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On the Uniqueness Classes of Solutions of Boundary Value Problems for Third-Order Equations of the Pseudo-Elliptic Type

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Abstract: The paper is devoted to solutions of the third order pseudo-elliptic type equations. An energy estimates for solutions of the equations considering transformation's character of the body form were established by using of an analog of the Saint-Venant principle. In consequence of this estimate, the uniqueness theorems were obtained for solutions of the first boundary value problem for third order equations in unlimited domains. The energy estimates are illustrated on two examples.

Keywords: equations of the pseudo-elliptic type of third order; energy estimate; analog of the Saint-Venant principle

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1. Introduction

In the 19th century, A.J.C. Barré de Saint-Venant studied the planar theory of elasticity. His principle is expressed as a prior estimate for a solution of a biharmonic equation satisfying homogeneous boundary conditions of the first boundary value problem in the part of the domain boundary (c.f., [1,2]). Many recent recent results are inspired by Saint-Venant principle (c.f., [3–5] and many others).

The energetic estimates were received first in [6,7]. These estimates do not take into account character of transformation of the body form at moving off from those part of the bound where exterior forces are applied. In the paper [8], a proof of the Saint-Venant principle in the planar theory of elasticity was obtained by different way. The energetic estimate was gained in the connection considered character of transformation of the body form. The uniqueness theorem for the first boundary value problem of the planar theory of elasticity in unlimited domains and also Pharagmen–Lindelöf type theorems were obtained as a corollary of the energetic estimate. The proofs of the Pharagmen–Lindelöf type theorems were done for equations of the theory of elasticity in [9] and for elliptic equations of higher order in the papers [2,6,7,10–14]. The Saint-Venant principle for a cylindrical body was studied in [15].

Boundary value problems have applications in fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. Boundary value problems of higher order is studied in papers [16,17]. An overview of some results on the class of functions with subharmonic behaviour and their invariance properties under conformal and quasiconformal mappings is presented in [18].

An analog of the Saint-Venant principle, uniqueness theorems in unlimited domains, and Pharagmen–Lindelöf type theorems in the theory of elasticity were derived for the system of equations in the case of space with boundary conditions of the first boundary value problem (c.f., [19,20]). Similar results were obtained for the mixed problems in [21].

We shall note else work [12,22], which by means of principle Saint-Venant's is studied asymptotic characteristic of the solutions of the third order equations of the composite type and dynamic systems.

Boundary value problems have applications in fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics.

2. Notations and Formulation of the Problem

Consider in the unlimited domain Q the equation

$$L_0 lu + L_1 u + Mu = f(x, y, t) \tag{1}$$

where

$$lu = u_t + \alpha^k(x)u_{x_k} + \alpha_0(x)u, \quad L_1 u = b^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i},$$

$$L_o u = u_t - a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a_0(x)u,$$

$$Mu = c^{pq}(x)u_{y_p y_q} + c^p(x)u_{y_p} + c_0(x)u.$$

We suppose here and later on that the summation is carried out by repeating indexes, all coefficients in (1) and their derivatives are bounded and measurable in any finite subdomain of the domain Q. Furthermore, we suppose that boundary of Q is smooth or piecewise-smooth. We assume that the operators L_o, M are uniformly elliptic, i.e.,

$$a^{ij} = a^{ji}, \quad \lambda_0|\xi|^2 \le a^{ij}\xi_i\xi_j \le \lambda_1|\xi|^2, \quad \text{for all} \quad (x, y, t) \in Q \cup \partial Q, \quad \text{for all} \quad \xi \in \mathbb{R}^{n+m+1}$$

$$c^{pq} = c^{qp}, \quad \mu_0|\xi|^2 \le a^{ij}\xi_i\xi_j \le \mu_1|\xi|^2, \quad \text{for all} \quad (x, y, t) \in Q \cup \partial Q, \quad \text{for all} \quad \xi \in \mathbb{R}^{n+m+1}. \tag{2}$$

Let $G = D \times \Omega$ and $\nu(x) = (\nu_{x_1}, \ldots, \nu_{x_n}, \nu_{y_1}, \ldots, \nu_{y_m}, \nu_t)$ is a vector of the inner normal of Q in the point (x, y, t).

