. \end{aligned}$$

Hence,

$$\zeta\left(\alpha(0,2)d(f\left(0\right), f\left(2\right)), \varphi\left(\mathcal{R}_f^q(0,2)\right)\right) \geq 0.$$

In all other cases, $\alpha(x,y) = 0$ and

$$\zeta\left(0, \varphi\left(\mathcal{R}_f^q(x,y)\right)\right) = \frac{1}{2}\varphi\left(\mathcal{R}_f^q(x,y)\right) \geq 0.$$

Thus, we obtain that f is an admissible hybrid $\mathcal{Z}$-contraction which satisfies the assumptions of Theorem 2 and then $x = \frac{1}{2}$ is the fixed point of f.

Remark 3. *If, in the above example, we consider $f(x) = \begin{cases} 1/3, & \text{if } x \in [0,1] \\ x/2, & \text{if } x \in (1,2] \end{cases}$, then f is not continuous, but $f^2(x) = \frac{1}{3}$ and for $x = \frac{1}{3} \in Fix_{f^2}(X)$, we have $\alpha\left(f\left(\frac{1}{3}\right).\frac{1}{3}\right) = \alpha\left(\frac{1}{3}, \frac{1}{3}\right) = 2 > 1$.*

Let Φ be the collection of all auxiliary functions $\phi : [0,\infty) \to [0,\infty)$ which are continuous and $\phi(t) = 0$ if and only if $t = 0$.

Theorem 4. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that there exist two functions $\phi_1, \phi_2 \in \Phi$, with $\phi_1(t) < t \leq \phi_2(t)$, for all $t > 0$, such that*

$$\phi_2\left(\alpha(x,y)d(fx,fy)\right) \leq \phi_1\left(\mathcal{R}_f^q(x,y)\right). \tag{16}$$

Furthermore, we suppose that:

- (i) *f is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
- (iii) *either, f is continuous or*
- (iv) *f^2 is continuous and $\alpha(fx,x) \geq 1$ for any $x \in Fix_{f^2}(X)$.*
- (v) *if $x^*, y^* \in Fix_f(X)$, then $\alpha(x^*, y^*) \geq 1$.*

Then, f has a unique fixed point.

Proof. Let $\zeta(t,s) = \phi_1(s) - \phi_2(t)$. According to Example 10, if $\phi_1, \phi_2 \in \Phi$ have the property $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$, then $\zeta \in \mathcal{Z}$. Thus, the desired results follow from Theorems 2 and 3. □

Theorem 5. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that there exists a function $\phi \in \Phi$, such that*

$$\alpha(x,y)d(fx,fy) \leq \mathcal{R}_f^q(x,y) - \phi\left(\mathcal{R}_f^q(x,y)\right). \tag{17}$$

Furthermore, we suppose that

- (i) *f is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
- (iii) *either, f is continuous or*
- (iv) *f^2 is continuous and $\alpha(fx,x) \geq 1$ for any $x \in Fix_{f^2}(X)$.*
- (v) *if $x^*, y^* \in Fix_f(X)$, then $\alpha(x^*, y^*) \geq 1$.*

Then, f has a unique fixed point.

Proof. Let $\zeta(t,s) = s - \phi(s)) - t$. According to Example 10, $\zeta \in \mathcal{Z}$. Thus, the desired results follow from Theorems 2 and 3. □

Theorem 6. *Let (X,d) be a complete b-metric space with constant $s \geq 1$, $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that there exists a function $\mu : [0,\infty) \to [0,\infty)$ such that $\int_0^\varepsilon \mu(u)du$ exists and $\int_0^\varepsilon \mu(u)du > \varepsilon$, for each $\varepsilon > 0$, with the property that*

$$\alpha(x,y)d(fx,fy) \leq \int_0^{\mathcal{R}_f^q(x,y)} \mu(u)du. \tag{18}$$

Furthermore, we suppose that

- (i) *f is triangular α-orbital admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
- (iii) *either, f is continuous or*
- (iv) *f^2 is continuous and $\alpha(fx,x) \geq 1$ for any $x \in Fix_{f^2}(X)$.*
- (v) *if $x^*, y^* \in Fix_f(X)$, then $\alpha(x^*,y^*) \geq 1$.*

Then, f has a unique fixed point.

Proof. Let $\zeta(t,s) = s - \int_0^t \mu(u)du$. According to Example 10, $\zeta \in \mathcal{Z}$. Thus, the desired results follow from Theorems 2 and 3. □

Let Φ be the class of auxiliary functions $\phi : [0,\infty) \to [0,\infty)$ that are continuous functions and $\mu(t) = 0$ if and only if, $t = 0$.

