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On the Product Rule for the Hyperbolic Scator Algebra

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Abstract: Scator set, introduced by Fernández-Guasti and Zaldívar, is endowed with a very peculiar non-distributive product. In this paper we consider the scator space of dimension $1+2$ and the so called fundamental embedding which maps the subset of scators with non-zero scalar component into 4-dimensional space endowed with a natural distributive product. The original definition of the scator product is induced in a straightforward way. Moreover, we propose an extension of the scator product on the whole scator space, including all scators with vanishing scalar component.

Keywords: scators; non-distributive algebras; Lorentz velocity addition formula; fundamental embedding

MSC: 30G35; 20M14

1. Introduction

The scator algebra was introduced by Fernández-Guasti and Zaldívar in a series of papers, starting from [1,2]. In this paper we confine ourselves to scators in the hyperbolic space of dimension $1+2$. To be more precise, we consider a real linear space $\mathbb{R}^{1+2}$ with a fixed basis $\overset{o}{\boldsymbol{e}}_0$, $\overset{o}{\boldsymbol{e}}_1$ and $\overset{o}{\boldsymbol{e}}_2$. An element $\overset{o}{a} \in \mathbb{R}^{1+2}$ will be denoted as

$$\overset{o}{a} = (a_0;\ a_1,\ a_2) = a_0\overset{o}{\boldsymbol{e}}_0 + a_1\overset{o}{\boldsymbol{e}}_1 + a_2\overset{o}{\boldsymbol{e}}_2 = a_0 + a_1\overset{o}{\boldsymbol{e}}_1 + a_2\overset{o}{\boldsymbol{e}}_2 \tag{1}$$

where a_0, a_1 and a_2 are real numbers. The coefficient a_0 is referred to as a scalar component and a_1, a_2 are called director components. The unit scalar $\overset{o}{\boldsymbol{e}}_0$ is usually omitted, see the last equality of Equation (1). The decomposition into scalar and director components is crucial for many properties of scators, including their multiplication. In particular, in the hyperbolic case we have $(\overset{o}{\boldsymbol{e}}_1)^2 = (\overset{o}{\boldsymbol{e}}_2)^2 = 1$.

The elliptic case (when squares of director basis vectors are negative) can be considered as a new realization of hypercomplex numbers [2,3], while the hyperbolic case has potential physical applications, usually related to deformations and generalizations of the special theory of relativity [4,5], see also [6–8].

Originally, the scator product has been defined as a map $\mathbb{S}^{1+2} \times \mathbb{S}^{1+2}$ into $\mathbb{S}^{1+2}$ (for details see Definition 1), where

$$\mathbb{S}^{1+2} = \mathbb{R}^{1+2} \setminus \{\overset{o}{a} :\ a_0 = 0 \ \text{and}\ \ a_1 a_2 \neq 0\}\,, \tag{2}$$

Usually, only the elements of $\mathbb{S}^{1+2}$ has been called scators, compare [1,2,9]. In this paper we prefer to use this name in a little bit more extended sense (scators as elements of $\mathbb{R}^{1+2}$) because one of our goals is to extend the definition of the scator product on the whole space $\mathbb{R}^{1+2}$, see Section 4.

Definition 1. *Scator product of two scators, $\overset{o}{a} = (a_0; a_1, a_2) \in \mathbb{S}^{1+2}$ and $\overset{o}{b} = (b_0; b_1, b_2) \in \mathbb{S}^{1+2}$, is denoted by $\overset{o}{a}\overset{o}{b} \equiv (u_0;\ u_1,\ u_2)$. In the hyperbolic case it is defined as follows.*