We break up the bound of Q. Denote

$$\sigma_0 = \{(x, y, t) \in \partial G \times (0, T) : \alpha^k \nu_k = 0\},$$

$$\sigma_1 = \{(x, y, t) \in \partial G \times (0, T) : \alpha^k \nu_k > 0\},$$

$$\sigma_2 = \{(x, y, t) \in \partial G \times (0, T) : \alpha^k \nu_k < 0\},$$

Consider in Q the boundary value problem

$$L_0 lu + L_1 u + Mu = f(x, y, t),$$

$$u|_{\partial Q} = 0, \quad \alpha^k u_{x_k}|_{\sigma_2} = 0. \tag{3}$$

Define the operator d:

$$du = (b^{ij} + \alpha^k a^{ij}_{x_k} - \alpha_0 a^{ij} + a^{ij}_t)u_{x_i x_j} + (b^i + \alpha_0 a^i - \alpha^i a^k_{x_k} + \alpha^i a_0 - a^i_t)u_{x_i} + (a_{0_t} - \alpha_0 a_0)u \equiv$$

$$d^{ij}u_{x_i x_j} + d^i u_{x_i} + du.$$

Assume that the condition

$$d^{ij} = d^{ji}, \quad \gamma_0|\xi|^2 \le d^{ij}\xi_i\xi_j \le \gamma_1|\xi|^2, \quad \text{for all} \quad (x, y, t) \in Q \cup \partial Q, \quad \text{for all} \quad \xi \in \mathbb{R}^{n+m+1} \tag{4}$$

holds.

Let

$$Q_\tau = Q \cap \{(x,y,t) : 0 < y_1 < \tau\}, \quad \partial G_\tau = \partial G \cap \{y : 0 < y_1 < \tau\},$$
$$\sigma_{0,\tau} = \{(x.y.t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k = 0\},$$
$$\sigma_{1,\tau} = \{(x,y,t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k > 0\},$$
$$\sigma_{2,\tau} = \{(x,y,t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k < 0\}.$$

For some $h > 0$, define

$$\sigma_{2,h,\tau} = \{(x,y,t) \in \sigma_{2,\tau} : \rho((x,y,t), \partial\sigma_{2,\tau}) > h\}, \quad \sigma_{2,\tau}^h = \sigma_{2,\tau} \setminus \sigma_{2,h,\tau}.$$

Let $E(Q_\tau)$ be a set of functions $v \in C^2\left(\overline{Q}_\tau\right)$ such that $v = 0$ in $\partial G_\tau \times (0,T)$ and $\alpha^k v_{x_k} = 0$ on $\sigma_{0,\tau} \cup \sigma_{1,\tau} \cup \sigma_{2,\tau}^h$ for some $h > 0$.

We denote as $H(Q_\tau)$ the Hilbert space obtained by closing $E(Q_\tau)$ with respect to the norm

$$\|u\|_{H(Q_\tau)} = \left\{ \int\limits_{Q_\tau} \left(d_1^{ij} u_{x_i} u_{x_j} + u_{y_p} u_{y_q} + u_t^2 + u^2 \right) dx\, dy\, dt - \int\limits_{\sigma_{2,\tau}} \alpha^k \nu_k a^{ij} u_{x_i} u_{x_j} ds \right\}^{\frac{1}{2}},$$

where

$$d_1^{ij} = -\frac{1}{2}\alpha^j a_{x_j}^{ij} - \frac{1}{2} a_t^{ij} + \alpha^j a^i + d^{ij} - \frac{1}{2\lambda_0} a^{ij},$$

$$d_1^{ij} = d_1^{ji}, \quad \beta_0 |\xi|^2 \le d_1^{ij} \xi_i \xi_j \le \beta_1 |\xi|^2, \quad \text{for all} \quad (x,y,t) \in Q \cup \partial Q, \quad \text{for all} \quad \xi \in \mathbb{R}^{n+m+1}.$$

