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Nonlocal Inverse Problem for a Pseudohyperbolic-Pseudoelliptic Type Integro-Differential Equations

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Abstract: The questions of solvability of a nonlocal inverse boundary value problem for a mixed pseudohyperbolic-pseudoelliptic integro-differential equation with spectral parameters are considered. Using the method of the Fourier series, a system of countable systems of ordinary integro-differential equations is obtained. To determine arbitrary integration constants, a system of algebraic equations is obtained. From this system regular and irregular values of the spectral parameters were calculated. The unique solvability of the inverse boundary value problem for regular values of spectral parameters is proved. For irregular values of spectral parameters is established a criterion of existence of an infinite set of solutions of the inverse boundary value problem. The results are formulated as a theorem.

Keywords: integro-differential equation; mixed type equation; spectral parameters; integral conditions; solvability

1. Statement of the Inverse Problem

From the point of applications, partial differential and integro-differential equations are of great interest [1,2]. The presence of the integral term in the differential equation plays an important role [3,4]. Also important to study the spectral questions of solvability of the differential and integro-differential equations [5–10]. In References [11–13], using the results of the theory of complete generalized Jordan sets it is considered the reduction of the partial differential equations with irreversible linear operator of finite index in the main differential expression to the regular problems.

Direct and inverse boundary value problems, where the type of differential equation in the domain under consideration changes, have important applications. Direct boundary value problems for differential and integro-differential equations of mixed type were studied in the works of many authors, in particular, in References [14–24]. In References [25,26] the inverse problems for second order mixed type differential equations were considered in rectangular domain. In this paper, we study the unique classical solvability of a nonlocal inverse boundary value problem of mixed pseudohyperbolic-pseudoelliptic integro-differential equation for regular values of spectral parameters. We also study the solvability conditions of the inverse boundary value problem for irregular values of spectral parameters.

In multidimensional domain $\Omega = \{-T < t < T,\, 0 < x_1, x_2, ..., x_m < l\}$ a mixed integro-differential equation of the following form is considered

$$\begin{cases} U_{tt} - \sum\limits_{i=1}^{m} \left[U_{ttx_ix_i} - U_{x_ix_i}\right] = \nu \int\limits_0^T K_1(t,s)\, U(s,x)\, ds + f_1(t)\, g_1(x), \ t > 0, \\ U_{tt} - \sum\limits_{i=1}^{m} \left[U_{ttx_ix_i} + \omega^2 U_{x_i x_i}\right] = \nu \int\limits_{-T}^{0} K_2(t,s)\, U(s,x)\, ds + f_2(t)\, g_2(x), \ t < 0, \end{cases} \tag{1}$$

where T and l are given positive real numbers, ω is positive spectral parameter, $x \in \mathbb{R}^m$, ν is real non-zero spectral parameter, $0 \neq K_j(t,s) = a_j(t)\, b_j(s),\ a_j,\ b_j \in C\,[-T;T]$, $0 \neq f_1 \in C\,[0;T]$, $0 \neq f_2 \in C\,[-T;0]$, $g_j \in C\,(\Omega_l^m)$ are redefinition functions, $\Omega_l^m = [0;l]^m$, $j = 1, 2$.

Problem 1. *Find in the domain Ω a triple of unknown functions*

$$U\,(t,x) \in C\,(\overline{\Omega}) \cap C^1\,(\Omega') \cap C^{2,2}(\Omega) \cap C^{2+2}_{t,x}(\Omega) \cap$$

$$\cap C^{2+2+0+\ldots+0}_{t,x_1,x_2,\ldots,x_m}(\Omega) \cap C^{2+0+2+0+\ldots+0}_{t,x_1,x_2,x_3,\ldots,x_m}(\Omega) \cap \ldots \cap C^{2+0+\ldots+0+2}_{t,x_1,\ldots,x_{m-1},x_m}(\Omega),$$

$$g_i(x) \in C\,(\Omega_l^m),\ i = 1,2,$$

satisfying the mixed integro-differential Equation (1) and the following nonlocal boundary conditions

$$\int_0^T U\,(t,x)\,d\,t = \varphi_1(x),\ \ x \in \Omega_l^m, \tag{2}$$

$$\int_{-T}^0 U\,(t,x)\,d\,t = \varphi_2(x),\ \ x \in \Omega_l^m, \tag{3}$$

$$\begin{gathered}
U\,(t,0,x_2,\,x_3,\,\ldots,\,x_m) = U\,(t,\,l,\,x_2,\,x_3,\,\ldots,\,x_m) = \\
= U\,(t,\,x_1,\,0,\,x_3,\,\ldots,\,x_m) = U\,(t,\,x_1,\,l,\,x_3,\,\ldots,\,x_m) = \ \ldots\ = \\
= U\,(t,\,x_1,\,\ldots,\,x_{m-1},\,0) = U\,(t,\,x_1,\,\ldots,\,x_{m-1},\,l) = \\
= U_{x_1x_1}(t,\,0,\,x_2,\,x_3,\,\ldots,\,x_m) = U_{x_1x_1}(t,\,l,\,x_2,\,x_3,\,\ldots,\,x_m) = \\
= U_{x_1x_1}(t,\,x_1,\,0,\,x_3,\,\ldots,\,x_m) = U_{x_1x_1}(t,\,x_1,\,l,\,x_3,\,\ldots,\,x_m) = \ \ldots\ = \\
= U_{x_1x_1}(t,\,x_1,\,\ldots,\,x_{m-1}\,,0) = U_{x_1x_1}(t,\,x_1,\,\ldots,\,x_{m-1},\,l) = \ \ldots\ = \\
= U_{x_mx_m}(t,\,0,\,x_2,\,x_3,\,\ldots,\,x_m) = U_{x_mx_m}(t,\,l,\,x_2,\,x_3,\,\ldots,\,x_m) = \\
= U_{x_mx_m}(t,\,x_1,\,0,\,x_3,\,\ldots,\,x_m) = U_{x_mx_m}(t,\,x_1,\,l,\,x_3,\,\ldots,\,x_m) = \ \ldots = \\
= U_{x_mx_m}(t,\,x_1,\,\ldots\ ,\,x_{m-1}\,,0) = U_{x_mx_m}(t,\,x_1,\,\ldots,\,x_{m-1},\,l) = 0,\ 0 \leq t \leq T,
\end{gathered} \tag{4}$$

and additional conditions

$$U\,(t_i,\,x) = \psi_i(x),\ \ i = 1,2,\ \ x \in \Omega_l^m, \tag{5}$$

where $\varphi_i(x)$, $\psi_i(x)$ are given smooth functions, $\varphi_i(0) = \varphi_i(l) = 0$, $\psi_i(0) = \psi_i(l) = 0$, $i = 1,2$, $t_1 \in (0;T)$, $t_2 \in (-T;0)$, $\Omega' = \Omega \cup \{x_1, x_2, ..., x_m = 0\} \cup \{x_1, x_2, ..., x_m = l\}$, $\Omega = \Omega_- \cup \Omega_+$, $\Omega_- = \{-T < t < 0,\ 0 < x_1, x_2, ..., x_m < l\}$, $\Omega_+ = \{0 < t < T,\ 0 < x_1, x_2, ..., x_m < l\}$, $\overline{\Omega} = \{-T \leq t \leq T,\ 0 \leq x_1, x_2, ..., x_m \leq l\}$.

2. Expansion of the Solution of the Direct Problem (1)–(4) into Fourier Series. Regular Case

The solution of the integro-differential Equation (1) in domain Ω is sought in the form of a Fourier series

$$U\,(t,\,x) = \sum_{n_1,\ldots,n_m=1}^{\infty} u_{n_1,\ldots,n_m}(t)\,\vartheta_{n_1,\ldots,n_m}(x), \tag{6}$$

where

$$u_{n_1,\ldots,n_m}(t) = \int_{\Omega_l^m} U\,(t,\,x)\,\vartheta_{n_1,\ldots,n_m}\,(x)\,d\,x, \tag{7}$$

$$\int\limits_{\Omega_l^m} U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \int\limits_0^l \ldots \int\limits_0^l U(t,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx_1\ldots dx_m$$

$$\vartheta_{n_1,\ldots,n_m}(x) = \left(\sqrt{\frac{2}{l}}\right)^m \sin\frac{\pi n_1}{l}x_1\ldots\sin\frac{\pi n_m}{l}x_m,$$

$$\Omega_l^m = [0;l]^m, n_1,\ldots,n_m = 1,2,\ldots$$

Also suppose that

$$g_i(x) = \sum_{n_1,\ldots,n_m=1}^{\infty} g_{i\,n_1,\ldots,n_m}\vartheta_{n_1,\ldots,n_m}(x), \tag{8}$$

where

$$g_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} g_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx, \quad i = 1,2.$$

Substituting series (6) and (8) into Equation (1), we obtain a countable system of integro-differential equations

$$u''_{n_1,\ldots,n_m}(t) - \lambda^2_{n_1,\ldots,n_m}u_{n_1,\ldots,n_m}(t) =$$

$$= \nu\int\limits_0^T a_1(t)\,b_1(s)\,u_{n_1,\ldots,n_m}(s)\,ds + f_1(t)\,g_{1\,n_1,\ldots,n_m}, \quad t>0, \tag{9}$$

$$u''_{n_1,\ldots,n_m}(t) + \lambda^2_{n_1,\ldots,n_m}\omega^2 u_{n_1,\ldots,n_m}(t) =$$

$$= \nu\int\limits_{-T}^0 a_2(t)\,b_2(s)\,u_{n_1,\ldots,n_m}(s)\,ds + f_2(t)\,g_{2\,n_1,\ldots,n_m}, \quad t<0, \tag{10}$$

where $\lambda^2_{n_1,\ldots,n_m} = \frac{\mu^2_{n_1,\ldots,n_m}}{1+\mu^2_{n_1,\ldots,n_m}}$, $\mu_{n_1,\ldots,n_m} = \frac{\pi}{l}\sqrt{n_1^2+\ \ldots\ +n_m^2}$.