- *For $a_0 \neq 0$ and $b_0 \neq 0$,*

$$\begin{aligned} u_0 &= a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0} \equiv a_0b_0\left(1+\frac{a_1b_1}{a_0b_0}\right)\left(1+\frac{a_2b_2}{a_0b_0}\right), \\ u_1 &= a_0b_1 + a_1b_0 + \frac{a_1a_2b_2}{a_0} + \frac{a_2b_1b_2}{b_0} \equiv a_0b_0\left(\frac{a_1}{a_0}+\frac{b_1}{b_0}\right)\left(1+\frac{a_2b_2}{a_0b_0}\right), \\ u_2 &= a_0b_2 + a_2b_0 + \frac{a_1a_2b_1}{a_0} + \frac{a_1b_1b_2}{b_0} \equiv a_0b_0\left(1+\frac{a_1b_1}{a_0b_0}\right)\left(\frac{a_2}{a_0}+\frac{b_2}{b_0}\right). \end{aligned} \tag{3}$$

- *For $a_0 = a_2 = 0$, $a_1 = 1$ and $b_0 \neq 0$,*

$$\overset{o}{\boldsymbol{e}}_1\overset{o}{\beta} = \overset{o}{\beta}\overset{o}{\boldsymbol{e}}_1 = \left(b_1;\ b_0,\ \frac{b_1b_2}{b_0}\right). \tag{4}$$

- *For $a_0 = a_1 = 0$, $a_2 = 1$ and $b_0 \neq 0$,*

$$\overset{o}{\boldsymbol{e}}_2\overset{o}{\beta} = \overset{o}{\beta}\overset{o}{\boldsymbol{e}}_2 = \left(b_2;\ \frac{b_1b_2}{b_0},\ b_1\right). \tag{5}$$

In particular, the hyperbolic basis satisfies

$$\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_1 = \overset{o}{\boldsymbol{e}}_2\overset{o}{\boldsymbol{e}}_2 = 1, \qquad \overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_2 = \overset{o}{\boldsymbol{e}}_2\overset{o}{\boldsymbol{e}}_1 = 0, \tag{6}$$

anticipating commutativity and non-associativity of the scator set under multiplication. In addition, the following property holds: $(\lambda\overset{o}{a})\overset{o}{b} = \overset{o}{a}(\lambda\overset{o}{b}) = \lambda(\overset{o}{a}\overset{o}{b})$. Generalization of Definition 1 to higher dimensions is quite natural, see [3,10].

Remark 2. *Using the notation*

$$\begin{aligned} (a_0;\ a_1,\ a_2) &= a_0(1;\ \beta_{a1},\ \beta_{a2}), \\ (b_0;\ b_1,\ b_2) &= b_0(1;\ \beta_{b1},\ \beta_{b2}), \end{aligned} \tag{7}$$

(compare [11]), we rewrite the generic case (i.e., Equation (3)) as:

$$\begin{aligned} u_0 &= a_0b_0(1+\beta_{a1}\beta_{b1})(1+\beta_{a2}\beta_{b2}), \\ u_1 &= a_0b_0(\beta_{a1}+\beta_{b1})(1+\beta_{a2}\beta_{b2}), \\ u_2 &= a_0b_0(1+\beta_{a1}\beta_{b1})(\beta_{a2}+\beta_{b2}), \end{aligned} \tag{8}$$

In other words, for $a_0 \neq 0$ and $b_0 \neq 0$ we have

$$\overset{o}{a}\overset{o}{b} = a_0b_0\,(1+\beta_{a1}\beta_{b1})\,(1+\beta_{a2}\beta_{b2})\left(1;\ \frac{\beta_{a1}+\beta_{b1}}{1+\beta_{a1}\beta_{b1}},\ \frac{\beta_{a2}+\beta_{b2}}{1+\beta_{a2}\beta_{b2}}\right). \tag{9}$$

The components of Equation (9) remind the Lorentz rule for relativistic sum of velocities (in the one-dimensional case), which motivates some physical applications [4,5]. In this paper we present a natural motivation for the complicated formulas of Definition 1. We also we propose a novel extension of Definition 1 on the case $a_0 = 0$ (for any values of a_1 and a_2).