Now consider bilinear form

$$a(u,v) = \int\limits_{Q_\tau} \Big[\alpha^k a^{ij} u_{x_i} v_{x_j x_k} + a^{ij} u_{x_i} v_{x_j t} + \left(\alpha^k a_{x_j}^{ij} - \alpha^i a^k \right) u_{x_i} v_{x_j} +$$
$$d^{ij} u_{x_i} v_{x_j} + \left(d^i - d_{x_j}^{ij} \right) u v_{x_i} + \left(a_{x_i}^{ij} + a^i + \alpha^i \right) u_{x_i} v_t + c^{pq} u_{y_p} v_{y_q} + \left(c^p - c_{y_q}^{pq} \right) u v_{y_p} +$$
$$u_t v_t + (\alpha_0 + a_0) u v_t + \left(c_{y_p}^p - c_0 - c_{y_p y_q}^{pq} + d + d_{x_i}^i + d_{x_i x_j}^{ij} \right) u v \, dx\, dy\, dt.$$

Definition 1. *If $u(x,y,t) \in H(Q_\tau)$ for any $\tau < +\infty$ and*

$$a(u,v) = \int\limits_{Q_\tau} f v \, dx\, dy\, dt \tag{5}$$

for an arbitrary function $v \in E(Q_\tau)$, $v|_{S_\tau} = 0$ where $S_\tau = Q \cap \{(x,y,t) : y_1 = \tau\}$, then the function $u(x,y,t)$ is said to be a generalized solution of the problem (1),(3) in the domain Q.

3. Energy Inequalities

Theorem 1. (Analog of the Saint-Venant principle)

Let $-1 \le a_{x_i}^{ij} + a^i + a_0 \le 0$; $\theta \equiv d_0 - \frac{1}{2} d_{x_i x_j}^{ij} + \frac{1}{2} d_{x_i}^i - \frac{1}{2} c_{y_p y_q}^{pq} + \frac{1}{2} c_{y_p}^p - c_0 \le \theta_0 < 0$, for all $(x,y,t) \in Q \cup \partial Q$.

If $u(x,y,t)$is generalized solution of the problem (1), (3) and $f(x,y,t) = 0$ at $y_1 \le \tau_2$, then for any τ_1 such that $0 \le \tau_1 \le \tau_2$, takes place

$$\int\limits_{Q_{\tau_1}} E(u) dx\, dy\, dt \le \Phi^{-1}(\tau_1, \tau_2) \int\limits_{Q_{\tau_2}} E(u) dx\, dy\, dt \tag{6}$$

where $E(u) = d^{ij} u_{x_i} u_{x_j} + c^{pq} u_{y_p} u_{y_q} + u_t^2 - \theta u^2$.

Here $\Phi(\tau,\tau_2)$ *is a solution of the problem*

$$\begin{gathered}\Phi' = -\mu(\tau)\Phi, \quad \tau_1 \le \tau \le \tau_2, \\ \Phi(\tau_2,\tau_2) = 1,\end{gathered} \tag{7}$$

$\mu(\tau)$ *is an arbitrary continuous function such that*

$$0 < \mu(\tau) \le \inf_N \left\{ \int_{S_\tau} E(v)dx\,dy'\,dt \left| \int_{S_\tau} P(v)dx\,dy'\,dt \right|^{-1} \right\}, \tag{8}$$

$$y' = (y_2, y_3, \dots, y_m),$$

$$P(v) = -c^{p1} v v_{y_p} + \frac{1}{2}\left(c^1 - c^{1q}_{y_q}\right) v^2, \tag{9}$$

N is the set of continuously differentiable functions in the neighborhood of $\overline{S_\tau}$ *which are equal to zero in* $\overline{S_\tau} \cap (\partial G_\tau \times (0,T))$.

Proof. Assume in (5) $v = u_m(\Psi(y_1) - 1)$ where $\Psi(y_1) = \Phi(\tau_1,\tau_2)$ if $0 \le y_1 \le \tau_1$, $\Psi(y_1) = \Phi(y_1,\tau_2)$ if $\tau_1 \le y_1 \le \tau_2$, and $\Psi(y_1) = 1$ if $\tau_2 \le y_1$.

$$u_m \in E(Q_\tau), \quad \|u_m - u\|_{H(Q_\tau)} \to 0, \quad u \in H(Q).$$

Then

$$a(u - u_m + u_m, u_m(\Psi - 1)) = 0 \text{ in } Q_{\tau_2}.$$

Therefore

$$a(u_m, u_m(\Psi - 1)) = \delta_m \text{ in } Q_{\tau_2} \tag{10}$$

where $\delta_m = -a(u - u_m, u_m(\Psi - 1))$.