By the aid of notations

$$\alpha_{n_1,\ldots,n_m} = \int\limits_0^T b_1(s)\,u_{n_1,\ldots,n_m}(s)\,ds, \tag{11}$$

$$\beta_{n_1,\ldots,n_m} = \int\limits_{-T}^0 b_2(s)\,u_{n_1,\ldots,n_m}(s)\,ds, \tag{12}$$

we rewrite the countable systems of Equations (9) and (10) as follows

$$u''_{n_1,\ldots,n_m}(t) - \lambda^2_{n_1,\ldots,n_m}u_{n_1,\ldots,n_m}(t) = \nu\,a_1(t)\,\alpha_{n_1,\ldots,n_m} + f_1(t)\,g_{1\,n_1,\ldots,n_m}, \quad t>0, \tag{13}$$

$$u''_{n_1,\ldots,n_m}(t) + \lambda^2_{n_1,\ldots,n_m}\omega^2 u_{n_1,\ldots,n_m}(t) = \nu\,a_2(t)\,\beta_{n_1,\ldots,n_m} + f_2(t)\,g_{2\,n_1,\ldots,n_m}, \quad t<0. \tag{14}$$

Countable systems of differential Equations (13) and (14) are solved by the method of variation of arbitrary constants:

$$u_{n_1,\ldots,n_m}(t) = A_{1\,n_1,\ldots,n_m}\exp\{\lambda_{n_1,\ldots,n_m}t\} + A_{2\,n_1,\ldots,n_m}\exp\{-\lambda_{n_1,\ldots,n_m}t\}+$$

$$+\eta_{1\,n_1,\ldots,n_m}(t), \quad t>0, \tag{15}$$

$$u_{n_1,\ldots,n_m}(t) = B_{1\,n_1,\ldots,n_m}\cos\lambda_{n_1,\ldots,n_m}\omega t + B_{2\,n_1,\ldots,n_m}\sin\lambda_{n_1,\ldots,n_m}\omega t+$$

$$+\eta_{2\,n_1,\ldots,n_m}(t), \quad t<0, \tag{16}$$

where $A_{i\,n_1,\ldots,n_m}$, $B_{i\,n_1,\ldots,n_m}$ $(i = 1, 2)$ are unknown constants to be uniquely determined,

$$\eta_{1\,n_1,\ldots,n_m}(t) = \nu\,\alpha_{n_1,\ldots,n_m}\,h_{1\,n_1,\ldots,n_m}(t) + g_{1\,n_1,\ldots,n_m}h_{2\,n_1,\ldots,n_m}(t),$$

$$\eta_{2\,n_1,\ldots,n_m}(t) = \nu\,\beta_{n_1,\ldots,n_m}\delta_{1\,n_1,\ldots,n_m}(t) + g_{2\,n_1,\ldots,n_m}\delta_{2\,n_1,\ldots,n_m}(t),$$

$$h_{1\,n_1,\ldots,n_m}(t) = \frac{1}{\lambda_{n_1,\ldots,n_m}}\int_0^t \sinh\lambda_{n_1,\ldots,n_m}(t-s)\,a_1(s)\,ds,$$

$$h_{2\,n_1,\ldots,n_m}(t) = \frac{1}{\lambda_{n_1,\ldots,n_m}}\int_0^t \sinh\lambda_{n_1,\ldots,n_m}(t-s)\,f_1(s)\,ds,$$

$$\delta_{1\,n_1,\ldots,n_m}(t) = \frac{1}{\lambda_{n_1,\ldots,n_m}\omega}\int_0^t \sin\lambda_{n_1,\ldots,n_m}\omega\,(t-s)\,a_2(s)\,ds,$$

$$\delta_{2\,n_1,\ldots,n_m}(t) = \frac{1}{\lambda_{n_1,\ldots,n_m}\omega}\int_0^t \sin\lambda_{n_1,\ldots,n_m}\omega\,(t-s)\,f_2(s)\,ds.$$

From the statement of the problem it follows that the continuous conjugation conditions are fulfilled: $U\,(0+0,\,x) = U\,(0-0,\,x)$ and $U'\,(0+0,\,x) = U'\,(0-0,\,x)$. So, taking the Formula (7) into account, we have

$$u_{n_1,\ldots,n_m}(0+0) = \int_{\Omega_l^m} U\,(0+0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx =$$

$$= \int_{\Omega_l^m} U\,(0-0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = u_{n_1,\ldots,n_m}(0-0). \tag{17}$$

Differentiating functions (7) once with respect to t, similarly to (17) we obtain

$$u'_{n_1,\ldots,n_m}(0+0) = \int_{\Omega_l^m} U_t\,(0+0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx =$$

$$= \int_{\Omega_l^m} U_t\,(0-0,\,x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = u'_{n_1,\ldots,n_m}(0-0). \tag{18}$$

Taking conditions (17) and (18) into account from representations (15) and (16) we obtain

$$A_{1\,n_1,\ldots,n_m} = \frac{1}{2}\left(B_{1\,n_1,\ldots,n_m} + \omega\,B_{2\,n_1,\ldots,n_m}\right),\ A_{2\,n_1,\ldots,n_m} = \frac{1}{2}\left(B_{1\,n_1,\ldots,n_m} - \omega\,B_{2\,n_1,\ldots,n_m}\right).$$

Then the functions (15) and (16) take the forms

$$u_{n_1,\ldots,n_m}(t) = B_{1\,n_1,\ldots,n_m}\cosh\lambda_{n_1,\ldots,n_m}t + \omega\,B_{2\,n_1,\ldots,n_m}\sinh\lambda_n t + \eta_{1\,n_1,\ldots,n_m}(t),\ \ t>0, \tag{19}$$

$$u_{n_1,\ldots,n_m}(t) = B_{1\,n_1,\ldots,n_m}\cos\lambda_{n_1,\ldots,n_m}\omega\,t + B_{2\,n_1,\ldots,n_m}\sin\lambda_{n_1,\ldots,n_m}\omega\,t + \eta_{2\,n_1,\ldots,n_m}(t),\ t<0. \tag{20}$$

Taking formula (7) into account we will rewrite conditions (2) and (3) in the following forms

$$\int_0^T u_{n_1,\ldots,n_m}(t)\,dt = \int_{\Omega_l^m}\int_0^T U\,(t,\,x)\,dt\,\vartheta_{n_1,\ldots,n_m}(x)\,dx =$$

$$= \int\limits_{\Omega_l^m} \varphi_1(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \varphi_{1\,n_1,\ldots,n_m}, \tag{21}$$

$$\int\limits_{-T}^{0} u_{n_1,\ldots,n_m}(t)\,dt = \int\limits_{\Omega_l^m}\int\limits_{-T}^{0} U(t,x)\,dt\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \tag{22}$$
$$= \int\limits_{\Omega_l^m} \varphi_2(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx = \varphi_{2\,n_1,\ldots,n_m}.$$

The coefficients $B_{1\,n_1,\ldots,n_m}$ and $B_{2\,n_1,\ldots,n_m}$ in (19) and (20) are unknown. To find them we use the conditions (21) and (22):

$$\int\limits_0^T u_{n_1,\ldots,n_m}(t)\,dt =$$
$$= \int\limits_0^T \left[B_{1n_1,\ldots,n_m}\cosh\lambda_{n_1,\ldots,n_m}t + \omega\,B_{2n_1,\ldots,n_m}\sinh\lambda_{n_1,\ldots,n_m}t + \eta_{1\,n_1,\ldots,n_m}(t)\right]\,dt =$$
$$= \frac{1}{\lambda_{n_1,\ldots,n_m}}\left[B_{1\,n_1,\ldots,n_m}\sin h\,\lambda_{n_1,\ldots,n_m}T + \omega\,B_{2\,n_1,\ldots,n_m}\left(\cos h\,\lambda_{n_1,\ldots,n_m}T - 1\right)\right] +$$
$$+\xi_{1\,n_1,\ldots,n_m} = \varphi_{1\,n_1,\ldots,n_m}, \tag{23}$$

$$\int\limits_{-T}^0 u_{n_1,\ldots,n_m}(t)\,dt =$$
$$= \int\limits_{-T}^0 \left[B_{1\,n_1,\ldots,n_m}\cos\lambda_{n_1,\ldots,n_m}\omega\,t + B_{2\,n_1,\ldots,n_m}\sin\lambda_{n_1,\ldots,n_m}\omega\,t + \eta_{2\,n_1,\ldots,n_m}(t)\right]\,dt =$$
$$= \frac{1}{\lambda_{n_1,\ldots,n_m}\omega}\left[B_{1\,n_1,\ldots,n_m}\sin\lambda_{n_1,\ldots,n_m}\omega\,T + B_{2\,n_1,\ldots,n_m}\left(\cos\lambda_{n_1,\ldots,n_m}\omega\,T - 1\right)\right] +$$
$$+\xi_{2\,n_1,\ldots,n_m} = \varphi_{2\,n_1,\ldots,n_m}, \tag{24}$$

where $\xi_{1\,n_1,\ldots,n_m} = \int\limits_0^T \eta_{1\,n_1,\ldots,n_m}(t)\,dt$, $\xi_{2\,n_1,\ldots,n_m} = \int\limits_{-T}^0 \eta_{2\,n_1,\ldots,n_m}(t)\,dt$.

Relations (23) and (24) are considered as a system of algebraic equations (SAE) with respect to unknown coefficients $B_{1\,n_1,\ldots,n_m}$ and $B_{2\,n_1,\ldots,n_m}$

$$\begin{cases} B_{1\,n_1,\ldots,n_m}\sinh\lambda_{n_1,\ldots,n_m}T + \omega\,B_{2\,n_1,\ldots,n_m}\left(\cosh\lambda_{n_1,\ldots,n_m}T - 1\right) = \\ = \lambda_{n_1,\ldots,n_m}\,\varphi_{1\,n_1,\ldots,n_m} - \lambda_{n_1,\ldots,n_m}\xi_{1\,n_1,\ldots,n_m}, \\ B_{1\,n_1,\ldots,n_m}\sin\lambda_{n_1,\ldots,n_m}\omega\,T + B_{2\,n_1,\ldots,n_m}\left(\cos\lambda_{n_1,\ldots,n_m}\omega\,T - 1\right) = \\ = \lambda_{n_1,\ldots,n_m}\varphi_{2\,n_1,\ldots,n_m}\omega - \lambda_{n_1,\ldots,n_m}\omega\;\xi_{2\,n_1,\ldots,n_m}. \end{cases}$$

If we assume that

$$\sigma_{n_1,\ldots,n_m} =$$
$$= \sinh\lambda_{n_1,\ldots,n_m}T\,\left(\cos\lambda_{n_1,\ldots,n_m}\omega\,T - 1\right) - \omega\,\sin\lambda_{n_1,\ldots,n_m}\omega\,T\,\left(\cosh\lambda_{n_1,\ldots,n_m}T - 1\right) \neq 0, \tag{25}$$

then SAE with respect to $B_{1\,n_1,\ldots,n_m}$ and $B_{2\,n_1,\ldots,n_m}$ is uniquely solvable. Solving this system from (19) and (20) we arrive at the following representations

$$u_{n_1,\ldots,n_m}(t,\omega) = \frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\left[\varphi_{1\,n_1,\ldots,n_m}M_{1\,n_1,\ldots,n_m}(t,\omega) + \varphi_{2\,n_1,\ldots,n_m}M_{2\,n_1,\ldots,n_m}(t,\omega) +\right.$$

$$+\xi_{1n_1,\ldots,n_m}M_{3n_1,\ldots,n_m}(t,\omega)+\xi_{2n_1,\ldots,n_m}M_{4n_1,\ldots,n_m}(t,\omega)\big]+\eta_{1n_1,\ldots,n_m}(t),\ \ t>0, \tag{26}$$

$$u_{n_1,\ldots,n_m}(t,\omega)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\ \big[\varphi_{1n_1,\ldots,n_m}N_{1n_1,\ldots,n_m}(t,\omega)+\varphi_{2n_1,\ldots,n_m}N_{2n_1,\ldots,n_m}(t,\omega)+$$

$$+\xi_{1n_1,\ldots,n_m}N_{3n_1,\ldots,n_m}(t,\omega)+\xi_{2n_1,\ldots,n_m}N_{4n_1,\ldots,n_m}(t,\omega)\big]+\eta_{2n_1,\ldots,n_m}(t),\ \ t<0, \tag{27}$$