2. Commutativity, Non-Distributivity and Generic Associativity of the Scator Product

The scator product, as defined above, is manifestly commutative, for any pair of scators. In general it is non-distributive, which is clearly underlined in the first papers on the scator algebra [1,2,10]. In order to explicitly show no-distributivity one can compute [11]:

$$\Delta(\overset{o}{a},\overset{o}{b};\overset{o}{c}) := (\overset{o}{a}+\overset{o}{b})\overset{o}{c} - \overset{o}{a}\overset{o}{c} - \overset{o}{b}\overset{o}{c} = \frac{(b_0a_1 - a_0b_1)(a_0b_2 - b_0a_2)}{a_0b_0(a_0+b_0)}(\frac{c_1c_2}{c_0}, c_2, c_1), \tag{10}$$

where we denote, as usual, $\overset{o}{c} = (c_0;\ c_1,\ c_2)$. We see, that except some special cases ($a_0b_1 = a_1b_0$ or $a_0b_2 = a_2b_0$) the scator product is not distributive.

Computing the product of three scators with non-vanishing scalar components, under assumption that the scalar components of $\overset{o}{a}\overset{o}{b}$, $\overset{o}{b}\overset{o}{c}$ and $\overset{o}{a}\overset{o}{c}$ do not vanish, we obtain

$$(\overset{o}{a}\overset{o}{b})\overset{o}{c} = \overset{o}{a}(\overset{o}{b}\overset{o}{c}) = \overset{o}{b}(\overset{o}{c}\overset{o}{a}) = (w_0;\ w_1,\ w_2) \equiv w_0(1;\ \beta_{w1},\ \beta_{w2})\ , \tag{11}$$

where

$$\begin{aligned} w_0 &= a_0b_0c_0(1+\beta_{a1}\beta_{b1}+\beta_{a1}\beta_{c1}+\beta_{b1}\beta_{c1})(1+\beta_{a2}\beta_{b2}+\beta_{a2}\beta_{c2}+\beta_{b2}\beta_{c2})\ , \\ w_1 &= a_0b_0c_0(\beta_{a1}+\beta_{b1}+\beta_{c1}+\beta_{a1}\beta_{b1}\beta_{c1})(1+\beta_{a2}\beta_{b2}+\beta_{a2}\beta_{c2}+\beta_{b2}\beta_{c2})\ , \\ w_2 &= a_0b_0c_0(1+\beta_{a1}\beta_{b1}+\beta_{a1}\beta_{c1}+\beta_{b1}\beta_{c1})(\beta_{a2}+\beta_{b2}+\beta_{c2}+\beta_{a2}\beta_{b2}\beta_{c2})\ . \end{aligned} \tag{12}$$

Therefore, in the generic case the scator product is associative but in special cases the associativity is broken. A general discussion of the associativity of the scator product can be found in [9]. The simplest example of the non-associativity is given by triple products of basis vectors.

$$(\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_1)\overset{o}{\boldsymbol{e}}_2 = \overset{o}{\boldsymbol{e}}_2\ , \qquad \overset{o}{\boldsymbol{e}}_1(\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_2) = 0\ , \qquad \text{hence} \qquad (\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_1)\overset{o}{\boldsymbol{e}}_2 \neq \overset{o}{\boldsymbol{e}}_1(\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_2)\ . \tag{13}$$

Note that both $\overset{o}{\boldsymbol{e}}_1$ and $\overset{o}{\boldsymbol{e}}_2$ are zero divisors.

3. Fundamental Embedding as a Natural Interpretation of the Scator Product

In Ref. [11] we introduced the so called fundamental embedding $F : \mathbb{S}^{1+2}_* \to \mathbb{A}_2$, where $\mathbb{S}^{1+2}_*$ is subset of the scator space containing scators with non-zero scalar component and the space $\mathbb{A}_2$ is the algebra over $\mathbb{R}$ generated by elements $\boldsymbol{e}_1, \boldsymbol{e}_2$ satisfying

$$\boldsymbol{e}_1\boldsymbol{e}_1 = \boldsymbol{e}_2\boldsymbol{e}_2 = 1\ , \qquad \boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_{12}\ . \tag{14}$$

Therefore, $\mathbb{A}_2$ is spanned by $1, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_{12}$. We assume that the product in the space $\mathbb{A}_2$ is commutative, associative and distributive over addition.