It is obvious that $\delta_m \to 0$ at $m \to +\infty$. Integrating by parts (10), we have

$$\int_{Q_{\tau_2}} E(u_m)(\Psi - 1)dx\,dy\,dt \le \int_{Q_{\tau_2}} P(u_m)\Psi' dx\,dy\,dt + \delta_m.$$

Hence

$$\int_{Q_{\tau_2}} E(u_m)(\Psi - 1)dx\,dy\,dt \le \int_{Q_{\tau_2}\setminus Q_{\tau_1}} P(u_m)\mu\Psi dx\,dy\,dt + \delta_m. \tag{11}$$

The estimation (6) follows from (8) and (11) at $m \to +\infty$. □

Now we will estimate $\mu(y_1)$ in case when S_τ can be included to the $(n+m)$-dimensional parallelepiped which smallest edge is equal to $\lambda(\tau)$. Suppose that

$$\max_{S_\tau}\left\{ \left(\frac{1}{2}c^1 - c^{1q}_{y_q}\right), 0 \right\} = \gamma(\tau), \quad \max_{S_\tau} c_{p1} = \beta(\tau).$$

Applying the Friedreich and Cauchy–Bunyakovsky inequalities, we have from (9)

$$\left| \int_{S_\tau} P(v)dx\,dy'\,dt \right| \le \left| \int_{S_\tau} c^{p1} v v_{y_p} dx\,dy'\,dt \right| + \left| \int_{S_\tau} \frac{1}{2}\left(c^1 - c^{1q}_{y_q}\right) v^2 dx\,dy'\,dt \right| \le$$

$$\beta(\tau)\left[\int\limits_{S_\tau} v^2 dx\, dy'\, dt\right]^{\frac{1}{2}}\left[\int\limits_{S_\tau} v_{y_p}^2 dx\, dy'\, dt\right]^{\frac{1}{2}} + \gamma(\tau)\int\limits_{S_\tau} v^2 dx\, dy'\, dt \leq$$

$$\left(\frac{\beta(\tau)\lambda(\tau)}{\pi\gamma_0} + \frac{\gamma(\tau)\lambda^2(\tau)}{\pi^2\gamma_0}\right)\int\limits_{S_\tau} E(v) dx\, dy'\, dt.$$

Therefore we can set

$$\mu(\tau) = \pi^2\gamma_0\left(\pi\beta(\tau)\lambda(\tau) + \lambda^2(\tau)\gamma(\tau)\right)^{-1}.$$

If $\left(c^1 - 2c_{y_q}^{1q}\right) \leq 0$ in S_τ, then $\gamma(\tau) = 0$. Consequently

$$\mu(\tau) = \frac{\pi\gamma_0}{\beta(\tau)\lambda(\tau)}. \tag{12}$$

Example 1.

1. *Let as $y_1 \geq \tau_1 \geq 0$, the domain Q lies inside the rotation body $|y'| \leq \frac{M}{2}(y_1+1)$, i.e., $\lambda(y_1) \leq M(y_1+1)$, $M > 0$. We have from (15)*

$$\mu(y_1) = \frac{\pi c(y_1)}{M(y_1+1)}, \quad c(y_1) = \frac{d_0}{\beta(y_1)}.$$

   Suppose that $c(x_1) = c = const > 0$.

   In this case, from the inequality (6) we have

$$\int\limits_{Q_{\tau_1}} E(u) dx\, dy\, dt \leq \Phi^{-1}(\tau_1, \tau_2)\int\limits_{Q_{\tau_2}} E(u) dx\, dy\, dt \leq \left(\frac{\tau_1+1}{\tau_2+1}\right)^{\pi c}\int\limits_{Q_{\tau_2}} E(u) dx\, dy\, dt.$$