where

$$M_{1n_1,\ldots,n_m}(t,\omega)=(\cos\lambda_{n_1,\ldots,n_m}\omega T-1)\cosh\lambda_{n_1,\ldots,n_m}t-\sin\lambda_{n_1,\ldots,n_m}\omega T\sinh\lambda_{n_1,\ldots,n_m}t,$$

$$M_{2n_1,\ldots,n_m}(t,\omega)=\omega^2(1-\cosh\lambda_{n_1,\ldots,n_m}T)\cosh\lambda_{n_1,\ldots,n_m}t+\omega\sinh\lambda_{n_1,\ldots,n_m}T\sinh\lambda_{n_1,\ldots,n_m}t,$$

$$M_{3n_1,\ldots,n_m}(t,\omega)=\ (1-\cos\lambda_{n_1,\ldots,n_m}\omega T)\cosh\lambda_{n_1,\ldots,n_m}t+\sin\lambda_{n_1,\ldots,n_m}\omega T\sinh\lambda_{n_1,\ldots,n_m}t,$$

$$M_{4n_1,\ldots,n_m}(t,\omega)=\omega^2(\cosh\lambda_{n_1,\ldots,n_m}T-1)\cosh\lambda_{n_1,\ldots,n_m}t-\omega\sinh\lambda_{n_1,\ldots,n_m}T\sinh\lambda_{n_1,\ldots,n_m}t,$$

$$N_{1n_1,\ldots,n_m}(t,\omega)=(\cos\lambda_{n_1,\ldots,n_m}\omega T-1)\cos\lambda_{n_1,\ldots,n_m}\omega t-\sin\lambda_{n_1,\ldots,n_m}\omega T\sin\lambda_{n_1,\ldots,n_m}\omega t,$$

$$N_{2n_1,\ldots,n_m}(t,\omega)=\omega\,(1-\cosh\lambda_{n_1,\ldots,n_m}T)\cos\lambda_{n_1,\ldots,n_m}\omega t+\omega\sinh\lambda_{n_1,\ldots,n_m}T\sin\lambda_{n_1,\ldots,n_m}\omega t,$$

$$N_{3n_1,\ldots,n_m}(t,\omega)=(1-\cos\lambda_{n_1,\ldots,n_m}\omega T)\cos\lambda_{n_1,\ldots,n_m}\omega t+\sin\lambda_{n_1,\ldots,n_m}\omega T\sin\lambda_{n_1,\ldots,n_m}\omega t,$$

$$N_{4n_1,\ldots,n_m}(t,\omega)=\omega^2(\cosh\lambda_{n_1,\ldots,n_m}T-1)\cos\lambda_{n_1,\ldots,n_m}\omega t-\omega\sinh\lambda_{n_1,\ldots,n_m}T\sin\lambda_{n_1,\ldots,n_m}\omega t.$$

Taking the following presentations

$$\eta_{1n_1,\ldots,n_m}(t)=\nu\,\alpha_{n_1,\ldots,n_m}h_{1n_1,\ldots,n_m}(t)+g_{1n_1,\ldots,n_m}h_{2n_1,\ldots,n_m}(t),$$

$$\eta_{2n_1,\ldots,n_m}(t)=\nu\,\beta_{n_1,\ldots,n_m}\delta_{1n_1,\ldots,n_m}(t)+g_{2n_1,\ldots,n_m}\delta_{2n_1,\ldots,n_m}(t)$$

into account representations (26) and (27) are written in the following forms

$$u_{n_1,\ldots,n_m}(t,\omega)=$$

$$=P_{1n_1,\ldots,n_m}(t,\omega)+\nu\,\alpha_{n_1,\ldots,n_m}P_{2n_1,\ldots,n_m}(t,\omega)+\nu\,\beta_{n_1,\ldots,n_m}P_{3n_1,\ldots,n_m}(t,\omega)+$$

$$+g_{1n_1,\ldots,n_m}P_{4n_1,\ldots,n_m}(t,\omega)+g_{2n_1,\ldots,n_m}P_{5n_1,\ldots,n_m}(t,\omega),\ \ t>0, \tag{28}$$

$$u_{n_1,\ldots,n_m}(t,\omega)=$$

$$=Q_{1n_1,\ldots,n_m}(t,\omega)+\nu\,\alpha_{n_1,\ldots,n_m}Q_{2n_1,\ldots,n_m}(t,\omega)+\nu\,\beta_{n_1,\ldots,n_m}Q_{3n_1,\ldots,n_m}(t,\omega)+$$

$$+g_{1n_1,\ldots,n_m}Q_{4n_1,\ldots,n_m}(t,\omega)+g_{2n_1,\ldots,n_m}Q_{5n_1,\ldots,n_m}(t,\omega),\ \ t<0, \tag{29}$$

where

$$P_{1n_1,\ldots,n_m}(t,\omega)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\ \big[\varphi_{1n_1,\ldots,n_m}M_{1n_1,\ldots,n_m}(t,\omega)+\varphi_{2n_1,\ldots,n_m}M_{2n_1,\ldots,n_m}(t,\omega)\big],$$

$$P_{2n_1,\ldots,n_m}(t,\omega)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}M_{3n_1,\ldots,n_m}(t,\omega)\int_0^T h_{1n_1,\ldots,n_m}(t)\,dt+h_{1n_1,\ldots,n_m}(t),$$

$$P_{3n_1,\ldots,n_m}(t,\omega)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}M_{4n_1,\ldots,n_m}(t,\omega)\int_{-T}^0 \delta_{1n_1,\ldots,n_m}(t)\,dt,$$

$$P_{4n_1,\ldots,n_m}(t,\omega)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}M_{3n_1,\ldots,n_m}(t,\omega)\int_0^T h_{2n_1,\ldots,n_m}(t)\,dt+h_{2n_1,\ldots,n_m}(t),$$

$$P_{5\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}M_{4\,n_1,\ldots,n_m}(t,\,\omega)\int\limits_{-T}^{0}\delta_{2\,n_1,\ldots,n_m}(t)\,d\,t,$$

$$Q_{1\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\left[\varphi_{1\,n_1,\ldots,n_m}N_{1\,n_1,\ldots,n_m}(t,\,\omega)+\varphi_{2\,n_1,\ldots,n_m}N_{2\,n_1,\ldots,n_m}(t,\,\omega)\right],$$

$$Q_{2\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}N_{3\,n_1,\ldots,n_m}(t,\,\omega)\int\limits_{0}^{T}h_{1\,n_1,\ldots,n_m}(t)\,d\,t,$$

$$Q_{3\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}N_{4\,n_1,\ldots,n_m}(t,\,\omega)\int\limits_{-T}^{0}\delta_{1\,n_1,\ldots,n_m}(t)\,d\,t+\delta_{1\,n_1,\ldots,n_m}(t),$$

$$Q_{4\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}N_{3\,n_1,\ldots,n_m}(t,\,\omega)\int\limits_{0}^{T}h_{2\,n_1,\ldots,n_m}(t)\,d\,t,$$

$$Q_{5\,n_1,\ldots,n_m}(t,\,\omega)=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}N_{4\,n_1,\ldots,n_m}(t,\,\omega)\int\limits_{-T}^{0}\delta_{2\,n_1,\ldots,n_m}(t)\,d\,t+\delta_{2\,n_1,\ldots,n_m}(t).$$

We substitute (28) and (29) into (11) and (12), respectively. Then we obtain a countable system of two algebraic equations (CSTAE)

$$\begin{cases}\alpha_{\,n_1,\ldots,n_m}(1-\nu\,E_{\,n_1,\ldots,n_m})-\beta_{\,n_1,\ldots,n_m}\nu\,F_{\,n_1,\ldots,n_m}(\omega)=\Phi_{\,n_1,\ldots,n_m}(\omega),\\ -\alpha_{\,n_1,\ldots,n_m}\nu\;H_{\,n_1,\ldots,n_m}(\omega)+\beta_{\,n_1,\ldots,n_m}\,(1-\nu\,G_{\,n_1,\ldots,n_m})=\Psi_{\,n_1,\ldots,n_m}(\omega)\,,\end{cases}\tag{30}$$

where

$$E_{\,n_1,\ldots,n_m}=\int\limits_{0}^{T}b_1\,(t)\,P_{2\,n_1,\ldots,n_m}\,(t)\,d\,t,\;F_{\,n_1,\ldots,n_m}\,(\omega)=\int\limits_{0}^{T}b_1(t)\,P_{3\,n_1,\ldots,n_m}\,(t\,,\,\omega)\,d\,t,$$

$$H_{\,n_1,\ldots,n_m}(\omega)=\int\limits_{-T}^{0}b_2(t)\,Q_{2\,n_1,\ldots,n_m}(t,\,\omega)\,d\,t,\;G_{\,n_1,\ldots,n_m}=\int\limits_{-T}^{0}b_2(t)\,Q_{3\,n_1,\ldots,n_m}\,(t)\,d\,t,$$

$$\Phi_{\,n_1,\ldots,n_m}(\omega)=\varphi_{1\,n_1,\ldots,n_m}P_{01\,n_1,\ldots,n_m}+$$
$$+\varphi_{2\,n_1,\ldots,n_m}P_{02\,n_1,\ldots,n_m}+g_{1\,n_1,\ldots,n_m}P_{03\,n_1,\ldots,n_m}+g_{2\,n_1,\ldots,n_m}P_{04\,n_1,\ldots,n_m},\tag{31}$$

$$\Psi_{\,n_1,\ldots,n_m}(\omega)=\varphi_{1\,n_1,\ldots,n_m}Q_{01\,n_1,\ldots,n_m}+$$
$$+\varphi_{2\,n_1,\ldots,n_m}Q_{02\,n_1,\ldots,n_m}+g_{1\,n_1,\ldots,n_m}Q_{03\,n_1,\ldots,n_m}+g_{2\,n_1,\ldots,n_m}Q_{04\,n_1,\ldots,n_m},\tag{32}$$

$$P_{0\,i\,n_1,\ldots,n_m}=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\int\limits_{-T}^{0}b_2(t)\,M_{i\,n_1,\ldots,n_m}(t,\,\omega)\,d\,t,\;\;i=1,\,2,$$

$$P_{0\,j\,n_1,\ldots,n_m}=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\int\limits_{-T}^{0}b_2(t)\,P_{1+j,\,n_1,\ldots,n_m}(t,\,\omega)\,d\,t,\;\;j=3,\,4,$$

$$Q_{0\,i\,n_1,\ldots,n_m}=\frac{\lambda_{\,n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}\int\limits_{-T}^{0}b_2(t)\,N_{i\,n_1,\ldots,n_m}(t,\,\omega)\,d\,t,\;\;i=1,\,2,$$

$$Q_{0\,j\,n_1,\ldots,n_m} = \frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}} \int\limits_{-T}^{0} b_2(t)\, Q_{1+j,n_1,\ldots,n_m}(t,\,\omega)\,d\,t, \quad j = 3,\,4.$$