In this paper we propose a slightly more general definition of the fundamental embedding, extending the former definition on the whole set $\mathbb{S}^{1+2}$, defined by Equation (2).

Definition 3. *The fundamental embedding $F : \mathbb{S}^{1+2} \to \mathbb{A}_2$ is defined as:*

$$\begin{aligned} &F(\overset{o}{a}) = a_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + \frac{a_1a_2}{a_0}\,\boldsymbol{e}_{12} \qquad (a_0 \neq 0)\ , \\ &F(a_1\overset{o}{\boldsymbol{e}}_1) = a_1\boldsymbol{e}_1\ , \\ &F(a_2\overset{o}{\boldsymbol{e}}_2) = a_2\boldsymbol{e}_2\ . \end{aligned} \tag{15}$$

Remark 4. *The planes $(a_0;\ 0,\ a_2)$ and $(a_0;\ a_1,\ 0)$ are invariants of the fundamental embedding:*

$$\begin{aligned} F(a_0 + a_1 \overset{o}{\boldsymbol{e}}_1) &= a_0 + a_1 \boldsymbol{e}_1 \ , \\ F(a_0 + a_2 \overset{o}{\boldsymbol{e}}_2) &= a_0 + a_2 \boldsymbol{e}_2 \ . \end{aligned} \tag{16}$$

For $a_0 \neq 0$ this is a special case of the first formula of Equation (15).

The space $\mathbb{A}_2$ may be understood as a commutative analogue of the geometric algebra or the Clifford algebra [12], where vectors commute only when they are parallel (and orthogonal vectors anti-commute). In this context we may interpret $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as vectors and $\boldsymbol{e}_{12}$ as a bivector or pseudoscalar. The set $F(\mathbb{S}_*^{1+2})$ has a natural group structure reminding a commutative analogue of the Clifford (or Lipschitz) group [13]. Indeed, the first equation of Equation (15) can be rewritten as

$$F(\overset{o}{a}) = a_0 \left(1 + \frac{a_1}{a_0}\boldsymbol{e}_1\right) \left(1 + \frac{a_2}{a_0}\boldsymbol{e}_2\right) \tag{17}$$

Operations in the space $\mathbb{A}_2$ give a natural interpretation and motivation for the definition in Equation (3) of the scator product. The following theorem holds.

Theorem 5. *If $a_0 \neq 0$, $b_0 \neq 0$, $a_1 b_1 + a_0 b_0 \neq 0$ and $a_2 b_2 + a_0 b_0 \neq 0$, then there exists $\overset{o}{c} \in \mathbb{S}_*^{1+2}$ such that*

$$F(\overset{o}{a})F(\overset{o}{b}) = F(\overset{o}{c}). \tag{18}$$

What is more, $\overset{o}{c}$ coincides with $\overset{o}{a}\overset{o}{b}$ given by Definition 1, i.e.,

$$F(\overset{o}{a}\overset{o}{b}) = F(\overset{o}{a})F(\overset{o}{b}) \ , \qquad \overset{o}{a}\overset{o}{b} = F^{-1}(F(\overset{o}{a})F(\overset{o}{b})) \ . \tag{19}$$