The following example is derived from [5,14,15].

Example 11. *(See, e.g., [5,14,15]) Let $\phi_i \in \Phi$ for $i = 1,2,3$ and $\sigma_j : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ for $j = 1,2,3,4,5,6$. Each of the functions defined below is an example of simulation functions:*

- $(E1)$ *$\sigma_2(t,s) = s - \phi_3(s) - t$ for all $t,s \geq 0$.*
- $(E2)$ *$\sigma_4(t,s) = f(s) - t$ for all $t,s \geq 0$, $t,s \geq 0$, where the function $f : [0,\infty) \to [0,\infty)$ is upper semi-continuous and such that $f(t) < t$ for all $t > 0$ and $f(0) = 0$.*
- $(E3)$ *$\sigma_5(t,s) = s - \dfrac{g(t,s)}{h(t,s)}$ for all $t,s \geq 0$, where $g,h : [0,\infty)^2 \to (0,\infty)$ are two continuous functions with respect to each variable such that $g(t,s) > h(t,s)$ for all $t,s > 0$.*
- $(E4)$ *$\sigma_6(t,s) = s\,\eta(s) - t$ for all $t,s \geq 0$, where $\eta : [0,\infty) \to [0,1)$ is a function with the property $\limsup_{t\to r^+} \eta(t) < 1$ for all $r > 0$*

Remark 4. *By using the examples above, we may derive more consequences of our results.*

4. Well Posedness and Ulam–Hyers Stability

Considered as a type of data dependence, the notion of Ulam stability was started by Ulam [19,20] and developed by Hyers [21], Rassias [22], etc. In this section, we investigate the general Ulam type stability in sense of a fixed point problem as well the well posedness of the fixed point problem.

Suppose that $f : X \to X$ is a self-mapping on a b-metric space (X,d) with the constant $s > 1$ and let us consider the following fixed point problem:

$$x = f(x). \tag{19}$$

Definition 8. *The fixed point problem (19) is well-posed if*

- (i) *$Fix_f(X) = \{x^*\}$;*
- (ii) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $d(x_n, f(x_n)) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.*

Theorem 7. *Let (X,d) be a complete b-metric space with constant $s>1$. Suppose that all the hypotheses of Theorem 3 hold, and $q>0$. Additionally, we suppose that for any sequence $(x_n)_{n\in\mathbb{N}}$, with $d(x_n, f(x_n)) \to 0$, as $n\to\infty$, we have $\alpha(x_n, x^*) \geq 1$, for all $n\in\mathbb{N}$, where $x^* \in Fix_f(X)$. If $\lambda_1+\lambda_5 < \frac{1}{\gamma^2(q)}$, where $\gamma(q) = \max\{1, 2^{q-1}s^q\}$, then the fixed point problem (19) is well-posed.*

Proof. Taking into account the supplementary condition, since $Fix_f(X) = x^*, u\sin g$ (6), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n,x^*)\, d(f(x_n), f(x^*)), \varphi(\mathcal{R}_f^q(x_n,x^*))) \\ &< \varphi(\mathcal{R}_f^q(x_n,x^*)) - \alpha(x_n,x^*)\, d(f(x_n), f(x^*)). \end{aligned}$$

We have

$$\begin{aligned} d(x_n,x^*) &\leq sd(x_n, f(x_n)) + sd(f(x_n), f(x^*)) \leq sd(x_n, f(x_n)) + s\alpha(x_n,x^*)d(f(x_n), f(x^*)) \\ &\leq sd(x_n, f(x_n)) + s\varphi(\mathcal{R}_f^q(x_n,x^*)) < sd(x_n, f(x_n)) + s\mathcal{R}_f^q(x_n,x^*) \\ &\leq s\left[\lambda_1 d^q(x_n,x^*) + \lambda_2 d^q(x_n, f(x_n)) + \lambda_3 d^q(x^*, f(x^*)) + \lambda_4\left(\tfrac{d(x^*,f(x^*))(1+d(x_n,f(x_n)))}{1+d(x_n,x^*)}\right)^q + \right. \\ &\qquad \left. +\lambda_5\left(\tfrac{d(x^*,f(x_n))(1+d(x_n,f(x^*)))}{1+d(x_n,x^*)}\right)^q\right]^{\frac{1}{q}} + sd(x_n, f(x_n)) \\ &= s\left[\lambda_1 d^q(x_n,x^*) + \lambda_2 d^q(x_n, f(x_n)) + \lambda_5 d^q(x^*, f(x_n))\right]^{\frac{1}{q}} + sd(x_n, f(x_n)) \\ &\leq s\left[\lambda_1 d^q(x_n,x^*) + \lambda_2 d^q(x_n, f(x_n)) + s^q\lambda_5\left(d(x^*,x_n) + d(x_n, f(x_n))\right)^q\right]^{\frac{1}{q}} + sd(x_n, f(x_n)) \\ &\leq s\left[\lambda_1 d^q(x_n,x^*) + \lambda_2 d^q(x_n, f(x_n)) + 2^{q-1}s^q\lambda_5 d^q(x^*,x_n) + 2^{q-1}s^q\lambda_5 d^q(x_n, f(x_n))\right]^{\frac{1}{q}} + \\ &\quad + sd(x_n, f(x_n)). \end{aligned}$$