For the unique solvability of CSTAE (30) the following condition is required

$$\Delta_{n_1,\ldots,n_m}(\nu) = \begin{vmatrix} 1 - \nu\, E_{n_1,\ldots,n_m} & -\nu\, F_{n_1,\ldots,n_m}(\omega) \\ -\nu\, H_{n_1,\ldots,n_m}(\omega) & 1 - \nu\, G_{n_1,\ldots,n_m} \end{vmatrix} =$$

$$= \left(E_{n_1,\ldots,n_m}\, G_{n_1,\ldots,n_m} - H_{n_1,\ldots,n_m}(\omega)\, F_{n_1,\ldots,n_m}(\omega)\right) \nu^2 -$$

$$- \left(E_{n_1,\ldots,n_m} + G_{n_1,\ldots,n_m}\right) \nu + 1 \neq 0. \tag{33}$$

A quadratic equation has no real roots, if its discriminant is negative. Therefore, from condition (33) we arrive at the following condition

$$\left(E_{n_1,\ldots,n_m} - G_{n_1,\ldots,n_m}\right)^2 + 4\, H_{n_1,\ldots,n_m}(\omega) F_{n_1,\ldots,n_m}(\omega) < 0. \tag{34}$$

Let condition (34) be fulfilled. Then we solve the CSTAE (30):

$$\alpha_{n_1,\ldots,n_m} = \frac{\Phi_{n_1,\ldots,n_m}(\omega) + \nu\, \left(\Psi_{n_1,\ldots,n_m}(\omega)\, F_{n_1,\ldots,n_m}(\omega) - \Phi_{n_1,\ldots,n_m}(\omega)\, G_{n_1,\ldots,n_m}\right)}{\Delta_{n_1,\ldots,n_m}(\nu)},$$

$$\beta_{n_1,\ldots,n_m} = \frac{\Psi_{n_1,\ldots,n_m}(\omega) + \nu\, \left(\Phi_{n_1,\ldots,n_m}(\omega)\, H_{n_1,\ldots,n_m}(\omega) - \Psi_{n_1,\ldots,n_m}(\omega)\, E_{n_1,\ldots,n_m}\right)}{\Delta_{n_1,\ldots,n_m}(\nu)}.$$

Substituting these solutions into (28) and (29), we obtain

$$u_{n_1,\ldots,n_m}(t,\omega,\nu) = P_{1\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ \frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)} \left[\Phi_{n_1,\ldots,n_m}(\omega)\, \left(1 - \nu\, G_{n_1,\ldots,n_m}\right) + \nu\, \Psi_{n_1,\ldots,n_m}(\omega)\, F_{n_1,\ldots,n_m}(\omega)\right] P_{2\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ \frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)} \left[\nu\, \Phi_{n_1,\ldots,n_m}(\omega)\, H_{n_1,\ldots,n_m}(\omega) + \Psi_{n_1,\ldots,n_m}(\omega)\, \left(1 - \nu\, E_{n_1,\ldots,n_m}\right)\right] P_{3\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ g_{1\,n_1,\ldots,n_m} P_{4\,n_1,\ldots,n_m}(t,\omega) + g_{2\,n_1,\ldots,n_m} P_{5\,n_1,\ldots,n_m}(t,\omega), \quad t > 0, \tag{35}$$

$$u_{n_1,\ldots,n_m}(t,\omega,\nu) = Q_{1\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ \frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)} \left[\Phi_{n_1,\ldots,n_m}(\omega)\, \left(1 - \nu\, G_{n_1,\ldots,n_m}\right) + \nu\, \Psi_{n_1,\ldots,n_m}(\omega)\, F_{n_1,\ldots,n_m}(\omega)\right] Q_{2\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ \frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)} \left[\nu\, \Phi_{n_1,\ldots,n_m}(\omega) H_{n_1,\ldots,n_m}(\omega) + \Psi_{n_1,\ldots,n_m}(\omega)\, \left(1 - \nu\, E_{n_1,\ldots,n_m}\right)\right] Q_{3\,n_1,\ldots,n_m}(t,\omega) +$$

$$+ g_{1\,n_1,\ldots,n_m} Q_{4\,n_1,\ldots,n_m}(t,\omega) + g_{2\,n_1,\ldots,n_m} Q_{5\,n_1,\ldots,n_m}(t,\omega), \quad t < 0, \tag{36}$$

Taking (31), (32) and the following relations

$$P_{1\,n_1,\ldots,n_m}(t,\omega) = \frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}} \left[\varphi_{1\,n_1,\ldots,n_m} M_{1\,n_1,\ldots,n_m}(t,\omega) + \varphi_{2\,n_1,\ldots,n_m} M_{2\,n_1,\ldots,n_m}(t,\omega)\right],$$

$$Q_{1\,n_1,\ldots,n_m}(t,\omega) = \frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}} \left[\varphi_{1\,n_1,\ldots,n_m} N_{1\,n_1,\ldots,n_m}(t,\omega) + \varphi_{2\,n_1,\ldots,n_m} N_{2\,n_1,\ldots,n_m}(t,\omega)\right]$$

into account the representations (35) and (36) we rewrite in the following views

$$u_{n_1,\ldots,n_m}(t,\,\omega,\,\nu) = \varphi_{1\,n_1,\ldots,n_m} V_{1\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) + \varphi_{2\,n_1,\ldots,n_m} V_{2\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) +$$

$$+ g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) + g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu), \quad t > 0, \tag{37}$$

$$u_{n_1,\ldots,n_m}(t,\omega,\nu)=\varphi_{1\,n_1,\ldots,n_m}W_{1\,n_1,\ldots,n_m}(t,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m}W_{2\,n_1,\ldots,n_m}(t,\omega,\nu)+$$
$$+g_{1\,n_1,\ldots,n_m}W_{3\,n_1,\ldots,n_m}(t,\omega,\nu)+g_{2\,n_1,\ldots,n_m}W_{4\,n_1,\ldots,n_m}(t,\omega,\nu),\ \ t<0, \tag{38}$$

where

$$V_{i\,n_1,\ldots,n_m}(t,\omega,\nu)=\frac{\lambda_{n_1,\ldots,n_m}}{\sigma_{n_1,\ldots,n_m}}M_{i\,n_1,\ldots,n_m}(t,\omega)+$$
$$+P_{0i\,n_1,\ldots,n_m}V_{01\,n_1,\ldots,n_m}(t,\omega,\nu)+Q_{0i\,n_1,\ldots,n_m}V_{02\,n_1,\ldots,n_m}(t,\omega,\nu),\ i=1,2,$$
$$V_{j\,n_1,\ldots,n_m}(t,\omega,\nu)=P_{(i+1)\,n_1,\ldots,n_m}(t,\omega)+$$
$$+P_{0j\,n_1,\ldots,n_m}V_{01\,n_1,\ldots,n_m}(t,\omega,\nu)+Q_{0j\,n_1,\ldots,n_m}V_{02\,n_1,\ldots,n_m}(t,\omega,\nu),\ j=3,4,$$
$$W_{i\,n_1,\ldots,n_m}(t,\omega,\nu)=\frac{\lambda}{\sigma_{n_1,\ldots,n_m}}N_{i\,n_1,\ldots,n_m}(t,\omega)+$$
$$+P_{0i\,n_1,\ldots,n_m}W_{01\,n_1,\ldots,n_m}(t,\omega,\nu)+Q_{0i\,n_1,\ldots,n_m}W_{02\,n_1,\ldots,n_m}(t,\omega,\nu),\ i=1,2,$$
$$W_{j\,n_1,\ldots,n_m}(t,\omega,\nu)=Q_{(i+1)\,n_1,\ldots,n_m}(t,\omega)+$$
$$+P_{0j\,n_1,\ldots,n_m}W_{01\,n_1,\ldots,n_m}(t,\omega,\nu)+Q_{0j\,n_1,\ldots,n_m}W_{02\,n_1,\ldots,n_m}(t,\omega,\nu),\ j=3,4,$$
$$V_{01\,n_1,\ldots,n_m}(t,\omega,\nu)+$$
$$=\frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)}\left[(1-\nu\,G_{n_1,\ldots,n_m})\,P_{2\,n_1,\ldots,n_m}(t,\omega)+\nu\,H_{n_1,\ldots,n_m}(\omega)P_{3\,n_1,\ldots,n_m}(t,\omega)\right],$$
$$V_{02\,n_1,\ldots,n_m}(t,\omega,\nu)=\nu\,P_{2\,n_1,\ldots,n_m}(t,\omega)+(1-\nu\,E_{n_1,\ldots,n_m})\,P_{3\,n_1,\ldots,n_m}(t,\omega),$$
$$W_{01\,n_1,\ldots,n_m}(t,\omega,\nu)=$$
$$=\frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)}\left[(1-\nu\,G_{n_1,\ldots,n_m})\,Q_{2\,n_1,\ldots,n_m}(t,\omega)+\nu\,H_{n_1,\ldots,n_m}(\omega)\,Q_{3\,n_1,\ldots,n_m}(t,\omega)\right],$$
$$W_{02\,n_1,\ldots,n_m}(t,\omega,\nu)=$$
$$=\frac{\nu}{\Delta_{n_1,\ldots,n_m}(\nu)}\left[\nu\,F_{n_1,\ldots,n_m}(\omega)\,Q_{2\,n_1,\ldots,n_m}(t,\omega)+(1-\nu\,E_{n_1,\ldots,n_m})\,Q_{3\,n_1,\ldots,n_m}(t,\omega)\right].$$

Now we substitute representations (37) and (38) into the Fourier series (6) and obtain the following formal solution of the direct problem (1)–(4)

$$U(t,x,\omega,\nu)=$$
$$=\sum_{n_1,\ldots,n_m=1}^{\infty}\vartheta_{n_1,\ldots,n_m}(x)\left[\varphi_{1\,n_1,\ldots,n_m}V_{1\,n_1,\ldots,n_m}(t,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m}V_{2\,n_1,\ldots,n_m}(t,\omega,\nu)+\right.$$
$$\left.+g_{1\,n_1,\ldots,n_m}V_{3\,n_1,\ldots,n_m}(t,\omega,\nu)+g_{2\,n_1,\ldots,n_m}V_{4\,n_1,\ldots,n_m}(t,\omega,\nu)\right],\ \ t>0, \tag{39}$$
$$U(t,x,\omega,\nu)=$$
$$=\sum_{n_1,\ldots,n_m=1}^{\infty}\vartheta_{n_1,\ldots,n_m}(x)\left[\varphi_{1\,n_1,\ldots,n_m}W_{1\,n_1,\ldots,n_m}(t,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m}W_{2\,n_1,\ldots,n_m}(t,\omega,\nu)+\right.$$
$$\left.+g_{1\,n_1,\ldots,n_m}W_{3\,n_1,\ldots,n_m}(t,\omega,\nu)+g_{2\,n_1,\ldots,n_m}W_{4\,n_1,\ldots,n_m}(t,\omega,\nu)\right],\ \ t<0. \tag{40}$$

3. Inverse Problem (1)–(5). The Regular Case of the Spectral Parameter ω

We use the additional conditions (5) and from the Fourier series (39) and (40) we obtain that

$$\psi_1(x)=U(t_1,x,\omega,\nu)=\sum_{n_1,\ldots,n_m=1}^{\infty}\vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} V_{1\,n_1,\ldots,n_m}(t_1,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m} V_{2\,n_1,\ldots,n_m}(t_1,\omega,\nu)+\right.$$

$$\left.+g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)+g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)\right],\ \ 0<t_1<T, \tag{41}$$

$$\psi_2(x)=U(t_2,x,\omega,\nu)=\sum_{n_1,\ldots,n_m=1}^{\infty}\vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} W_{1\,n_1,\ldots,n_m}(t_2,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m} W_{2\,n_1,\ldots,n_m}(t_2,\omega,\nu)+\right.$$

$$\left.+g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu)+g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)\right],\ \ -T<t_2<0. \tag{42}$$