Proof. First, let us observe that

$$\left(1 + \frac{a_1}{a_0}\boldsymbol{e}_1\right)\left(1 + \frac{b_1}{b_0}\boldsymbol{e}_1\right) = 1 + \frac{a_1 b_1}{a_0 b_0} + \left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)\boldsymbol{e}_1 = (1 + \beta_{a1}\beta_{b1}) + (\beta_{a1} + \beta_{b1})\,\boldsymbol{e}_1 \ , \tag{20}$$

where we used the notation in Equation (7). Therefore, the product

$$F(\overset{o}{a})F(\overset{o}{b}) = a_0 b_0 \left(1 + \frac{a_1}{a_0}\boldsymbol{e}_1\right)\left(1 + \frac{b_1}{b_0}\boldsymbol{e}_1\right)\left(1 + \frac{a_2}{a_0}\boldsymbol{e}_2\right)\left(1 + \frac{b_2}{b_0}\boldsymbol{e}_2\right) \tag{21}$$

can be easily represented in the form of Equation (17). Indeed,

$$F(\overset{o}{a})F(\overset{o}{b}) = a_0 b_0 \left(1 + \beta_{a1}\beta_{b1} + (\beta_{a1} + \beta_{b1})\,\boldsymbol{e}_1\right)\left(1 + \beta_{a2}\beta_{b2} + (\beta_{a2} + \beta_{b2})\,\boldsymbol{e}_2\right) \tag{22}$$

Denoting $\overset{o}{c} := (c_0;\ c_1,\ c_2)$, where

$$\begin{aligned} c_0 &= a_0 b_0 (1 + \beta_{a1}\beta_{b1})(1 + \beta_{a2}\beta_{b2}) \ , \\ c_1 &= a_0 b_0 (\beta_{a1} + \beta_{b1})(1 + \beta_{a2}\beta_{b2}) \ , \\ c_2 &= a_0 b_0 (1 + \beta_{a1}\beta_{b1})(\beta_{a2} + \beta_{b2}) \ , \end{aligned} \tag{23}$$

we get Equation (18). The conditions $a_1 b_1 + a_0 b_0 \neq 0$ and $a_2 b_2 + a_0 b_0 \neq 0$ assure that $c_0 \neq 0$, so $\overset{o}{c} \in \mathbb{S}_*^{1+2}$, as needed. One can easily see that the above $\overset{o}{c}$ coincides with $\overset{o}{u}$ defined by Equation (8), which ends the proof. □

Remark 6. *Theorem 5 does not extend on all scators from the space $\mathbb{S}^{1+2}$. For instance,*

$$\begin{aligned} F(\overset{o}{\boldsymbol{e}}_1\overset{o}{b}) &= F(\overset{o}{\boldsymbol{e}}_1)F(\overset{o}{b}) \qquad (\textit{if } b_0 \neq 0 \textit{ and } b_1 \neq 0)\,, \\ F(\overset{o}{\boldsymbol{e}}_1\overset{o}{b}) &\neq F(\overset{o}{\boldsymbol{e}}_1)F(\overset{o}{b}) \qquad (\textit{for } \overset{o}{b} = b_0 + b_2\boldsymbol{e}_2, \quad b_0 \neq 0)\,. \end{aligned} \tag{24}$$

It is worth noting that conditions $a_1b_1 + a_0b_0 \neq 0$ and $a_2b_2 + a_0b_0 \neq 0$ in Theorem 5 are essential and necessary. If, for instance, $a_1b_1 + a_0b_0 = 0$, then $\overset{o}{a}\overset{o}{b}$ and $F(\overset{o}{a}\overset{o}{b})$ are proportional to $\boldsymbol{e}_1$, while $F(\overset{o}{a})F(\overset{o}{b})$ is, in general, a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_{12}$, compare Equation (22).