In this way, we obtain

$$\begin{aligned} d^q(x_n,x^*) &\leq 2^{q-1}s^q\lambda_1 d^q(x_n,x^*) + 2^{q-1}s^q\lambda_2 d^q(x_n, f(x_n)) + 2^{2q-2}s^{2q}\lambda_5 d^q(x^*,x_n) + \\ &\quad + 2^{2q-2}s^{2q}\lambda_5 d^q(x_n, f(x_n)) + 2^{q-1}s^q d^q(x_n, f(x_n)) \end{aligned}$$

or

$$\left(1 - 2^{q-1}s^q\lambda_1 - 2^{2q-2}s^{2q}\lambda_5\right) d^q(x_n,x^*) \leq 2^{q-1}s^q\left(1+\lambda_2+2^{q-1}s^q\lambda_5\right)^q d^q(x_n, f(x_n)).$$

From here, we obtain

$$d^q(x_n,x^*) \leq \frac{(1+\lambda_2+\gamma(q)\lambda_5)\gamma(q)}{1-\gamma(q)\lambda_1-\gamma^2(q)\lambda_5} d^q(x_n, f(x_n)).$$

Letting $n\to\infty$ in the above inequality and keeping in mind that $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n,x^*) = 0,$$

that is, the fixed point Equation (19) is well-posed. □

Definition 9. *The fixed point problem (19) is called generalized Ulam–Hyers stable if and only if there exists $\rho : [0,\infty) \to [0,\infty)$ is increasing, continuous in 0 and $\rho(0) = 0$, such that for each $\varepsilon > 0$ and for each $y^* \in X$, which satisfy the inequality*

$$d(y, f(y)) \leq \varepsilon, \tag{20}$$

there exists a solution x^* of the fixed point problem (19) such that

$$d(y^*,x^*) \leq \rho(\varepsilon).$$

If there exists $c > 0$ such that $\rho(t) := c \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point problem (19) is said to be Ulam–Hyers stable.

Before stating our theorem, we underline that Ulam–Hyers stability can be potentially applicable to the study of switched dynamics, see e.g., [23], and the related references therein.

Theorem 8. *Let (X, d) be a complete b-metric space with constant $s > 1$. Suppose that all the hypotheses of Theorem 3 hold, and $q > 0$. Additionally, we suppose that $\alpha(y^*, x^*) \geq 1$, for all $y^* \in X$ verifying (20) and $x^* \in Fix_f(X)$. If $\lambda_1 + \lambda_5 < \frac{1}{\gamma^2(q)}$, where $\gamma(q) = \max\{1, 2^{q-1}s^q\}$, then the fixed point problem (19) is Ulam–Hyers stable.*

Proof. Using (6),

$$\begin{aligned} 0 &\leq \zeta(\alpha(y^*, x^*)\, d(f(y^*), f(x^*)), \varphi(\mathcal{R}_f^q(y^*, x^*))) \\ &< \varphi(\mathcal{R}_f^q(y^*, x^*)) - \alpha(y^*, x^*)\, d(f(y^*), f(x^*)) \end{aligned}$$