Assume that the functions $\psi_i(x)$ are expanded in Fourier series

$$\psi_i(x)=\sum_{n_1,\ldots,n_m=1}^{\infty}\psi_{i\,n_1,\ldots,n_m}\vartheta_{n_1,\ldots,n_m}(x), \tag{43}$$

where $\psi_{i\,n_1,\ldots,n_m}=\int\limits_{\Omega_l^m}\psi_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,dx,\ \ i=1,2,\ \ n_1,\ldots,n_m=1,2,\ \ldots$

Then, taking into account (43), from (41) and (42) we obtain

$$\psi_{1\,n_1,\ldots,n_m}=\varphi_{1\,n_1,\ldots,n_m} V_{1\,n_1,\ldots,n_m}(t_1,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m} V_{2\,n_1,\ldots,n_m}(t_1,\omega,\nu)+$$

$$+g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)+g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu),\ \ 0<t_1<T,$$

$$\psi_{2\,n_1,\ldots,n_m}=\varphi_{1\,n_1,\ldots,n_m} W_{1\,n_1,\ldots,n_m}(t_2,\omega,\nu)+\varphi_{2\,n_1,\ldots,n_m} W_{2\,n_1,\ldots,n_m}(t_2,\omega,\nu)+$$

$$+g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu)+g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu),\ \ -T<t_2<0.$$

Hence we find a system of two algebraic equations for finding the coefficients of the redefinition functions $g_{1\,n_1,\ldots,n_m}$ and $g_{2\,n_1,\ldots,n_m}$

$$\begin{cases} g_{1\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)+g_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)= \\ =\psi_{1\,n_1,\ldots,n_m}-\varphi_{1\,n_1,\ldots,n_m} V_{1\,n_1,\ldots,n_m}(t_1,\omega,\nu)-\varphi_{2\,n_1,\ldots,n_m} V_{2\,n_1,\ldots,n_m}(t_1,\omega,\nu), \\ g_{1\,n_1,\ldots,n_m} W_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu)+g_{2\,n_1,\ldots,n_m} W_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)= \\ =\psi_{2\,n_1,\ldots,n_m}-\varphi_{1\,n_1,\ldots,n_m} W_{1\,n_1,\ldots,n_m}(t_2,\omega,\nu)-\varphi_{2\,n_1,\ldots,n_m} W_{2\,n_1,\ldots,n_m}(t_2,\omega,\nu). \end{cases}$$

Solving this system of algebraic equations, we obtain

$$g_{1\,n_1,\ldots,n_m}(\omega,\nu)=$$

$$=\frac{1}{r_{01\,n_1,\ldots,n_m}}\left[\psi_{1\,n_1,\ldots,n_m} W_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)-\psi_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)+\right.$$

$$\left.+\varphi_{1\,n_1,\ldots,n_m} r_{11\,n_1,\ldots,n_m}+\varphi_{2\,n_1,\ldots,n_m} r_{12\,n_1,\ldots,n_m}\right], \tag{44}$$

$$g_{2\,n_1,\ldots,n_m}(\omega,\nu)=$$

$$=\frac{1}{r_{01\,n_1,\ldots,n_m}}\left[-\psi_{1\,n_1,\ldots,n_m} W_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu)+\psi_{2\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)+\right.$$

$$\left.+\varphi_{1\,n_1,\ldots,n_m} r_{21\,n_1,\ldots,n_m}+\varphi_{2\,n_1,\ldots,n_m} r_{22\,n_1,\ldots,n_m}\right], \tag{45}$$

where $r_{01\,n_1,\ldots,n_m}=$

$$=V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)-V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu)\neq 0,$$

$$r_{11\,n_1,\ldots,n_m}=-V_{1\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)+V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{1\,n_1,\ldots,n_m}(t_2,\omega,\nu),$$

$$r_{12\,n_1,\ldots,n_m}=-V_{2\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{4\,n_1,\ldots,n_m}(t_2,\omega,\nu)+V_{4\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{2\,n_1,\ldots,n_m}(t_2,\omega,\nu),$$

$$r_{21\,n_1,\ldots,n_m} = -V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{1\,n_1,\ldots,n_m}(t_2,\omega,\nu) + V_{1\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu),$$

$$r_{22\,n_1,\ldots,n_m} = -V_{3\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{2\,n_1,\ldots,n_m}(t_2,\omega,\nu) + V_{2\,n_1,\ldots,n_m}(t_1,\omega,\nu)\,W_{3\,n_1,\ldots,n_m}(t_2,\omega,\nu).$$

Substituting representations (44) and (45) into the Fourier series (8), we obtain

$$g_1(x,\,\omega,\,\nu)) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)\,\left[\psi_{1\,n_1,\ldots,n_m} W_{4\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu) - \right.$$

$$\left. -\psi_{2\,n_1,\ldots,n_m} V_{4\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu) + \varphi_{1\,n_1,\ldots,n_m} r_{11\,n_1,\ldots,n_m} + \varphi_{2\,n_1,\ldots,n_m} r_{12\,n_1,\ldots,n_m}\right], \tag{46}$$

$$g_2(x,\,\omega,\,\nu)) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)\,\left[-\psi_{1\,n_1,\ldots,n_m} W_{3\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu) + \right.$$

$$\left. +\psi_{2\,n_1,\ldots,n_m} V_{3\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu) + \varphi_{1\,n_1,\ldots,n_m} r_{21\,n_1,\ldots,n_m} + \varphi_{2\,n_1,\ldots,n_m} r_{22\,n_1,\ldots,n_m}\right]. \tag{47}$$

Now we substitute representations (44) and (45) into the main series (39) and (40):

$$U(t,\,x,\,\omega,\,\nu) =$$

$$= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)\,\left[\varphi_{1\,n_1,\ldots,n_m} D_{11\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{12\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{13\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{14\,n_1,\ldots,n_m}(t,\omega,\nu)\right],\ \ t > 0, \tag{48}$$

$$U(t,x,\omega,\nu) =$$

$$= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x)\,\left[\varphi_{1\,n_1,\ldots,n_m} D_{21\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{22\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{23\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{24\,n_1,\ldots,n_m}(t,\omega,\nu)\right],\ \ t < 0, \tag{49}$$

where

$$D_{1i\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = V_{i\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) +$$

$$+\frac{r_{1i\,n_1,\ldots,n_m}}{r_{01\,n_1,\ldots,n_m}} V_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) + \frac{r_{2i\,n_1,\ldots,n_m}}{r_{01\,n_1,\ldots,n_m}} V_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu),\ i = 1,\,2,$$

$$D_{13\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \times$$

$$\times \left[V_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,W_{4\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu) - V_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,W_{3\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu)\right],$$

$$D_{14\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \times$$

$$\times \left[-V_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,V_{4\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu) + V_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,V_{3\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu)\right],$$

$$D_{2i\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = W_{i\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) +$$

$$+\frac{r_{1i\,n_1,\ldots,n_m}}{r_{01\,n_1,\ldots,n_m}} W_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) + \frac{r_{2i\,n_1,\ldots,n_m}}{r_{01\,n_1,\ldots,n_m}} W_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu),\ i = 1,\,2,$$

$$D_{23\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \times$$

$$\times \left[W_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,W_{3\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu) - W_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,W_{4\,n_1,\ldots,n_m}(t_2,\,\omega,\,\nu)\right],$$

$$D_{24\,n_1,\ldots,n_m}(t,\,\omega,\,\nu) = \frac{1}{r_{01\,n_1,\ldots,n_m}} \times$$

$$\times \left[-W_{3\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,V_{4\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu) + W_{4\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,V_{3\,n_1,\ldots,n_m}(t_1,\,\omega,\,\nu)\right].$$

4. Convergence of Series (46)–(49)

We show that under certain conditions with respect to the functions $\varphi_i(x)$ and $\psi_i(x)$ $(i = 1, 2)$ the series (46)–(49) converge absolutely and uniformly in the domain $\overline{\Omega}$. Indeed, according to the statement of the problem the functions $D_{ij\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)$ $(i = 1, 2;\ j = \overline{1, 4})$ uniformly bounded on the segment $[-T;\,T]$. So $\left|\,D_{ij\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,\right| < \infty$ for all $i = 1, 2,\ \ j = \overline{1, 4}$. Since $0 < \lambda_{n_1,\ldots,n_m} < 1$, then for any positive integers $n_1,\,\ldots,\,n_m$ there exist finite constant numbers C_{0i} $(i = 1, 2)$, that there take place the following estimates

$$\begin{gathered}\max_{n_1,\ldots,n_m\in N}\left\{\max_{t\in[0;\,T]}\left|\,D_{1j\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,\right|;\ \max_{t\in[-T;\,0]}\left|\,D_{2j\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,\right|\right\}\le C_{01},\\ \max_{n_1,\ldots,n_m\in N}\left\{\max_{t\in[0;\,T]}\left|\,D''_{1j\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,\right|;\ \max_{t\in[-T;\,0]}\left|\,D''_{2j\,n_1,\ldots,n_m}(t,\,\omega,\,\nu)\,\right|\right\}\le C_{02},\end{gathered} \tag{50}$$

$j = \overline{1, 4}$.

Condition A. We suppose that the functions $\varphi_i,\ \psi_i \in C^2[0;\,l]^m,\ \ i = 1, 2$ on the domain $[0;\,l]^m$ have piecewise continuous third order derivatives. Then by integrating in parts the following integrals three times with respect to the variable x_1

$$\varphi_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \varphi_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\ \psi_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \psi_i(x)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\ \ i = 1, 2$$

we derive that

$$\varphi_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\varphi'''_{i\,n_1,\ldots,n_m}}{n_1^3},\ \psi_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\psi'''_{i\,n_1,\ldots,n_m}}{n_1^3}, \tag{51}$$

where

$$\varphi'''_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^3\varphi_i(x)}{\partial\,x_1^3}\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\ \psi'''_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^3\psi_i(x)}{\partial\,x_1^3}\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x. \tag{52}$$

By integrating in parts the integrals (52) three times with respect to the variable x_2 we obtain that

$$\varphi'''_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\varphi^{(6)}_{i\,n_1,\ldots,n_m}}{n_2^3},\ \psi'''_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\psi^{(6)}_{i\,n_1,\ldots,n_m}}{n_2^3}, \tag{53}$$

where

$$\varphi^{(6)}_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^6\varphi_i(x)}{\partial\,x_1^3\partial\,x_2^3}\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\ \psi^{(6)}_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^6\psi_i(x)}{\partial\,x_1^3\partial\,x_2^3}\vartheta_{n_1,\ldots,n_m}(x)\,d\,x.$$