4. Extension of the Scator Product on All Scators with Vanishing Scalar Component

Definition 1 contains only two special cases when one factor can have vanishing scalar component. We may shortly say, that Definition 1 is applicable for all scators belonging to $\mathbb{S}^{1+2}$. In this section we propose an extension of this definition on the whole scator space $\mathbb{R}^{1+2}$. First, we consider the case when the first factor is of the form

$$\overset{o}{a} = a_1\overset{o}{\boldsymbol{e}}_1 + a_2\overset{o}{\boldsymbol{e}}_2, \tag{25}$$

while the second factor is assumed to belong to $\mathbb{S}^{1+2}_*$ (i.e., $b_0 \neq 0$):

$$\overset{o}{b} = b_0 + b_1\overset{o}{\boldsymbol{e}}_1 + b_2\overset{o}{\boldsymbol{e}}_2. \tag{26}$$

We base our treatment on discrete symmetries of the scator set, namely, reflection symmetries. Let us introduce

$$\overset{o}{a}_\varepsilon = \varepsilon + a_1\overset{o}{\boldsymbol{e}}_1 + a_2\overset{o}{\boldsymbol{e}}_2, \tag{27}$$

where we understand ε as a very small real positive number. Finally it will approach zero. We will consider products of the $\overset{o}{b}$ scator with both ε and $-\varepsilon$ versions of $\overset{o}{a}$. In this way, since product is non-distributive, we have to obtain two disparate results (divergent when $\varepsilon \to 0$) with troublesome terms having opposite signs. Therefore, to find the proper (finite) product we define it as

$$\overset{o}{a}\overset{o}{b} = \frac{\overset{o}{a}_\varepsilon\overset{o}{b} + \overset{o}{a}_{-\varepsilon}\overset{o}{b}}{2} = (a_1b_1 + a_2b_2;\ a_1b_0 + \frac{a_2b_1b_2}{b_0},\ a_2b_0 + \frac{a_1b_1b_2}{b_0}), \tag{28}$$

where, after straightforward calculation, we observe troublesome terms cancellation. This averaging procedure seems to be promising, as is confirmed by the following calculation. where two "deficient" scators are involved:

$$\overset{o}{a} = a_1\overset{o}{\boldsymbol{e}}_1 + a_2\overset{o}{\boldsymbol{e}}_2, \qquad \overset{o}{b} = b_1\overset{o}{\boldsymbol{e}}_1 + b_2\overset{o}{\boldsymbol{e}}_2\,. \tag{29}$$

Now we have four combinations of divergent terms which also can be cancelled by taking an appropriate sum. In the identical manner as before, we propose in the case of Equation (29) the following formula:

$$\overset{o}{a}\overset{o}{b} = \frac{\overset{o}{a}_\varepsilon\overset{o}{b}_\delta + \overset{o}{a}_{-\varepsilon}\overset{o}{b}_\delta + \overset{o}{a}_\varepsilon\overset{o}{b}_{-\delta} + \overset{o}{a}_{-\varepsilon}\overset{o}{b}_{-\delta}}{4} = a_1b_1 + a_2b_2\,, \tag{30}$$

where we use a standard convention $a_1b_1 + a_2b_2 \equiv (a_1b_1 + a_2b_2;\ 0,\ 0)$.

Finally, we will define the fundamental embedding for the scators of the form $\overset{o}{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2$, compare Equation (25). We use approach similar to what we performed above, i.e.,

$$F(\overset{o}{a}) = \frac{1}{2}(F(\overset{o}{a}_\varepsilon) + F(\overset{o}{a}_{-\varepsilon})) = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 = \overset{o}{a}\,. \tag{31}$$

Hence, taking into account Remark 4, we see that all coordinate axes and planes perpendicular to them, including the plane $(0;\ a_1,\ a_2)$, are invariant with respect to the fundamental embedding F.

5. Conclusions

We studied multiplication in the scator space of dimension $1+2$. Multiplication of scators is usually restricted to a subset which consists mostly of scators with non-vanishing scalar component. In this paper we proposed and discussed an extension of the scator product on the whole scator space. Theorem 5 gives a motivation and interpretation for the commonly used definition of this product (Definition 1) which otherwise is rather not very obvious. In the generic case, the scator product is induced by another product (in another space) which is commutative, associative and distributive over addition.

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