2. *Consider an example of Q for which*

$$\lambda(y_1) \leq \pi c\left[(y_1+1)^{k-1}\right]^{-1}, k = const > 0.$$

   It is clear that if $k > 1$, the domain Q is narrowing at $x_1 \to +\infty$. If $k = 1$, then $\lambda(x_1) \leq \pi c$ and this case includes domains lying in the band with the width πc. If $0 < k < 1$, then Q can be extended respectively at $x_1 \to +\infty$. For this kind of domains, we can assume

$$\mu(y_1) \leq (y_1+1)^{k-1}.$$

   Then the estimate (6) is valid for considered domains if

$$\Phi^{-1}(\tau_1, \tau_2) = 2\exp\left[-(\tau_2+1)^k + (\tau_1+1)^k\right].$$

As a corollary of the Saint-Venant principle, we have the uniqueness theorem for the problem (1), (3) in unlimited domain Q for classes of functions increasing in infinity depending from $\lambda(\tau)$.

Theorem 2. *Let $f(x,y,t) = 0$ in Q and conditions of theorem 1 hold. If $u(x,y,t)$ is a generalized solution of the problem (1), (3) in Q and for a sequence $\tau_m \to +\infty$ at $m \to +\infty$ and some $r_* = const > 0$,*

$$\int\limits_{Q_{\tau_m}} E(u)dx\,dy\,dt \leq \varepsilon(\tau_m)\Phi(r_*, \tau_m) \tag{13}$$

where $\varepsilon(\tau_m) \to 0$ at $\tau_m \to +\infty$, then $u = 0$ in Q_{r_*}.

Proof. We have from (6) considering (13)

$$\int\limits_{Q_{r_*}} E(u)dx\,dy\,dt \leq \Phi^{-1}(r_*, \tau_m) \int\limits_{Q_{\tau_2}} E(u)dx\,dy\,dt \leq \varepsilon(\tau_m) \to 0$$

at $\tau_m \to +\infty$. Hence $u = 0$ in Ω_{d_*}.

Further for any fixed $r_1 > r_*$, we have

$$\Phi(r_*, \tau_m) = e^{\int\limits_{r_*}^{\tau_m} \mu(s)ds} = e^{\int\limits_{r_1}^{\tau_m} \mu(s)ds}\, e^{\int\limits_{r_*}^{r_1} \mu(s)ds} = c\Phi(r_1, \tau_m)$$

Therefore

$$\int\limits_{Q_{e_1}} E(u)dx\,dy\,dt \leq \Phi^{-1}(r_1, \tau_m) \int\limits_{Q_{\tau_m}} E(u)dx\,dy\,dt \leq \Phi^{-1}(r_1, \tau_m)\varepsilon(\tau_m)\Phi(r_*, \tau_m) =$$

$$c^{-1}\varepsilon(\tau_m) \to 0 \text{ as } \tau_m \to +\infty.$$

Hence, $u = 0$ in Q_{r_1}. Since r_1 was chosen arbitrary, $u = 0$ in Q. □

4. Conclusions

In the present paper, the analogy of the Saint-Venant principle is established for the generalized solution of the third order pseudoelliptical type equation. Furthermore, uniqueness theorems are obtained for solutions of the first boundary value problem in classes of functions increasing in infinity depending on the geometric characteristics of the domain $Q = D \times \Omega \times (0, T)$, were $D \subset \mathbb{R}^n_+ = \{y : y_1 > 0\}$, Ω is bounded domain. Boundary value problems for the third order pseudoelliptical type equations in bounded domains were considered in [13].

The main goal of our research on these problems consists of the following parts:

(1) Establish energy estimates (analogous to the Saint-Venant's principle) that allow us to determine the widest class of uniqueness of solutions to the problem depending on the geometric characteristics of the domain.
(2) Construction of the solution of the problem under study on an unbounded domain in classes of functions growing at infinity.
(3) Establish estimates for solutions of the problem and its derivatives at infinitely remote boundary points.

The first part of our research on these problems is given in this paper. The remaining two parts will be studied in the future, which will be performed on the basis of this paper. Therefore, the results of this article are necessary and relevant for further qualitative research to solve third-order equations in the vicinity of irregular boundary points.

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Abbreviations

The following abbreviations are used in this manuscript:

A.R.K	Abdukomil Risbekovich Khashimov
D.S.	Dana Smetanová

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