Continuing this process, by induction we obtain

$$\varphi^{(3\,m-3)}_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\varphi^{(3\,m)}_{i\,n_1,\ldots,n_m}}{n_m^3},\ \psi^{(3\,m-3)}_{i\,n_1,\ldots,n_m} = -\left(\frac{l}{\pi}\right)^3 \frac{\psi^{(3\,m)}_{i\,n_1,\ldots,n_m}}{n_m^3}, \tag{54}$$

where

$$\varphi^{(3\,m)}_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{3\,m}\varphi_i(x)}{\partial\,x_1^3\partial\,x_2^3\ldots\,\partial\,x_m^3}\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x,\ \psi^{(3\,m)}_{i\,n_1,\ldots,n_m} = \int\limits_{\Omega_l^m} \frac{\partial^{3\,m}\psi_i(x)}{\partial\,x_1^3\partial\,x_2^3\ldots\,\partial\,x_m^3}\vartheta_{n_1,\ldots,n_m}(x)\,d\,x.$$

Here the Bessel inequalities are true

$$\sum_{n_1,\ldots,n_m=1}^{\infty}\left[\varphi_{i\,n_1,\ldots,n_m}^{(3m)}\right]^2 \le \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_i(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2 dx, \tag{55}$$

$$\sum_{n_1,\ldots,n_m=1}^{\infty}\left[\psi_{i\,n_1,\ldots,n_m}^{(3m)}\right]^2 \le \left(\frac{2}{l}\right)^m \int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_i(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2 dx. \tag{56}$$

From (51), (53) and (54) implies that

$$\varphi_{i\,n_1,\ldots,n_m}=\left(\frac{l}{\pi}\right)^{3m}\frac{\varphi_{i\,n_1,\ldots,n_m}^{(3m)}}{n_1^3\ldots n_m^3},\ \psi_{i\,n_1,\ldots,n_m}=\left(\frac{l}{\pi}\right)^{3m}\frac{\psi_{i\,n_1,\ldots,n_m}^{(3m)}}{n_1^3\ldots n_m^3},\ i=1,2. \tag{57}$$

Taking formulas (50), (55)–(57) into account and applying the Cauchy-Schwarz inequality and Bessel inequality, for series (48) and (49) we obtain

$$|U(t,x,\omega,\nu)|\le\sum_{n_1,\ldots,n_m=1}^{\infty}|u_{n_1,\ldots,n_m}(t,\omega,\nu)|\cdot|\vartheta_{n_1,\ldots,n_m}(x)|\le$$

$$\le\left(\sqrt{\frac{2}{l}}\right)^m C_{01}\sum_{n_1,\ldots,n_m=1}^{\infty}\left[|\varphi_{1\,n_1,\ldots,n_m}|+|\varphi_{2\,n_1,\ldots,n_m}|+|\psi_{1\,n_1,\ldots,n_m}|+|\psi_{2\,n_1,\ldots,n_m}|\right]\le$$

$$\le\gamma_1\left[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3\ldots n_m^3}\left|\varphi_{1n_1,\ldots,n_m}^{(3m)}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3\ldots n_m^3}\left|\varphi_{2\,n_1,\ldots,n_m}^{(3m)}\right|+\right.$$

$$\left.+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3\ldots n_m^3}\left|\psi_{1n_1,\ldots,n_m}^{(3m)}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3\ldots n_m^3}\left|\psi_{2\,n_1,\ldots,n_m}^{(3m)}\right|\right]\le$$

$$\le\left(\sqrt{\frac{2}{l}}\right)^m\gamma_1\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^6\ldots n_m^6}}\left[\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\right.$$

$$+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+$$

$$\left.+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right]<\infty, \tag{58}$$

where $\gamma_1=\left(\sqrt{\frac{2}{l}}\right)^m C_{01}\left(\frac{l}{\pi}\right)^{3m}$.

It follows from estimate (58) that the series (48) and (49) converge absolutely and uniformly in the domain $\bar{\Omega}$ under conditions (25) and (33).

From the convergence of series (48) and (49), in particular, it follows that the series (46) and (47) converge absolutely and uniformly in the domain Ω_l^m.

5. Possibility of Term Differentiation of the Series (48) and (49)

Functions (48) and (49) formally differentiate the required number of times

$$U_{tt}(t,x,\omega,\nu)=$$

$$= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x) \left[\varphi_{1\,n_1,\ldots,n_m} D''_{11\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D''_{12\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D''_{13\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D''_{14\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t>0, \tag{59}$$

$$U_{tt}(t,x,\omega,\nu) =$$

$$= \sum_{n_1,\ldots,n_m=1}^{\infty} \vartheta_{n_1,\ldots,n_m}(x) \left[\varphi_{1\,n_1,\ldots,n_m} D''_{21\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D''_{22\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D''_{23\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D''_{24\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t<0, \tag{60}$$

$$U_{x_1x_1}(t,x,\omega,\nu) = -\sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^2 \vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} D_{11\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{12\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{13\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{14\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t>0, \tag{61}$$

$$U_{x_1x_1}(t,x,\omega,\nu) = -\sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^2 \vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} D_{21\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{22\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{23\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{24\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t<0, \tag{62}$$

$$U_{x_2x_2}(t,x,\omega,\nu) = -\sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_2}{l}\right)^2 \vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} D_{11\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{12\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{13\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{14\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t>0, \tag{63}$$

$$U_{x_2x_2}(t,x,\omega,\nu) = -\sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_2}{l}\right)^2 \vartheta_{n_1,\ldots,n_m}(x)\times$$

$$\times\left[\varphi_{1\,n_1,\ldots,n_m} D_{21\,n_1,\ldots,n_m}(t,\omega,\nu) + \varphi_{2\,n_1,\ldots,n_m} D_{22\,n_1,\ldots,n_m}(t,\omega,\nu) + \right.$$

$$\left. +\psi_{1\,n_1,\ldots,n_m} D_{23\,n_1,\ldots,n_m}(t,\omega,\nu) + \psi_{2\,n_1,\ldots,n_m} D_{24\,n_1,\ldots,n_m}(t,\omega,\nu)\right], \quad t<0. \tag{64}$$

The expansions of the following functions into Fourier series are defined in the domain Ω_l^m in a similar way

$$U_{x_3x_3}(t,x,\omega,\nu), \ldots, U_{x_mx_m}(t,x,\omega,\nu), U_{ttx_1x_1}(t,x,\omega,\nu), U_{ttx_2x_2}(t,x,\omega,\nu), \ \ldots, U_{ttx_mx_m}(t,x,\omega,\nu).$$

The convergence of series (59) and (60) is proved similarly to the proof of the convergence of series (48) and (49). Let us show the convergence of series (61)–(64). Taking into account Formulas (50), (55)–(57) and applying the Cauchy-Schwarz inequality and Bessel inequality, we obtain

$$|U_{x_1x_1}(t,x,\omega,\nu)| \le \sum_{n_1,\ldots,n_m=1}^{\infty} \left(\frac{\pi n_1}{l}\right)^2 |u_{n_1,\ldots,n_m}(t,\omega,\nu)| \cdot |\vartheta_{n_1,\ldots,n_m}(x)| \le$$

$$\le \left(\sqrt{\frac{2}{l}}\right)^m \left(\frac{\pi}{l}\right)^2 C_{01} \sum_{n_1,\ldots,n_m=1}^{\infty} n_1^2 \left[|\varphi_{1\,n_1,\ldots,n_m}| + |\varphi_{2\,n_1,\ldots,n_m}| + |\psi_{1\,n_1,\ldots,n_m}| + |\psi_{2\,n_1,\ldots,n_m}|\right] \le$$

$$\le \gamma_2 \left[\sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1 n_2^3 \ldots n_m^3} \left|\varphi_{1n_1,\ldots,n_m}^{(3m)}\right| + \sum_{n_1,\ldots,n_m=1}^{\infty} \frac{1}{n_1 n_2^3 \ldots n_m^3} \left|\varphi_{2\,n_1,\ldots,n_m}^{(3m)}\right| + \right.$$

$$+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1 n_2^3\ldots n_m^3}\left|\psi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1 n_2^3\ldots n_m^3}\left|\psi^{(3m)}_{2n_1,\ldots,n_m}\right|\Bigg]\leq$$

$$\leq\left(\sqrt{\frac{2}{l}}\right)^m\gamma_2\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^2 n_2^6\ldots n_m^6}}\left[\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\right.$$

$$+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+$$

$$\left.+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right]<\infty,$$

where $\gamma_2=\left(\sqrt{\frac{2}{l}}\right)^m C_{01}\left(\frac{l}{\pi}\right)^{3m-2}$;

$$|U_{x_2x_2}(t,x,\omega,\nu)|\leq\sum_{n_1,\ldots,n_m=1}^{\infty}\left(\frac{\pi n_2}{l}\right)^2|u_{n_1,\ldots,n_m}(t,\omega,\nu)|\cdot|\vartheta_{n_1,\ldots,n_m}(x)|\leq$$

$$\leq\left(\sqrt{\frac{2}{l}}\right)^m\left(\frac{\pi}{l}\right)^2C_{01}\sum_{n_1,\ldots,n_m=1}^{\infty}n_2^2\left[|\varphi_{1n_1,\ldots,n_m}|+|\varphi_{2n_1,\ldots,n_m}|+|\psi_{1n_1,\ldots,n_m}|+|\psi_{2n_1,\ldots,n_m}|\right]\leq$$

$$\leq\gamma_2\Bigg[\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3 n_2 n_3^3\ldots n_m^3}\left|\varphi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3 n_2 n_3^3\ldots n_m^3}\left|\varphi^{(3m)}_{2n_1,\ldots,n_m}\right|+$$

$$+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3 n_2 n_3^3\ldots n_m^3}\left|\psi^{(3m)}_{1n_1,\ldots,n_m}\right|+\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^3 n_2 n_3^3\ldots n_m^3}\left|\psi^{(3m)}_{2n_1,\ldots,n_m}\right|\Bigg]\leq$$

$$\leq\left(\sqrt{\frac{2}{l}}\right)^m\gamma_2\sqrt{\sum_{n_1,\ldots,n_m=1}^{\infty}\frac{1}{n_1^6 n_2^2 n_3^6\ldots n_m^6}}\left[\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\right.$$

$$+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\varphi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_1(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}+$$

$$\left.+\sqrt{\int\limits_{\Omega_l^m}\left[\frac{\partial^{3m}\psi_2(x)}{\partial x_1^3\partial x_2^3\ldots\partial x_m^3}\right]^2dx}\right]<\infty,$$

The convergence of Fourier series for functions $U_{x_3x_3}(t,x,\omega,\nu),\ldots,$ $U_{x_mx_m}(t,x,\omega,\nu)$, $U_{ttx_1x_1}(t,x,\omega,\nu)$, $U_{ttx_2x_2}(t,x,\omega,\nu)$, $\ldots$, $U_{ttx_mx_m}(t,x,\omega,\nu)$ is proved in a similar way in the domain Ω_l^m.

Therefore, the functions $U(t,x,\omega,\nu)$, $g_1(x,\omega,\nu)$ and $g_2(x,\omega,\nu)$ defined by series (46)–(49) satisfy the conditions of the given problem.