$$\begin{aligned} d(y^*, x^*) = d(y^*, f(x^*)) &\leq s d(f(y^*), f(x^*)) + s d(y^*, f(y^*)) \\ &\leq s\alpha(y^*, x^*) d(f(y^*), f(x^*)) + s d(y^*, f(y^*)) \\ &\leq s\varphi(\mathcal{R}_f^q(y^*, x^*)) + s\varepsilon < s\mathcal{R}_f^q(y^*, x^*) + s\varepsilon \\ &\leq s\left[\lambda_1 d^q(y^*, x^*) + \lambda_2 d^q(y^*, f(y^*)) + \lambda_3 d^q(x^*, f(x^*)) + \right. \\ &\qquad \left. \lambda_4 \left(\tfrac{d(x^*, f(x^*))(1 + d(x^*, f(x^*)))}{1 + d(y^*, x^*)}\right)^q + \lambda_5 \left(\tfrac{d(x^*, f(y^*))(1 + d(y^*, f(x^*)))}{1 + d(y^*, x^*)}\right)^q\right]^{\frac{1}{q}} + s\varepsilon \\ &= s\left[\lambda_1 d^q(y^*, x^*) + \lambda_2 \varepsilon^q + \lambda_5 d^q(x^*, f(y^*))\right]^{\frac{1}{q}} + s\varepsilon \\ &\leq s\left[\lambda_1 d^q(y^*, x^*) + \lambda_2 \varepsilon^q + s^q \lambda_5 (d(y^*, x^*) + d(y^*, f(y^*)))^q\right]^{\frac{1}{q}} + s\varepsilon \\ &\leq s\left[\lambda_1 d^q(y^*, x^*) + \lambda_2 \varepsilon^q + 2^{q-1}s^q \lambda_5 d^q(y^*, x^*) + 2^{q-1}s^q \lambda_5 d^q(y^*, f(y^*))\right]^{\frac{1}{q}} + s\varepsilon \\ &\leq s\left[\lambda_1 d^q(y^*, x^*) + \lambda_2 \varepsilon^q + 2^{q-1}s^q \lambda_5 d^q(y^*, x^*) + 2^{q-1}s^q \lambda_5 \varepsilon^q\right]^{\frac{1}{q}} + s\varepsilon. \end{aligned}$$

In this way, we obtain

$$\begin{aligned} d^q(y^*, x^*) &\leq 2^{q-1}s^q \lambda_1 d^q(y^*, x^*) + 2^{q-1}s^q \lambda_2 \varepsilon^q + 2^{2q-2}s^{2q} \lambda_5 d^q(y^*, x^*) + \\ &+ 2^{2q-2}s^{2q} \lambda_5 \varepsilon^q + 2^{q-1}s^q \varepsilon^q \end{aligned}$$

or

$$\left(1 - 2^{q-1}s^q \lambda_1 - 2^{2q-2}s^{2q} \lambda_5\right) d^q(y^*, x^*) \leq 2^{q-1}s^q \left(1 + \lambda_2 + 2^{q-1}s^q \lambda_5\right)^q \varepsilon^q.$$

From here, we obtain

$$d^q(y^*, x^*) \leq \frac{(1 + \lambda_2 + \gamma(q)\lambda_5)\gamma(q)}{1 - \gamma(q)\lambda_1 - \gamma^2(q)\lambda_5} \varepsilon^q.$$

Hence,

$$d^q(y^*, x^*) \leq c\varepsilon^q,$$

where $c = \frac{(1+\lambda_2+\gamma(q)\lambda_5)\gamma(q)}{1-\gamma(q)\lambda_1-\gamma^2(q)\lambda_5}$, for any $q > 0$ and $\lambda_1, \lambda_5 \in [0, 1)$ such that $\lambda_1 + \lambda_5 < \frac{1}{\gamma^2(q)}$. □

5. Conclusions

In this paper, we unify, extend, and improve several existing fixed point theorems by introducing the notion of admissible hybrid $\mathcal{Z}$-contraction in the setting of complete b-metric spaces. Consequently, all presented results valid in the setting of complete metric space by letting $s = 1$. On the other hand, unifying several existing results in the literature, we have used admissible mappings, simulation functions, and hybrid contractions. We need to underline the fact that, by setting admissible function α in a proper way, one can get several new consequences of the existence results in the setting of (i) standard metric space, (ii) metric space endowed a partial order on it, and (iii) cyclic contraction. One can easily get these consequences by using the techniques in [4]. Furthermore, for the different examples of simulation functions (as we showed in Theorems 5 and 6), one can get more new corollaries. Lastly, by regarding hybrid contraction approaches, one can get several more consequences, by following the techniques in [21,24–26].

Besides expressing a more generalized result in the setting of a complete b-metric space, we also present some applications for our obtained results. In particular, we shall consider the well-posedness and the Ulam–Hyers stability of the fixed point problem. We note that the word 'hybrid' has been used in different ways, in particular, in applicable nonlinear fields, see, e.g., [27,28].

Author Contributions: Writing—original draft, I.C.C.; Writing—review and editing, E.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors thank the anonymous referees for their remarkable comments, suggestions, and ideas that help to improve this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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