To establish the uniqueness of the function $U(t,x,\omega,\nu)$ we show that, under the zero integral conditions $\int\limits_0^T U(t,x,\omega,\nu)\,dt=0$, $\int\limits_{-T}^0 U(t,x,\omega,\nu)\,dt=0$, $0\leq x\leq l$ the inverse boundary value problem (1)–(5) has only a trivial solution. We suppose that $\varphi_i(x)\equiv 0$, $\psi_i(x)\equiv 0$. Then $\varphi_{i\,n_1,\ldots,n_m}=0$, $\psi_{i\,n_1,\ldots,n_m}=0$ and from formulas (48) and (49) in the domain Ω_l^m implies that

$$\int\limits_{\Omega_l^m} U\,(t,\,x,\,\omega,\,\nu)\,\vartheta_{n_1,\ldots,n_m}(x)\,d\,x=0.$$

Hence, by virtue of completeness of systems of the eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi\,n_1}{l}\,x_1\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi\,n_2}{l}\,x_2\right\}$, $\ldots$, $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi\,n_m}{l}\,x_m\right\}$ in the space $L_2\left(\Omega_l^m\right)$ we deduce that $U\,(t,\,x,\,\omega,\,\nu)\equiv 0$ for all $x\in\Omega_l^m$ and $t\in[-T;\,T]$.

Therefore, under conditions (25) and (33), the inverse problem has a unique pair of solutions in the domain Ω_l^m.

6. Calculation of Values of Spectral Parameters

Let condition (25) be violated, that is, we suppose that

$$\sigma_{n_1,\ldots,n_m}=\sinh\lambda_{n_1,\ldots,n_m}T\,\left(\cos\lambda_{n_1,\ldots,n_m}\omega\,T-1\right)-$$
$$-\,\omega\,\sin\lambda_{n_1,\ldots,n_m}\omega\,T\,\left(\cosh\lambda_{n_1,\ldots,n_m}T-1\right)=0 \tag{65}$$

for some values of ω, where $\lambda^2_{n_1,\ldots,n_m}=\frac{\mu^2_{n_1,\ldots,n_m}}{1+\mu^2_{n_1,\ldots,n_m}}$, $\mu_{n_1,\ldots,n_m}=\frac{\pi}{l}\sqrt{n_1^2+\ldots+n_m^2}$.

From equality (65) with respect to the spectral parameter ω we arrive at the quadratic trigonometric equation

$$\left(a_{n_1,\ldots,n_m}+1\right)\,\tan^2\frac{y_{n_1,\ldots,n_m}}{2}+2\,b_{n_1,\ldots,n_m}\,\omega\,\tan\frac{y_{n_1,\ldots,n_m}}{2}+\left(a_{n_1,\ldots,n_m}-1\right)=0,$$

where

$$y_{n_1,\ldots,n_m}=\lambda_{n_1,\ldots,n_m}\omega\,T,\;a_{n_1,\ldots,n_m}=\sinh\lambda_{n_1,\ldots,n_m}T,$$
$$b_{n_1,\ldots,n_m}=\coth\lambda_{n_1,\ldots,n_m}T-\sinh^{-1}\lambda_{n_1,\ldots,n_m}T.$$

The set of positive solutions of this equation with respect to the spectral parameter ω for some $k_1,\,\ldots,\,k_m$ is denoted by $\Im_1$. We call the numbers $\omega\in\Im_1$ as irregular, since because the condition (25) is violated for them. The set $\Lambda_1=(0;\,\infty)\setminus\Im_1$ is called the set of regular values of the spectral parameter ω, for which condition (25) is fulfilled. If condition (34) is violated, then the kernels of the mixed integro-differential Equation (1) have at most two values of ν_1 and ν_2. We call these real nonzero numbers as an irregular kernel numbers of the mixed integro-differential Equation (1) and denote their set $\{\nu_1,\,\nu_2\}$ by $\Im_2$. We take away the values ν_1 and ν_2 of the spectral parameter ν from the set of nonzero real numbers $(-\infty;\,0)\cup(0;\,\infty)$. The resulting set $\Lambda_3=(-\infty;\,0)\cup(0;\,\infty)\setminus\{\nu_1,\,\nu_2\}$ is called the set of regular values of the parameter ν. For all values of $\nu\in\Lambda_2$ condition (33) is satisfied.

We use the following notations for sets

$$\aleph_1=\{(\omega,\,\nu)\;|\,\omega\in\Lambda_1;\,\nu\in\Lambda_3\}\,;\,\aleph_2=\{\,(\omega,\,\nu)\;|\,\omega\in\Im_1;\,\nu\in(-\infty;\,0)\cup\,(0;\,\infty)\,\}\,,$$

$$\aleph_3=\{\,(\omega,\,\nu)\;|\,\omega\in\Lambda_1;\,\nu\in\Im_2\,\}\,.$$

For $(\omega,\,\nu)\in\aleph_1$ formulas (46)–(49) hold. This is the case when all values of the spectral parameters ω and ν are regular. Therefore, in this case, the unique solution of the inverse boundary value problem (1)–(5) in the domain Ω_l^m is represented in the form of series (46)–(49).

7. Expansion of the Solution of the Direct Problem (1)–(4) in a Fourier Series. Irregular Case of a Spectral Parameter ω

For some $k_1, \ldots, k_m$ and $(\nu, \omega) \in \aleph_2$, where $\aleph_2 = \{(\omega, \nu) \mid \omega \in \Im_1;\ \nu \in (-\infty; 0) \cup (0; \infty)\}$ we first find a formal solution of the direct problem (1)–(4). In this case, instead of (28) and (29), we have the representations

$$u_{k_1,\ldots,k_m}(t,\omega) = C_{1k_1,\ldots,k_m} \cosh \lambda_{k_1,\ldots,k_m} t + \omega C_{2k_1,\ldots,k_m} \sinh \lambda_{k_1,\ldots,k_m} t +$$

$$+ \nu\, \alpha_{k_1,\ldots,k_m} h_{1k_1,\ldots,k_m}(t) + g_{1k_1,\ldots,k_m} h_{2k_1,\ldots,k_m}(t), \quad t > 0, \tag{66}$$

$$u_{k_1,\ldots,k_m}(t,\omega) = C_{1k_1,\ldots,k_m} \cos \lambda_{k_1,\ldots,k_m} \omega\, t + C_{2k_1,\ldots,k_m} \sin \lambda_{k_1,\ldots,k_m} \omega\, t +$$

$$+ \nu\, \beta_{k_1,\ldots,k_m} \delta_{1k_1,\ldots,k_m}(t) + g_{2k_1,\ldots,k_m} \delta_{2k_1,\ldots,k_m}(t), \quad t < 0, \tag{67}$$

where $C_{i\,k_1,\ldots,k_m}$ $(i = 1, 2)$ are arbitrary constants.

Substituting (66) into (11) and (67) into (12), we obtain

$$\tau_{i\,k_1,\ldots,k_m} \alpha_{k_1,\ldots,k_m} = C_{1k_1,\ldots,k_m} \chi_{i\,1k_1,\ldots,k_m} + C_{2k_1,\ldots,k_m} \chi_{i\,2k_1,\ldots,k_m} + g_{i\,k_1,\ldots,k_m} \chi_{i\,4k_1,\ldots,k_m}, \tag{68}$$

where

$$\tau_{i\,k_1,\ldots,k_m} = 1 - \nu\, \chi_{i3k_1,\ldots,k_m} \neq 0, \quad i = 1, 2, \tag{69}$$

$$\chi_{11k_1,\ldots,k_m} = \int_0^T b_1(s) \cosh \lambda_{k_1,\ldots,k_m} s\, ds, \ \chi_{12k_1,\ldots,k_m} = \omega \int_0^T b_1(s) \sinh \lambda_{k_1,\ldots,k_m} s\, ds,$$

$$\chi_{13k_1,\ldots,k_m} = \int_0^T b_1(s)\, h_{1k_1,\ldots,k_m}(s)\, ds, \ \chi_{14k_1,\ldots,k_m} = \int_0^T b_1(s)\, h_{2k_1,\ldots,k_m}(s)\, ds,$$

$$\chi_{21k_1,\ldots,k_m} = \int_{-T}^0 b_2(s) \cos \lambda_{k_1,\ldots,k_m} \omega\, s\, ds, \ \chi_{22k_1,\ldots,k_m} = \int_{-T}^0 b_2(s) \sin \lambda_{k_1,\ldots,k_m} \omega\, s\, ds,$$

$$\chi_{23k_1,\ldots,k_m} = \int_{-T}^0 b_2(s)\, \delta_{1k_1,\ldots,k_m}(s)\, ds, \ \chi_{24k_1,\ldots,k_m} = \int_{-T}^0 b_2(s)\, \delta_{2k_1,\ldots,k_m}(s)\, ds.$$

We show that condition (69) is always fulfilled, that is,

$$1 - \nu\, \chi_{13k_1,\ldots,k_m} \neq 0, \ 1 - \nu\, \chi_{23k_1,\ldots,k_m} \neq 0.$$

First, we suppose that simultaneously take place

$$1 - \nu\, \chi_{13k_1,\ldots,k_m} = 0, \ 1 - \nu\, \chi_{23k_1,\ldots,k_m} = 0. \tag{70}$$

Then we come to the conclusion that $\chi_{13k_1,\ldots,k_m} = \nu^{-1}$, $\chi_{23k_1,\ldots,k_m} = \nu^{-1}$, that is, $\chi_{13k_1,\ldots,k_m} = \chi_{23k_1,\ldots,k_m}$. It cannot be, since because $\chi_{13k_1,\ldots,k_m}$ and $\chi_{13k_1,\ldots,k_m}$ are different quantities. Therefore, (70) does not hold.

Now suppose that

$$1 - \nu\, \chi_{13k_1,\ldots,k_m} = 0, \ 1 - \nu\, \chi_{23k_1,\ldots,k_m} \neq 0. \tag{71}$$

Then we have consider the quadratic equation

$$(1 - \nu\, \chi_{13k_1,\ldots,k_m})(1 - \nu\, \chi_{23k_1,\ldots,k_m}) =$$

$$= \nu^2 \chi_{13k_1,\ldots,k_m} \chi_{23k_1,\ldots,k_m} - \nu\, (\chi_{13k_1,\ldots,k_m} + \chi_{23k_1,\ldots,k_m}) + 1 = 0.$$

Solving this equation we derive the roots: $\nu_1 = \frac{1}{\chi_{13\,k_1,\ldots,k_m}}$, $\nu_2 = \frac{1}{\chi_{23\,k_1,\ldots,k_m}}$. But, by our assumption: $1 - \nu\,\chi_{23\,k_1,\ldots,k_m} \neq 0$. We came to a contradiction. Therefore, (71) does not hold. Similarly, it can be shown that there is no

$$1 - \nu\,\chi_{13\,k_1,\ldots,k_m} \neq 0, \quad 1 - \nu\,\chi_{23\,k_1,\ldots,k_m} = 0.$$

Therefore, the condition (69) is always fulfilled. Then from (68) we find that

$$\alpha_{k_1,\ldots,k_m} = C_{1\,k_1,\ldots,k_m}\bar{\chi}_{1\,1\,k_1,\ldots,k_m} + C_{2\,k_1,\ldots,k_m}\bar{\chi}_{1\,2\,k_1,\ldots,k_m} + g_{1\,k_1,\ldots,k_m}\bar{\chi}_{1\,3\,k_1,\ldots,k_m}, \tag{72}$$

$$\beta_{k_1,\ldots,k_m} = C_{1\,k_1,\ldots,k_m}\bar{\chi}_{2\,1\,k_1,\ldots,k_m} + C_{2\,k_1,\ldots,k_m}\bar{\chi}_{2\,2\,k_1,\ldots,k_m} + g_{2\,k_1,\ldots,k_m}\bar{\chi}_{2\,3\,k_1,\ldots,k_m}, \tag{73}$$

where

$$\bar{\chi}_{i\,j\,k_1,\ldots,k_m} = \frac{\chi_{i\,j\,k_1,\ldots,k_m}}{\tau_{i\,k_1,\ldots,k_m}},\ \bar{\chi}_{i\,3\,k_1,\ldots,k_m} = \frac{\chi_{i\,4\,k_1,\ldots,k_m}}{\tau_{i\,k_1,\ldots,k_m}},\quad i = 1, 2,\ j = 1, 2.$$

Substitution of values (72) into (66) and (73) into (67) gives us the following representations

$$u_{k_1,\ldots,k_m}(t, \omega, \nu) = C_{1\,k_1,\ldots,k_m}\gamma_{11\,k_1,\ldots,k_m}(t, \omega, \nu) + C_{2\,k_1,\ldots,k_m}\gamma_{12\,k_1,\ldots,k_m}(t, \omega, \nu) +$$
$$+ g_{1\,k_1,\ldots,k_m}(\omega, \nu)\,\gamma_{13\,k_1,\ldots,k_m}(t, \omega, \nu),\quad t > 0, \tag{74}$$

$$u_{k_1,\ldots,k_m}(t, \omega, \nu) = C_{1\,k_1,\ldots,k_m}\gamma_{21\,k_1,\ldots,k_m}(t, \omega, \nu) + C_{2\,k_1,\ldots,k_m}\gamma_{22\,k_1,\ldots,k_m}(t, \omega, \nu) +$$
$$+ g_{2\,k_1,\ldots,k_m}(\omega, \nu)\,\gamma_{23\,k_1,\ldots,k_m}(t, \omega, \nu),\quad t < 0, \tag{75}$$

where

$$\gamma_{11\,k_1,\ldots,k_m}(t, \omega, \nu) = \cosh\lambda_{k_1,\ldots,k_m}t + \nu\,h_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{11\,k_1,\ldots,k_m},$$
$$\gamma_{12\,k_1,\ldots,k_m}(t, \omega, \nu) = \omega\,\sinh\lambda_{k_1,\ldots,k_m}t + \nu\,h_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{12\,k_1,\ldots,k_m},$$
$$\gamma_{13\,k_1,\ldots,k_m}(t, \omega, \nu) = h_{2\,k_1,\ldots,k_m}(t) + \nu\,h_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{13\,k_1,\ldots,k_m},$$
$$\gamma_{21\,k_1,\ldots,k_m}(t, \omega, \nu) = \cos\lambda_{k_1,\ldots,k_m}\omega\,t + \nu\,\delta_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{21\,k_1,\ldots,k_m},$$
$$\gamma_{21\,k_1,\ldots,k_m}(t, \omega, \nu) = \sin\lambda_{k_1,\ldots,k_m}\omega\,t + \nu\,\delta_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{22\,k_1,\ldots,k_m},$$
$$\gamma_{23\,k_1,\ldots,k_m}(t, \omega, \nu) = \delta_{2\,k_1,\ldots,k_m}(t) + \nu\,\delta_{1\,k_1,\ldots,k_m}(t)\,\bar{\chi}_{23\,k_1,\ldots,k_m}.$$

Then from (74) and (75) yields that the solution of the direct problem (1)–(4) in the domain Ω_l^m for $(\nu, \omega) \in \aleph_2$ can be represented as the following Fourier series

$$U(t, x, \omega, \nu) = \sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\big[C_{1\,k_1,\ldots,k_m}\gamma_{11\,k_1,\ldots,k_m}(t, \omega, \nu) +$$
$$+ C_{2\,k_1,\ldots,k_m}\gamma_{12\,k_1,\ldots,k_m}(t, \omega, \nu) + g_{1\,k_1,\ldots,k_m}(\omega, \nu)\,\gamma_{13\,k_1,\ldots,k_m}(t, \omega, \nu)\big],\quad t > 0, \tag{76}$$

$$U(t, x, \omega, \nu) = \sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\big[C_{1\,k_1,\ldots,k_m}\gamma_{21\,k_1,\ldots,k_m}(t, \omega, \nu) +$$
$$+ C_{2\,k_1,\ldots,k_m}\gamma_{22\,k_1,\ldots,k_m}(t, \omega, \nu) + g_{2\,k_1,\ldots,k_m}(\omega, \nu)\,\gamma_{23\,k_1,\ldots,k_m}(t, \omega, \nu)\big],\quad t < 0, \tag{77}$$

where $C_{i\,k_1,\ldots,k_m}$ $(i = 1, 2)$ are arbitrary constants.

8. Inverse Problem (1)–(5). Irregular Case of a Spectral Parameter ω

We apply the additional conditions (5) and from the Fourier series (76) and (77) we obtain that

$$\psi_i(x, \omega, \nu) = U(t_i, x, \omega, \nu) = \sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\big[C_{1\,k_1,\ldots,k_m}\gamma_{i1\,k_1,\ldots,k_m}(t_i, \omega, \nu) +$$

$$+C_{2k_1,\ldots,k_m}\gamma_{i2k_1,\ldots,k_m}(t_i,\omega,\nu)+g_{ik_1,\ldots,k_m}(\omega,\nu)\,\gamma_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)\Big],\ i=1,2. \tag{78}$$

Taking the expansions (43) and $\gamma_{i3k_1,\ldots,k_m}(t,\omega,\nu)\neq 0,\ i=1,2$ into account from the relations (78) we derive

$$g_{ik_1,\ldots,k_m}(\omega,\nu)=\psi_{1k_1,\ldots,k_m}(\omega,\nu)\cdot\bar{\gamma}_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)-$$

$$-C_{1k_1,\ldots,k_m}\cdot\bar{\gamma}_{i1k_1,\ldots,k_m}(t_i,\omega,\nu)-C_{2k_1,\ldots,k_m}\cdot\bar{\gamma}_{i2k_1,\ldots,k_m}(t_i,\omega,\nu),\ i=1,2, \tag{79}$$

where

$$\bar{\gamma}_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)=\left(\gamma_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)\right)^{-1},\ i=1,2,$$

$$\bar{\gamma}_{ijk_1,\ldots,k_m}(t_i,\omega,\nu)=\frac{\gamma_{ijk_1,\ldots,k_m}(t_i,\omega,\nu)}{\gamma_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)},\ i,j=1,2.$$

Substituting representations (79) into the Fourier series (8), we obtain

$$g_i(x,\omega,\nu)=\sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\Big[\psi_{1k_1,\ldots,k_m}(\omega,\nu)\cdot\bar{\gamma}_{i3k_1,\ldots,k_m}(t_i,\omega,\nu)-$$

$$-C_{1k_1,\ldots,k_m}\cdot\bar{\gamma}_{i1k_1,\ldots,k_m}(t_i,\omega,\nu)-C_{2k_1,\ldots,k_m}\cdot\bar{\gamma}_{i2k_1,\ldots,k_m}(t_i,\omega,\nu)\Big],\ i=1,2. \tag{80}$$

Substitution of the representations (79) into the series (76) and (77) gives

$$U(t,x,\nu)=\sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\Big[\psi_{1k_1,\ldots,k_m}Z_{11k_1,\ldots,k_m}(t,\omega,\nu)+$$

$$+C_{1k_1,\ldots,k_m}Z_{12k_1,\ldots,k_m}(t,\omega,\nu)+C_{2k_1,\ldots,k_m}Z_{13k_1,\ldots,k_m}(t,\omega,\nu)\Big],\ t>0, \tag{81}$$

$$U(t,x,\nu)=\sum_{k_1,\ldots,k_m=1}^{\infty}\vartheta_{k_1,\ldots,k_m}(x)\,\Big[\psi_{2k_1,\ldots,k_m}Z_{21k_1,\ldots,k_m}(t,\omega,\nu)+$$

$$+C_{1k_1,\ldots,k_m}Z_{22k_1,\ldots,k_m}(t,\omega,\nu)+C_{2k_1,\ldots,k_m}Z_{23k_1,\ldots,k_m}(t,\omega,\nu)\Big],\ t<0, \tag{82}$$

where

$$Z_{i1k_1,\ldots,k_m}(t,\omega,\nu)=\gamma_{i3k_1,\ldots,k_m}(t,\omega,\nu)\cdot\bar{\gamma}_{i3k_1,\ldots,k_m}(t_i,\omega,\nu),$$

$$Z_{i2k_1,\ldots,k_m}(t,\omega,\nu)=\gamma_{i1k_1,\ldots,k_m}(t,\omega,\nu)-\gamma_{i3k_1,\ldots,k_m}(t,\omega,\nu)\cdot\bar{\gamma}_{i1k_1,\ldots,k_m}(t_i,\omega,\nu),$$

$$Z_{i3k_1,\ldots,k_m}(t,\omega,\nu)=\gamma_{i2k_1,\ldots,k_m}(t,\omega,\nu)-\gamma_{i3k_1,\ldots,k_m}(t,\omega,\nu)\cdot\bar{\gamma}_{i2k_1,\ldots,k_m}(t_i,\omega,\nu),\ i=1,2.$$

By virtue of the fact that $Z_{ijk_1,\ldots,k_m}(t,\omega,\nu)$ $(i=1,2;\ j=1,2,3)$ are uniformly bounded functions and the conditions **A** are satisfied for the functions $\psi_i(x)$, the arbitrary constants $C_{ik_1,\ldots,k_m}$ can be chosen such that the series (80)–(82) converge absolutely and uniformly. The proof of this statement is carried out in exactly the same way as in the case of regular values of spectral parameters.

9. Statement of the Theorem. Conclusions

The questions of solvability of a nonlocal inverse boundary value problem for a mixed pseudohyperbolic-pseudoelliptic integro-differential Equation (1) with spectral parameters ω and ν are considered. Using the method of the Fourier series in the form (6), a system of countable systems of ordinary integro-differential Equations (9) and (10) is obtained. To determine arbitrary integration constants, a system of algebraic equations is obtained. From this system, regular and irregular values of the spectral parameter ω were calculated (condition (25)). From the condition (34) we calculate regular and irregular values of the spectral parameter ν. The following theorem is proved.

Theorem 1. *Let conditions* **A** *be fulfilled. Then for values* $(\nu,\,\omega)\in\aleph_1$ *the inverse problem* (1)–(5) *is uniquely solvable in the domain* Ω_l^m *and this solution is represented in the form of series* (46)–(49). *And for values* $(\nu,\,\omega)\in\aleph_2$ *the inverse problem* (1)–(5) *in the domain* Ω_l^m *has an infinite number of solutions. These solution is represented in the form of series* (80)–(82). *Moreover, a necessary conditions for the existence of solutions of the problem are:* $\varphi_1(x)\equiv 0,\ \varphi_2(x)\equiv 0$.

In the case of all possible values $(\nu\,,\omega)\in\aleph_3$, where $\aleph_3=\{(\omega,\nu)\,|\,\omega\in\Lambda_1;\,\nu\in\Im_2\}$, the questions of solvability of the inverse problem (1)–(5) are studied in a similar way.

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